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ON REGULAR QB-IDEALS

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Abstract. Let I be a regular ideal of a ring R. It is shown that I is a QB-ideal if and only if for all finitely generated projective right R-module A with $AI = A$, if B_1 and B_2 are any right R-modules such that $A \oplus B_1 \cong A \oplus B_2$, then there exists a pair of orthogonal ideals I_1 and I_2 and $B_1 \oplus C_1 \cong B_2 \oplus C_2$ such that $C_1 I_1 \cong C_1$ and $CI_2 \cong C_2$.

1. Introduction

The theory of QB-rings has been developed by Ara, Pedersen and Perera to provide an infinite analogue of rings with stable range one. Following Ara et al. [2], we say that a ring R is a QB-ring when $aR + bR = R$ with $a, b \in R$ implies that $a + by \in R_q^{-1}$ for a $y \in R$. Let I be an ideal of a ring R. I is a QB-ideal of R if and only if whenever $xa - x - a + b = 0$ for x, a and b in I, there exists $y \in I$ such that $1 - (a - yb) \in R_q^{-1}$ (see [2] and [11]). Clearly, every ideal of a QB-ring R is a QB-ideal. An element $x \in R$ is regular in case there exists $y \in R$ such that $x = xyx$. We say that an ideal I of a ring R is regular if every element in I is regular. Let $M(R) = \{x \in R | RxR \text{ be a regular ideal}\}$. In view of [5, Theorem 1], $M(R)$ is the maximal regular ideal of R.

So far, most of investigation of the QB-ideals is only in an exchange ring. In this paper, we obtain a new characterization of a regular QB-ideal for an arbitrary ring. It is shown that a regular ideal I of a ring R is a QB-ideal if and only if for all finitely generated projective right R-module A with $AI = A$, if B_1 and B_2 are any right R-modules such that $A \oplus B_1 \cong A \oplus B_2$, then there exists a pair of orthogonal ideals I_1 and I_2 and $B_1 \oplus C_1 \cong B_2 \oplus C_2$ such that $C_1 I_1 \cong C_1$ and $CI_2 \cong C_2$.

Throughout the paper, all rings are associative with identity. We say that $x, y \in R$ are centrally orthogonal, in symbols $x \perp y$, if $xRy = 0$ and $yRx = 0$. We use

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R_q^{-1} to denote the set $\{u \in R \mid \exists a, b \in R$ such that $(1-ua)\perp(1-bu)\}$. If I_1 and I_2 are ideals of R, then $I_1 \perp I_2$ means that $x \perp y$ for all $x \in I_1, y \in I_2$, and we say that I_1 and I_2 are orthogonal ideals. The notation $M \lesssim^{\oplus} N$ means that M is isomorphic to a direct summand of N.

Lemma 1. *Let I be a regular ideal of R. Then the following are equivalent:*

(1) *I is a QB-ideal.*

(2) *eRe is a QB-ring for all idempotents $e \in I$.*

Proof. (1) $\Rightarrow$ (2) Given $ax+b=e$ with $a,x,b \in eRe, e \in I$, then $(a+1-e)(x+1-e)+b=1$ in R. As $a+1-e \in 1+I$, we have $y \in R$ such that $a+1-e+by \in R_q^{-1}$. By [2,Proposition 2.2], there exists $u \in R$ such that $\big(1-(a+1-e+by)u\big)\perp\big(1-(a+1-e+by)u\big)$. That is, $\big(1-(a+1-e+by)u\big)R\big(1-u(a+1-e+by)\big)=0$ and $\big(1-u(a+1-e+by)\big)R\big(1-(a+1-e+by)u\big)=0$; hence, $\big(e-(a+by)(ue)\big)(eRe)\big(e-(eue)(a+b(eye)\big)=0$ and $\big(e-(eue)(a+b(eye))\big)(eRe)\big(e-(a+by)(ue)\big)=0$. Furthermore, we get

$$
\begin{aligned}
&\big(e-(a+by)(ue)\big)(eRe)\big(e-(eue)(a+b(eye)\big)\\
=\ &\big(e-(a+by)(eue+(1-e)ue)\big)(eRe)\big(e-(eue)(a+b(eye)\big)\\
=\ &\big(e-(a+by)(eue)-(a+by)(1-e)ue)\big)(eRe)\big(e-(eue)(a+b(eye)\big)\\
=\ &0.
\end{aligned}
$$

