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ASYMPTOTIC REPRESENTATIONS OF THE PROPORTION OF THE SAMPLE BELOW THE SAMPLE MEAN FOR ϕ-MIXING RANDOM VARIABLES

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Abstract. Let $\{\mathrm{X}_i;\ -\infty < i < \infty\}$ be a stationary sequence of random variables. Let $\mathrm{F}_n(x)$ be the corresponding empirical distribution function of $\mathrm{X}_1, \ldots, \mathrm{X}_n$, and let $\bar{\mathrm{X}} = \sum_{i=1}^n \mathrm{X}_i/n$ be the sample mean. In this paper, we derive the asymptotic almost sure representation, the central limit theorem, a law of iterated logarithm, a Wiener precess embedding and an invariant principle for $\mathrm{F}_n(\bar{\mathrm{X}})$ under different ϕ–mixing conditions.

1. Introduction

Let $\{\mathrm{X}_i;\ -\infty < i < \infty\}$ be a stationary sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let $\mathcal{M}_{-\infty}^k$ and $\mathcal{M}_{k+n}^\infty$ be respectively the σ–fields generated by $\{\mathrm{X}_i;\ i \le k\}$ and $\{\mathrm{X}_i;\ i \ge k+n\}$. We say that $\{X_i; -\infty < i < \infty\}$ is ϕ-mixing if $\mathrm{A} \in \mathcal{M}_{-\infty}^k$ and $\mathrm{B} \in \mathcal{M}_{k+n}^\infty$, for all $k\ (-\infty < k < \infty)$ and $n\ (\ge 1)$,

$$|\mathrm{P}(\mathrm{B}|\mathrm{A}) - \mathrm{P}(\mathrm{B})| \le \phi(n), \quad \phi(n) \ge 0 \quad \text{for all } n \ge 1 \tag{1}$$

where $\phi(n) \downarrow$ in n and $\lim_{n\to\infty} \phi(n) = 0$.

Throughout this work, $\{\mathrm{X}_i;\ -\infty < i < \infty\}$ is a stationary sequence of ϕ–mixing random variables from a distribution function F. For $n \ge 1$, let $\mathrm{F}_n(x) = n^{-1} \sum_{i=1}^n c(x - \mathrm{X}_i)$ be the empirical distribution function based on $\mathrm{X}_1, \ldots, \mathrm{X}_n$, where $c(u) = 1$ if $u \ge 0$ and $c(u) = 0$ otherwise, and let $\bar{\mathrm{X}} = \sum_{i=1}^n \mathrm{X}_i/n$ be the sample mean. The simplest test of the null hypothesis (H_0) that n observations are from a distribution which is symmetric about a specified value ξ is the sign test

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which is defined $F_n(\xi) = n^{-1}\sum_{i=1}^{n} c(\xi - X_i)$. $nF_n(\xi)$ has a binomial distribution with parameter n and $\theta = F(\xi)$, and $EX_i = \xi$ under the null hypothesis. A practical difficulty in using the sign test is that the assumed center ξ must be known. In testing the null hypothesis (H_0') that the observations are from a distribution symmetric about a unknown center ξ, one can estimate ξ by the sample mean $\bar{X}$ and use a modified sign test $F_n(\bar{X}) = n^{-1}\sum_{i=1}^{n} c(\bar{X} - X_i)$. The statistic $T_n = F_n(\bar{X})$ represents the proportion of the sample below the sample mean which is often used in estimating a functional $\theta = F(\xi)$ where $\xi = E(X_1)$ if both F and ξ are unknown or in testing F is symmetric about a unknown location ξ against certain classes of alternatives. Related work for which such statistics are appropriate may be found, for example, in Blomqvist (1950), David (1962), Mustafi (1968), Gastwirth (1971), Ghosh (1971), Flinger and Wolfe (1979) and Ralescu and Puri (1984). For sequence of independent and identically distributed random variables (i.i.d.r.v.), Ghosh (1971) obtained the following result :

Assume that $0 < \mathrm{Var}(X_1) < \infty$ and $0 < F'(\xi) < \infty$. Then

$$F_n(\bar{X}) = F_n(\xi) + (\bar{X} - \xi)F'(\xi) + R_n \tag{2}$$

where $R_n = o_p(n^{-\frac{1}{2}})$, and the limiting distribution of $n^{-\frac{1}{2}}[F_n(\bar{X}) - F(\xi)]$ is normal with mean zero and variance $\mathrm{Var}[c(\xi - X_1) + (X_1 - \xi)F'(\xi)] > 0$. Ralescu and Puri (1984) established, under stronger assumptions, the improved stronger rate, namely $O(n^{-\frac{3}{4}}(\log n)^{\frac{1}{2}})(\log\log n)^{\frac{1}{4}})$, and the law of iterated logarithm. The object of the present investigation is to extend these results for ϕ–mixing processes which include independent, m–dependent, autoregressive and moving average processes as special cases.