Also we have $\big((1-e)-(1-e)u\big)(eRe)\big(e-(eue)(a+b(eye)\big)=0$; hence, $\big(-(a+by)(1-e)ue\big)(eRe)\big(e-(eue)(a+b(eye)\big)=0$. Clearly, we see that

$$
\begin{aligned}
&\big(e-(a+by)(eue)\big)(eRe)\big(e-(eue)(a+b(eye)\big)\\
\subseteq\ &\big(e-(a+by)(eue)-(a+by)(1-e)ue)\big)(eRe)\big(e-(eue)(a+b(eye)\big)\\
+\ &\big(-(a+by)(1-e)ue\big)(eRe)\big(e-(eue)(a+b(eye)\big),
\end{aligned}
$$

so we deduce that $\big(e-(a+by)(eue)\big)(eRe)\big(e-(eue)(a+b(eye)\big)=0$. Likewise, $\big(e-(eue)(a+b(eye)\big)(eRe)\big(e-(a+by)(eue)\big)=0$. Thus $a+b(eye) \in (eRe)_q^{-1}$, as required.

(2)$\Rightarrow$(1) Given $aR+bR=R$ with $a \in 1+I$ and $b \in R$, since I is regular, there exists $e=e^2 \in I$ such that $1-a=(1-a)e$; hence, $a(1-e)=1-e$. Suppose that $ar+bs=(1-a)e$ for some $r,s \in R$. Then $eae(e+ere)+ebse=eae(e+ere)+ea(1-e)re+ebse=e$. As eRe is a QB-ring, we can find $z \in eRe$ such that $eae+ebsez=u \in (eRe)_q^{-1}$. Set $w=(1-e)ae+(1-e)bsez$. By [2, Proposition 2.2], we have $v \in eRe$ such that $(e-uv)\perp(e-vu)$. Clearly,

$$\begin{aligned}
&1-(a+bsez)(v-wv+1-e)\\
&=1-a(1-e)-(a+bsez)(v-wv)\\
&=e+(a+bsez)wv-(a+bsez)v\\
&=e+awv-(a+bsez)v\\
&=e+wv-(a+bsez)v\\
&=e+\big(w-(a+bsez)\big)v\\
&=e+\big((1-e)ae+(1-e)bsez-(ae+bsez+a(1-e))\big)v\\
&=e-(eae+ebsez)v\\
&=e-uv.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&1-(v-wv+1-e)(a+bsez)\\
&=1-(v-wv+1-e)(ae+bsez)-(v-wv+1-e)a(1-e)\\
&=e-(v-wv+1-e)(ae+bsez)\\
&=e-w-(e-w)v(ae+bsez)\\
&=(e-w)-(e-w)vu\\
&=(e-w)(e-vu).
\end{aligned}$$

It follows from $(e-uv)\perp(e-vu)$ that $\big(1-(a+bsez)(v-wv+1-e)\big)\perp(v-wv+1-e)\big(1-(a+bsez)\big)$. Therefore $a+bsez \in R_q^{-1}$, as desired. ■

Lemma 2. *Let I be a regular ideal of R. Then the following are equivalent:*

(1) *I is a QB-ideal.*

(2) *For any idempotent $e \in I$, $M = A_1 \oplus H = A_2 \oplus K$ with $A_1 \cong eR \cong A_2$ implies that there exists a pair of orthogonal ideals I_1 and I_2 and $M = E \oplus B_1 \oplus H = E \oplus B_2 \oplus K$ such that $B_1I_1 = B_1$ and $B_2I_2 = B_2$.*

Proof. (1)⇒(2) Let $e \in I$ be an idempotent. By Lemma 1, $\varphi : End_R(eR) \cong eRe$ is a regular QB-ring. Given any right R-module decomposition $M = A_1 \oplus H = A_2 \oplus K$ with $A_1 \cong eR \cong A_2$. Using the decomposition $M = A_1 \oplus H \cong eR \oplus H$, we obtain projections $p_1 : M \to eR, p_2 : M \to H$ and injections $q_1 : eR \to M, q_2 : H \to M$ such that $p_1q_1 = 1, q_1p_1 + q_2p_2 = 1_M$ and $Kerp_1 = H$. Using the decomposition $M = A_2 \oplus K \cong eR \oplus K$, we obtain a projection $f : M \to eR$ and an injection $g : eR \to M$ such that $fg = 1$ and $Kerf = K$. From $(fq_1)(p_1g) + fq_2p_2g = f(q_1p_1 + q_2p_2)g = fg = 1$ in $End_R(eR)$, we can find some $u\in End_R(eR)_q^{-1}$ such that $fq_1+fq_2p_2gy=u$ for a $y\in End_R(eR)$. That is, $f(q_1+q_2p_2gy)=u$. Choose a quasi-inverse v for u and set $\alpha=vu,\beta=uv$. Let $\psi=q_1+q_2p_2gy$. Then $f\psi=u$ and $p_1\psi=1$. Let $D_1=ker\alpha p_1, D_2=ker\beta f$ and $E=\psi\alpha(eR)$. If $m\in E\cap D_1$, then $m=\psi\alpha(x)$ for some $x\in eR$. Hence $0=\alpha p_1(m)=\alpha p_1\psi\alpha(x)=vuvu(x)=vu(x)=\alpha(x)$, and then $m=0$. This means that $E\cap D_1=0$. Given any $m\in M$, we have $m=\psi\alpha p_1(m)+\big(m-\psi\alpha p_1(m)\big)\in E+D_1$. Thus $M=E\oplus D_1$. Likewise, $M=E\oplus D_2$. Let $B_1=p_1(e-\alpha)(eR)$ and $B_2=f(e-\beta)(eR)$. One easily checks that $D_1=B_1\oplus H$ and $D_2=B_2\oplus K$. Thus $M=E\oplus B_1\oplus H=E\oplus B_2\oplus K$. Let $I_1=R\big(e-\varphi(\alpha)\big)R$ and $I_2=R\big(e-\varphi(\beta)\big)R$. Then I_1 and I_2 are ideals of R. As $\big(e-\varphi(\alpha)\big)\perp\big(e-\varphi(\beta)\big)$, we deduce that $I_1\perp I_2$. Moreover, we have $B_1I_1=B_1$ and $B_2I_2=B_2$.

(2)⇒(1) Let $e\in I$ be an idempotent. Suppose that $a_1(eRe)+a_2(eRe)=eRe$ with $a_1,a_2\in eRe$. Set $M=eR\oplus eR$. Then we have a split epimorphism $\psi: M\to eR$ given by $\psi(s,t)=a_1s+a_2t$ for any $s\in eR, t\in eR$; hence, $M=A_2\oplus K$, where $K=ker\psi$ and $A_2\cong eR$. Therefore we get a pair of orthogonal ideals I_1 and I_2 and $M=E\oplus B_1\oplus eR=E\oplus B_2\oplus K$ such that $B_1I_1=B_1$ and $B_2I_2=B_2$. Let $\varphi: M=eR\oplus eR\to eR$ be the projection onto the first factor. Write $E_1=\varphi(E)$ and $B_1'=\varphi(B_1)$. Then $eR=E_1\oplus B_1'$. Let $h: eR=E_1\oplus B_1'\to E_1$ be the projection onto E_1. Then $h\in End_R(eR)$ is an idempotent. As $\alpha: End_R(eR)\cong eRe$, $\alpha(h)\in eRe$ is an idempotent. In addition, $e-\alpha(h)\in I_1$. Write $E_2=\psi(E)$ and $B_2'=\psi(B_2)$. We have $eR=E_2\oplus B_2'$. Let $k: eR=E_2\oplus B_2'\to E_2$ be the projection onto E_2. Then $k\in End_R(eR)$ is an idempotent, and that $e-\alpha(k)\in I_2$. Hence $\big(e-\alpha(h)\big)\perp\big(e-\alpha(k)\big)$ because $I_1\perp I_2$.