In what follows we make the following assumptions in different situations on the mixing coefficient $\phi(n)$.

$$\sum_{n=1}^{\infty} [\phi(n)]^{\frac{1}{2}} < \infty, \tag{3}$$

$$\phi(n) = O(n^{-5}) \quad \text{as } n \to \infty, \tag{4}$$

and for some $t > 0$

$$\sum_{n=1}^{\infty} e^{tn}\phi(n) < \infty. \tag{5}$$

For $n \geq 1$, let g_n be a real-valued Borel measurable function on the real line. Consider the double sequence of random variables

$$Y_{ni} = g_n(X_i),\ i = 1, \ldots, n; \quad P\{Y_{ni} = 1\} = 1 - P\{Y_{ni} = 0\} = p_n, \tag{6}$$

where $0 < p_n < 1$. Define

$$\mathrm{S}_n = \mathrm{Y}_{n1} + \cdots + \mathrm{Y}_{nn}. \tag{7}$$

Then we have the following.

Lemma 1.1. *If* $0 < p_n \le kn^{-\frac{3}{8}} \log n$ $(k > 0)$*, then under (4), for every* $s > 0$*, and* $\mathrm{C} > 0$*, there exists positive* $n_0(s)$ *and* C_s *such that* $n \ge n_0(s)$*,*

$$\mathrm{P}\{\mathrm{S}_n - np_n > \mathrm{C}n^{\frac{3}{8}} \log n\} \le \mathrm{C}_s n^{-s}. \tag{8}$$

Proof. By the Chebyshev inequality, for every $h > 0$

$$\begin{aligned} &\mathrm{P}\{\mathrm{S}_n - np_n > \mathrm{C}n^{\frac{3}{8}} \log n\} \\ &= \mathrm{P}\{\exp(-hnp_n - h\mathrm{C}n^{\frac{3}{8}} \log n + h\mathrm{S}_n) > 1\} \\ &\le \exp(-hnp_n - h\mathrm{C}n^{\frac{3}{8}} \log n)\mathrm{E}\{\exp(h\mathrm{S}_n)\}. \end{aligned} \tag{9}$$

Choose $k_n = [n^{\frac{1}{8}}(\log n)^{-\frac{1}{6}}]$, where $[x]$ denotes the largest integer $\le x$. Let

$$\mathrm{S}_n^{(j)} = \mathrm{Y}_{nj} + \mathrm{Y}_{n,j+k_n} + \cdots + \mathrm{Y}_{n,j+m_n^{(j)}k_n}, \quad 1 \le j \le k_n \tag{10}$$

where $m_n^{(j)}$ is the largest integer for which $j + m_n^{(j)}k_n \le n$. Then S_n can be written as

$$\mathrm{S}_n = \mathrm{S}_n^{(1)} + \cdots + \mathrm{S}_n^{(k_n)}. \tag{11}$$

We note that

$$m_n^{(j)} \le m_n^{(1)} \quad \text{and} \quad n - k_n \le m_n^{(1)}k_n \le n - 1, \quad j = 1, \ldots, k_n. \tag{12}$$

Then, by the Jensen inequality and the stationarity of $\{\mathrm{Y}_i;\ -\infty < i < \infty\}$, we have for every $h > 0$

$$\begin{aligned} \mathrm{E}\left\{\exp\left(h\mathrm{S}_n\right)\right\} &= \mathrm{E}\left\{\exp\left(h\textstyle\sum_{j=1}^{k_n} \mathrm{S}_n^{(j)}\right)\right\} \\ &\le \mathrm{E}\left\{k_n^{-1}\textstyle\sum_{j=1}^{k_n} \exp\left(h\mathrm{S}_n^{(j)}\right)\right\}^{k_n} \\ &\le k_n^{-1}\textstyle\sum_{j=1}^{k_n} \mathrm{E}\left\{\exp\left(hk_n\mathrm{S}_n^{(j)}\right)\right\} \\ &\le \mathrm{E}\left\{\exp\left(hk_n\mathrm{S}_n^{(1)}\right)\right\}. \end{aligned} \tag{13}$$

Now, for $0 \leq j \leq m_n^{(1)}$, by an elementary computation,

$$\begin{aligned} &\mathrm{E}\left\{\exp\left(hk_n \mathrm{Y}_{n,1+jk_n}\right) \mid \mathcal{M}_{-\infty}^{1+(j-1)k_n}\right\} \\ \leq\ & 1 + \left[\exp\left(hk_n\right) - 1\right]\left[p_n + \phi(k_n)\right]. \end{aligned} \tag{14}$$

Hence

$$\begin{aligned} \mathrm{E}\left\{\exp\left(hk_n \mathrm{S}_n^{(1)}\right)\right\} &= \mathrm{E}\left\{\mathrm{E}\left\{\cdots\left\{\mathrm{E}\left\{\exp\left(hk_n \mathrm{S}_n^{(1)}\right) \mid \mathcal{M}_{-\infty}^{1+(m_n^{(1)}-1)k_n}\right\} \mid \right.\right.\right. \\ &\qquad \left.\left.\left. \mathcal{M}_{-\infty}^{1+(m_n^{(1)}-2)k_n}\right\} \mid \cdots \mid \mathcal{M}_{-\infty}^{1}\right\}\right\} \\ &\leq \left\{1 + \left[p_n + \phi(k_n)\right]\left[\exp(hk_n) - 1\right]\right\}^{m_n^{(1)}+1} \\ &\leq \exp\left\{\left(m_n^{(1)} + 1\right)\left[hp_n k_n + h\phi(k_n)k_n \right.\right. \\ &\qquad \left.\left. + \tfrac{1}{2}p_n\left(1 - p_n\right)h^2 k_n^2 + \mathrm{O}\left(p_n h^3 k_n^3\right)\right]\right\} \\ &\leq \exp\left\{nhp_n + nh\phi(k_n) + np_n h^2 k_n + \mathrm{O}\left(np_n h^3 k_n^2\right)\right\}. \end{aligned} \tag{15}$$

Choose $h = 2sn^{-\frac{3}{8}}/\mathrm{C}$. Then $nh\phi(k_n) = \mathrm{O}((\log n)^{\frac{5}{6}})$ as $n \to \infty$. From (9), (13) and (15), it follows that there exists positive $n_0(s)$ and C_s such that $n \geq n_0(s)$

$$\begin{aligned} \mathrm{P}\left\{\mathrm{S}_n - np_n > \mathrm{C}n^{\frac{3}{8}}\log n\right\} &\leq \exp(-2s\log n + \mathrm{O}((\log n)^{\frac{5}{6}})) \\ &\leq \mathrm{C}_s n^{-s}. \end{aligned} \tag{16}$$

This completes the proof.

Lemma 1.2. *If* $0 < p_n < kn^{-\frac{1}{2}}(\log\log n)$ $(k > 0)$, *then under (5)* $(t \geq \frac{3}{4})$, *for every* $s > 0$ *and* $\mathrm{C} > 0$, *there exists positive* $n_0(s)$ *and* C_s *such that* $n \geq n_0(s)$,

$$\mathrm{P}\left\{\mathrm{S}_n - np_n \geq \mathrm{C}n^{\frac{1}{4}}(\log n)^{\frac{3}{2}}\right\} \leq \mathrm{C}_s n^{-s}. \tag{17}$$

Proof. On choosing $k_n = [\log n] + 1$, $h = \frac{2s}{\mathrm{C}}n^{-\frac{1}{4}}(\log n)^{-\frac{1}{2}}$, the proof follows on the same line as in Lemma 1.1.

Applying Lemma 1.1 and Lemma 1.2 , we can prove the following two propositions.