Obviously, $\psi\mid_{E\oplus B_2}: E\oplus B_2\to eR$ is an isomorphism. Let $\theta=(\psi\mid_{E\oplus B_2})^{-1}$, and let $i: E\oplus B_2\to M=eR\oplus eR$ be the injection. Since $\alpha(k)\in eRe$ is an idempotent, we may assume that $i\theta\big(\alpha(k)\big)=(x_1,x_2)$ with $x_1\in eR\alpha(k)$ and $x_2\in eR\alpha(k)$. Then $\alpha(k)=\psi i\theta\big(\alpha(k)\big)=\psi(x_1,x_2)=a_1x_1+a_2x_2$. Inasmuch as $E_1=\varphi(E)$ and $E_2=\psi(E)$, we get an isomorphism $\varphi\theta: E_2\to E_1$. Evidently, $E_2=k(eR)=k(e)eR=\alpha(k)R$. Likewise, $E_1=\alpha(h)R$. So we have $r\in R$ such that $\alpha(h)=\varphi\theta\big(\alpha(k)r\big)=\alpha(h)\varphi\theta\big(\alpha(k)\big)\alpha(k)r\alpha(h)$. Clearly, $x_1=\varphi i\theta\big(\alpha(k)\big)=\varphi\theta\big(\alpha(k)\big)=\alpha(h)\varphi\theta\big(\alpha(k)\big)\alpha(k)$. Set $y_1=\alpha(k)r\alpha(h)$. Then $\alpha(h)=x_1y_1$. Moreover, $y_1x_1=\alpha(k)r\alpha(h)\varphi\theta\big(\alpha(k)\big)=(\varphi\theta)^{-1}\big(\alpha(h)\big)\varphi\theta\big(\alpha(k)\big)$ $=(\varphi\theta)^{-1}\big(\alpha(h)\varphi\theta(\alpha(k))\big)=(\varphi\theta)^{-1}\big(\varphi\theta(\alpha(k))\big)=\alpha(k)$. Hence $(e-y_1x_1)\perp(e-x_1y_1)$. That is, $x_1\in(eRe)_q^{-1}$. In addition, $(a+bx_2y_1)x_1=ax_1+bx_2\alpha(k)=\alpha(h)=y_1x_1$. So $x_1(a+bx_2y_1)x_1=x_1y_1x_1=x_1\alpha(h)=x_1$. As $x_1\in(eRe)_q^{-1}$, we deduce that $a+bx_2y_1\in(eRe)_q^{-1}$ from [2, Remark 2.10]. It follows by Lemma 1 that I is a QB-ideal. ■

We use $\mathcal{V}(R)$ to denote the monoid of isomorphism classes of finitely generated projective right R-modules. An order-ideal in $\mathcal{V}(R)$ is a submonoid S of $\mathcal{V}(R)$ that is order-hereditary. If I is an ideal of R, we denote by $\mathcal{V}(I)$ the monoid of isomorphism classes of finitely generated projective right R-modules A such that $AI = A$. Following P. Ara et al.[2], we say that two order-ideals S_1 and S_2 of $\mathcal{V}(R)$ are orthogonal provided that $S_1 \cap S_2 = 0$. We denote it by $S_1 \perp S_2$.