Proposition 1.3. *Let* $\{a_n\}$ *be a sequence of positive constants such that* $a_n \sim kn^{-\frac{3}{8}}\log n$, *as* $n \to \infty$, *for some constant* k. *Then under* (4),

$$\begin{aligned} &\sup_{|x-\xi|\leq a_n} \left|\left[\mathrm{F}_n(x) - \mathrm{F}(x)\right] - \left[\mathrm{F}_n(\xi) - \mathrm{F}(\xi)\right]\right| \\ &= \mathrm{O}(n^{-\frac{5}{8}}\log n) \quad \text{a.s. as } n \to \infty \end{aligned} \tag{18}$$

Proof. Let $\{b_n\}$ be a sequence of positive integers such that $b_n \sim n^{\frac{1}{4}}$ as $n \to \infty$. For $n \geq 1$, let

$$\eta_{r,n} = \xi + r a_n b_n^{-1} \quad \text{for} \quad r = 0, \pm 1, \dots, b_n. \tag{19}$$

For $x \in [\xi - a_n, \xi + a_n]$, there exists an integer r, $-b_n \leq r \leq b_n - 1$, such that $x \in [\eta_{r,n}, \eta_{r+1,n}]$. Since F_n and F are no-decreasing in x, it is clear that

$$\begin{aligned} &[\mathrm{F}_n(x) - \mathrm{F}(x)] - [\mathrm{F}_n(\xi) - \mathrm{F}(\xi)] \\ \leq\ &[\mathrm{F}_n(\eta_{r+1,n}) - \mathrm{F}_n(\xi)] - [\mathrm{F}(\eta_{r,n}) - \mathrm{F}(\xi)] \\ =\ &[\mathrm{F}_n(\eta_{r+1,n}) - \mathrm{F}_n(\xi)] - [\mathrm{F}(\eta_{r+1,n}) - \mathrm{F}(\xi)] \\ &+ [\mathrm{F}(\eta_{r+1,n}) - \mathrm{F}(\eta_{r,n})]. \end{aligned} \tag{20}$$

Similarly

$$\begin{aligned} &[\mathrm{F}_n(x) - \mathrm{F}(x)] - [\mathrm{F}_n(\xi) - \mathrm{F}(\xi)] \\ \geq\ &[\mathrm{F}_n(\eta_{r,n}) - \mathrm{F}_n(\xi)] - [\mathrm{F}(\eta_{r,n}) - \mathrm{F}(\xi)] \\ &- [\mathrm{F}(\eta_{r+1,n}) - \mathrm{F}(\eta_{r,n})]. \end{aligned} \tag{21}$$

In (20) and (21), since F is sufficiently smooth in a fixed neighborhood of ξ, it follows that

$$\begin{aligned} \mathrm{F}(\eta_{r+1,n}) - \mathrm{F}(\eta_{r,n}) &= \mathrm{O}(a_n b_n^{-1}) \\ &= \mathrm{O}(n^{-\frac{5}{8}} \log n) \quad \text{as } n \to \infty. \end{aligned} \tag{22}$$

From (20), (21) and (22), it follows that

$$\begin{aligned} &\sup_{|x-\xi| \leq a_n} |\, [\mathrm{F}_n(x) - \mathrm{F}(x)] - [\mathrm{F}_n(\xi) - \mathrm{F}(\xi)] \,| \\ \leq\ &\max_{|r| \leq b_n} |\, [\mathrm{F}_n(\eta_{r,n}) - \mathrm{F}_n(\xi)] - [\mathrm{F}(\eta_{r,n}) - \mathrm{F}(\xi)] \,| \\ &+ \mathrm{O}(n^{-\frac{5}{8}} \log n) \quad \text{as } n \to \infty. \end{aligned} \tag{23}$$

It will therefore suffice to show that

$$\sum_{n=1}^{\infty} \mathrm{P}\left\{ \max_{|r| \leq b_n} |[\mathrm{F}_n(\eta_{r,n}) - \mathrm{F}_n(\xi)] - [\mathrm{F}(\eta_{r,n}) - \mathrm{F}(\xi)]| > n^{-\frac{5}{8}} \log n \right\} < \infty. \tag{24}$$

To prove this, we remark that for $1 \leq r \leq b_n$

$$[\mathrm{F}_n(\eta_{r,n} - \mathrm{F}_n(\xi)] - [\mathrm{F}(\eta_{r,n}) - \mathrm{F}(\xi)] = \frac{1}{n} \sum_{i=1}^{n} \mathrm{Y}_{n,i} - p_n \tag{25}$$

where $Y_{n,i} = c(\eta_{r,n} - X_i) - c(\xi - X_i)$, $i = 1, \ldots, n$, $p_n = F(\eta_{r,n}) - F(\xi)$ for which

$$\text{(26)} \qquad P\{Y_{n,i} = 1\} = 1 - P\{Y_{n,i} = 0\} = p_n \quad \text{and} \quad 0 < p_n < n^{-\frac{3}{8}} \log n.$$

Hence, by (26) and Lemma 1.1, we have on choosing $s = 2$ that there exists positive n_0 and C such that $n \geq n_0$

$$\text{(27)} \qquad P\{|[F_n(\eta_{r,n}) - F(\eta_{r,n})] - [F_n(\xi) - F(\xi)]| > n^{-\frac{5}{8}} \log n\} \leq Cn^{-2}.$$

Remark that from the proof of Lemma 1.1, it is clear that n_0 and C do not depend on r. The same inequality also holds for $-b_n \leq r \leq 1$. Hence, for $n \geq n_0$

$$\text{(28)} \quad P\left\{\max_{|r| \leq b_n} |\, [F_n(\eta_{r,n}) - F(\eta_{r,n}] - [F_n(\xi) - F(\xi)]\, | > n^{-\frac{5}{8}} \log n\right\} \leq Cn^{-\frac{7}{4}}.$$

(24) follows from (28).

Proposition 1.4. *Let $\{a_n\}$ be a sequence of positive constants such that $a_n \sim kn^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}}$, as $n \to \infty$, for some constant k. Then under (3) $(t \geq \frac{3}{4})$,*

$$\text{(29)} \qquad \begin{aligned} &\sup_{|x-\xi| \leq a_n} |\, [F_n(x) - F(x)] - [F_n(\xi) - F(\xi)]\, | \\ &= O(n^{-\frac{3}{4}}(\log n)^{\frac{3}{2}}) \quad \text{a.s. as } n \to \infty. \end{aligned}$$

Proof. Let $b_n \sim n^{\frac{1}{4}}(\log\log n)/(\log n)^{\frac{3}{2}}$ and $\eta_{r,n} = \xi + ra_n b_n^{-1}$. By using Lemma 1.2, the proof follows along the same line as in Proposition 1.3.

Proposition 1.3 and Proposition 1.4 are parallel to the result of Bahadur (1966). Using the method of Bahadur, we can derive the asymptotic normality for sample quantiles under mixing conditions.

The following lemma is due to Heyde and Scott (1973).

Lemma 1.5. *Let $\{Y_i; -\infty < i < \infty\}$ be ϕ–mixing sequence of random variables with $EY_1 = 0$ and mixing coefficient $\phi(n)$. Then, under (3), there exists a standard Wiener process $\{W(t); 0 \leq t < \infty\}$ such that*

$$\text{(30)} \qquad \frac{\sum_{i=1}^{n} Y_i}{\sigma_1} = W(n) + o((n \log\log n)^{\frac{1}{2}}) \quad \text{a.s. as } n \to \infty$$

provided that $\sigma_1^2 > 0$, where $\sigma_1^2 = EY_1^2 + 2\sum_{k=2}^{\infty} E(Y_1 Y_k)$.

2. Results and Proofs

It is assumed that $\mathrm{F}(x)$ is absolutely continuous in some neighborhood of ξ, and has a continuous density function $f(x)$, such that $0 < f(\xi) < \infty$, $f'(\xi)$ exists and $\mathrm{EX}_1^2 < \infty$. For convenience, we let $\mathrm{T}_n = \mathrm{F}_n(\bar{\mathrm{X}})$ and $\theta = \mathrm{F}(\xi)$. Our main theorems are the following.

Theorem 2.1. *Under* (4),

$$\mathrm{T}_n - \theta = n^{-1}\sum_{i=1}^{n} \mathrm{Z}_i + \mathrm{O}(n^{-\frac{5}{8}}\log n) \quad \text{a.s. as } n \to \infty \tag{31}$$

where $\mathrm{Z}_i = c(\xi - \mathrm{X}_i) - \theta + (\mathrm{X}_i - \xi)f(\xi),$, $i = 1, \ldots, n$.

Proof. By Lemma 1.5, there exists a standard Wiener process $\{\mathrm{W}(t);\, 0 \le t < \infty\}$ such that

$$\frac{\sum_{i=1}^{n}(\mathrm{X}_i - \xi)}{\sigma_1} = \mathrm{W}(n) + \mathrm{o}((n \log\log n)^{\frac{1}{2}}) \quad \text{a.s. as } n \to \infty. \tag{32}$$

From (32) and the classical law of iterated logarithm, it follows that

$$|\bar{\mathrm{X}} - \xi| \le 2\sigma_1 n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}} \le kn^{-\frac{2}{8}}\log n \quad \text{a.s. as } n \to \infty. \tag{33}$$

Now, using Proposition 1.3, we have

$$\mathrm{F}_n(\bar{\mathrm{X}}) = \mathrm{F}(\bar{\mathrm{X}}) + \mathrm{F}_n(\xi) - \mathrm{F}(\xi) + \mathrm{O}(n^{-\frac{5}{8}}\log n) \quad \text{a.s. as } n \to \infty. \tag{34}$$

By Theorem C page 45 in Serfling (1980) and (33),

$$\mathrm{F}(\bar{\mathrm{X}}) = \mathrm{F}(\xi) + (\bar{\mathrm{X}} - \xi)f(\xi) + \mathrm{O}(n^{-1}\log\log n) \quad \text{a.s. as } n \to \infty. \tag{35}$$

From (34) and (35), we have

$$\mathrm{F}_n(\bar{\mathrm{X}}) = \mathrm{F}(\xi) + (\bar{\mathrm{X}} - \xi)f(\xi) + \mathrm{F}_n(\xi) - \mathrm{F}(\xi) + \mathrm{O}(n^{-\frac{5}{8}}\log n) \tag{36}$$

and (31) follows.

Theorem 2.2. *Under* (5), $(t \ge \frac{3}{4})$,

$$\mathrm{T}_n - \theta = n^{-1}\sum_{i=1}^{n} \mathrm{Z}_i + \mathrm{O}(n^{-\frac{3}{4}}(\log n)^{\frac{3}{2}}) \quad \text{a.s. as } n \to \infty \tag{37}$$

where Z_i *is defined as in Theorem 2.1*

Proof. By applying Proposition 1.4, the proof follows on the same line as in Theorem 2.1.

2. Remarks and Applications

Define

$$\sigma^2 = \mathrm{E}\{\mathrm{Z}_1^2\} + 2\sum_{k=2}^{\infty} \mathrm{E}\{\mathrm{Z}_1\mathrm{Z}_k\} \tag{38}$$

where Z_i is defined as in Theorem 2.1. We remark that $\mathrm{EX}_1^2 < \infty$ implies $\mathrm{EZ}_1^2 < \infty$. By Lemma 1 on page 170 in Billingsley (1968),

$$\mathrm{E}\{\mathrm{Z}_1\mathrm{Z}_{k+1}\} \le 2\phi(k)^{\frac{1}{2}}\mathrm{E}\{\mathrm{Z}_1^2\}. \tag{39}$$

Therefore, under (3), $\sigma^2 < \infty$.

Theorem 3.1. *Under* (4),

$$\sqrt{n}\left(\frac{\mathrm{T}_n - \theta}{\sigma}\right) \xrightarrow{\mathcal{D}} \mathrm{N}(0,1) \quad \text{as} \quad n \to \infty, \tag{40}$$

provided $\sigma^2 > 0$, *where* σ^2 *is defined as in (38).*

Proof. By Theorem 2.1,

$$\sqrt{n}\left(\frac{\mathrm{T}_n - \theta}{\sigma}\right) = \frac{\sum_{i=1}^{n} \mathrm{Z}_i}{\sqrt{n}\sigma} + \mathrm{O}\left(n^{-\frac{1}{8}}\log n\right) \quad \text{a.s. as } n \to \infty. \tag{41}$$

(40) is a consequence of (49) and Lemma 1.5.

In the context of the law of iterated logarithm, Ralescu and Puri (1984), under i.i.d. case, has shown that $\mathrm{R}_n = \mathrm{O}(n^{-\frac{3}{4}}(\log n)^{\frac{1}{2}} \cdot\cdot (\log\log n)^{\frac{1}{4}})$ a.s. as $n \to \infty$. Since we do not need such a strong rate on R_n and (31), (32) suffice our purpose.

Theorem 3.2. *Under* (4), *there exists a standard Wiener Process* $\{\mathrm{W}(t); 0 \le t < \infty\}$ *such that*

$$n\left(\frac{\mathrm{T}_n - \theta}{\sigma}\right) = \mathrm{W}(n) + \mathrm{o}(n^{\frac{1}{2}}(\log\log n)^{\frac{1}{2}}) \quad \text{a.s. as } n \to \infty \tag{42}$$

provided $\sigma^2 > 0$. *Proof.* By Theorem 2.1,

$$n\left(\frac{\mathrm{T}_n - \theta}{\sigma}\right) = \frac{\sum_{i=1}^{n} \mathrm{Z}_i}{\sigma} + \mathrm{O}\left(n^{\frac{3}{8}}\log n\right) \quad \text{a.s. as } n \to \infty, \tag{43}$$

and (42) follows from (43) and Lemma 1.5.

As an immediate consequence of (42), we establish the law of iterated logarithm for T_n. **Theorem 3.3.** *Under* (4),

(44) $$\overline{\lim_{n\to\infty}} \frac{n^{\frac{1}{2}}(\mathrm{T}_n - \theta)}{\sqrt{2\sigma^2 \log\log n}} = 1 \quad \text{a.s.}$$

and

(45) $$\underline{\lim_{n\to\infty}} \frac{n^{\frac{1}{2}}(\mathrm{T}_n - \theta)}{\sqrt{2\sigma^2 \log\log n}} = -1 \quad \text{a.s.}$$

Proof.

(46) $$\begin{aligned}\frac{n^{\frac{1}{2}}(\mathrm{T}_n - \theta)}{\sqrt{2\sigma^2 \log\log n}} &= \frac{\mathrm{W}(n)}{\sqrt{2n\log\log n}} + \mathrm{o}(1) \\ &= \frac{\sum_{i=1}^{n}(\mathrm{W}(i) - \mathrm{W}(i-1))}{\sqrt{2n\log\log n}} + \mathrm{o}(1) \quad \text{a.s. as } n \to \infty\end{aligned}$$

(44) and (45) follow from (46) and the classical law of iterated logarithm.

For each $n \geq 1$, let W_n be a random variable on $[0,1]$ defined as follows :

(47) $$\mathrm{W}_n(0) = 0, \qquad \mathrm{W}_n(\frac{i}{n}) = \frac{i(\mathrm{T}_i - \theta)}{n^{\frac{1}{2}}\sigma}$$

and $\mathrm{W}_n(t)$, $0 \leq t \leq 1$, defined elsewhere, by linear interpolation. Then we have the following theorem.

Theorem 3.4. *Under* (4),

(48) $$\mathrm{W}_n \longrightarrow \mathrm{W}_1 \quad \text{in} \quad (\mathrm{C}[0,1], d)$$

where $\mathrm{C}[0,1]$ *is the space of all continuous functions on* $[0,1]$, d *is the uniform metric, and* W_1 *is a standard Wiener process on* $[0,1]$.

Proof. Let, for each $n \geq 1$,

(49) $$\mathrm{R}_n = \frac{n^{\frac{1}{2}}}{\sigma}\left(\mathrm{T}_n - \theta - n^{-1}\sum_{j=1}^{n}\mathrm{Z}_j\right).$$

Define

$$(50)\qquad W_n^*(0)=0,\ W_n^*(t)=\left(\frac{1}{\sigma\sqrt{n}}\right)\sum_{j=1}^{[nt]} Z_j+(nt-[nt])\left(\frac{1}{\sigma\sqrt{n}}\right)Z_{[nt]+1}$$

for $0<t\le 1$. The usual functional central limit theorem for ϕ-mixing processes (viz., Theorem 20.1 Billingsley (1968)) holds, so that

$$(51)\qquad W_n^*\longrightarrow W_1\quad \text{in } (C[0,1],d).$$

Therefore, it suffices to prove that

$$(52)\qquad d(W_n,W_n^*)\xrightarrow{\mathcal{P}}0\quad \text{as } n\to\infty.$$

But

$$(53)\qquad d(W_n,W_n^*)\le 3\max_{1\le i\le n}\frac{i^{\frac12}|R_i|}{n^{\frac12}}\quad\text{and}\quad \lim_{n\to\infty}|R_n|=0\quad\text{a.s.}$$

For any $\varepsilon>0$, $\delta>0$, by the Egoroff's theorem, there exists a set A such that $P(A)\ge 1-\frac{\delta}{2}$ and R_n converges to zero uniformly on A, so that there exists a positive integer n_0 such that $|R_n(\omega)|<\frac{\varepsilon}{3}$ for all $\omega\varepsilon A$ if $n\ge n_0$. Now

$$(54)\qquad \begin{aligned}&P\left\{3\sum_{1\le i\le n}\frac{i^{\frac12}|R_i|}{n^{\frac12}}>\varepsilon\right\}\\ &\le P\left\{\max_{1\le i\le n_0}\frac{i^{\frac12}|R_i|}{n^{\frac12}}>\frac{\varepsilon}{3}\right\}+P\left\{\max_{n_0\le i\le n}\frac{i^{\frac12}|R_i|}{n^{\frac12}}>\frac{\varepsilon}{3}\right\}.\end{aligned}$$

However,

$$(55)\qquad \begin{aligned}&P\left\{\sum_{n_0\le i\le n}\frac{i^{\frac12}|R_i|}{n^{\frac12}}>\frac{\varepsilon}{3}\right\}\\ \le\ &P\left(\left\{\max_{n_0\le i\le n}\frac{i^{\frac12}|R_i|}{n^{\frac12}}>\frac{\varepsilon}{3}\right\}\cap A\right)+P(A^c)\\ \le\ &P\left(\left\{\max_{n_0\le i\le n}|R_i|>\frac{\varepsilon}{3}\right\}\cap A\right)+\frac{\delta}{2}\\ =\ &\frac{\delta}{2}\end{aligned}$$

where A^c is the complement of set A. On the other hand, the first term on the right hand side of (54) is bounded by

$$(56)\qquad P\left\{\max_{1\le i\le n_0}|R_i|>\frac{\varepsilon n^{\frac12}}{3n_0^{\frac12}}\right\}\le\sum_{i=1}^{n_0}P\left\{|R_i|>\frac{\varepsilon n^{\frac12}}{3n_0^{\frac12}}\right\}.$$

It is plain that there exists $n_1 \geq n_0$ such that if $n \geq n_1$, $\mathrm{P}\{|\mathrm{R}_i| > \varepsilon n^{\frac{1}{2}}/3n_0^{\frac{1}{2}}\} > \delta/2n_0$, $1, \ldots, n_0$. Hence, for $n \geq n_1$

$$\mathrm{R}\left\{\max_{1\leq i\leq n_0} \frac{i^{\frac{1}{2}}|\mathrm{R}_i|}{n^{\frac{1}{2}}} > \frac{\varepsilon}{3}\right\} < \frac{\delta}{2}. \tag{57}$$

From (55) and (57) we obtain $\mathrm{P}\{3 \max_{1\leq i\leq n} \frac{i^{\frac{1}{2}}|\mathrm{R}_i|}{n^{\frac{1}{2}}} > \varepsilon\} < \delta$ if $n \geq n_1$, (57) follows. This proves the theorem.

Remark 3.5. Theorem 3.4 clearly generalizes Theorem 3.1, and by Theorem 1.5.1 in Csörgö and Révész (1981), it implies that for $x > 0$

$$\begin{aligned} \lim_{n\to\infty} \mathrm{P}\left\{\max_{1\leq i\leq n} i(\mathrm{T}_i - \theta) > x\sigma n^{\frac{1}{2}}\right\} &= \mathrm{P}\left\{\sup_{0\leq t\leq 1} \mathrm{W}_1(t) \geq x\right\} \\ &= 2[1-\phi(x)] \end{aligned} \tag{58}$$

Remark 3.6. Let $\{\mathrm{N}_k;\, k \geq 1\}$ be a sequence of nonnegative integer-valued random variables, and $k^{-1}\mathrm{N}_k \to \lambda$ in probability as $k \to \infty$, where λ is a positive random variable defined on the same space $(\Omega, \mathcal{F}, \mathrm{P})$. Then, by Theorem 2 of Mogyorodi (1965) and Proposition 1.3, under (4),

$$\frac{\mathrm{N}_k^{\frac{1}{2}}(\mathrm{T}_n - \theta)}{\sigma} \xrightarrow{\mathcal{D}} \mathrm{N}(0,1) \qquad \text{as } n \to \infty \tag{59}$$

provided $\sigma^2 > 0$. Also, as an immediate consequence of Lemma 1.5 and Theorem 20.3 of Billingsley (1968), we have

$$\mathrm{W}_{\mathrm{N}_k} \longrightarrow \mathrm{W}_1 \quad \text{in } (\mathrm{C}[0,1], d) \tag{60}$$

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