Lemma 3. *Let I be a regular ideal of a ring R, and let $e \in I$ an idempotent. For any right R-modules A, B_1 and B_2, if $eR \oplus B_1 \cong A \oplus B_2$ then we have a refinement matrix*

$$\begin{array}{cc} & \begin{array}{cc} eR & B_1 \end{array} \\ \begin{array}{c} A \\ B_2 \end{array} & \left(\begin{array}{cc} A' & B_1' \\ B_2' & C' \end{array} \right). \end{array}$$

That is, $A \cong A' \oplus B_1', eR \cong A' \oplus B_2', B_1 \cong B_1' \oplus C'$ and $B_2 \cong B_2' \oplus C'$.

Proof. Suppose that $\psi : eR \oplus B_1 \cong A \oplus B_2$. Given decompositions $N := eR \oplus B_1 = \psi^{-1}(A) \oplus \psi^{-1}(B_2)$. Since I is a regular ideal and $e = e^2 \in I$, eRe is a regular ring; hence eR as a right R-module has the finite exchange property. Thus we can find some $B_1' \lesssim^{\oplus} \psi^{-1}(A)$ and $C' \lesssim^{\oplus} \psi^{-1}(B_2)$ such that $N = eR \oplus B_1' \oplus C'$. So $A \cong \psi^{-1}(A) = A' \oplus B_1'$ and $B_2 \cong \psi^{-1}(B_2) = B_2' \oplus C'$ for some right R-modules A' and B_2'. It follows from $N = \psi^{-1}(A) \oplus \psi^{-1}(B_2) = A' \oplus B_1' \oplus B_2' \oplus C' = eR \oplus B_1' \oplus C'$ that $eR \cong A' \oplus B_2'$. In addition, we claim that $B_1 \cong B_1' \oplus C'$ because $N = eR \oplus B_1 = eR \oplus B_1' \oplus C'$. ∎

Theorem 4. *Let I be a regular ideal of R. Then the following are equivalent:*

(1) *I is a QB-ideal.*

(2) *For any idempotent $e \in I$, $[eR] + b_1 = [eR] + b_2$ in $\mathcal{V}(R)$ implies that there exist orthogonal order-ideal S_1 and S_2 in $\mathcal{V}(R)$ and elements c_1, c_2, such that $c_1 \in S_1, c_2 \in S_2$ and $b_1 + c_1 = b_2 + c_2$.*

(3) *For all idempotents $e \in I$, if B_1 and B_2 are any right R-modules such that $eR \oplus B_1 \cong eR \oplus B_2$ then there exists a pair of orthogonal ideals I_1 and I_2 and $B_1 \oplus C_1 \cong B_2 \oplus C_2$ such that $C_1 I_1 = C_1$ and $C I_2 = C_2$.*

Proof. (1) $\Rightarrow$ (2) Choose representations B_1 and B_2 for b_1 and b_2 such that $M := A_1 \oplus B_1 = A_1 \oplus B_2$ with $A_1 \cong eR \cong A_2$. By Lemma 2, there exists a pair of orthogonal ideals I_1 and I_2 and $M = E \oplus C_1 \oplus B_1 = E \oplus C_2 \oplus B_2$ such that $C_1 I_1 = C_1$ and $C_2 I_2 = C_2$. Since $A_1, B_1 \in \mathcal{V}(R)$, we have $M \in \mathcal{V}(R)$; hence, $E, C_1, C_2 \in \mathcal{V}(R)$. Let $c_1 = [C_1]$ and $c_2 = [C_2]$. We get $b_1 + c_1 = b_2 + c_2$. Let $S_i = \mathcal{V}(I_i)$. Then $\mathcal{V}(I_1)$ and $\mathcal{V}(I_2)$ are orthogonal order-ideals of $\mathcal{V}(R)$. Furthermore, we have $c_i \in S_i$ for $i = 1, 2$.

(2) $\Rightarrow$ (3) Let $e \in I$ be an idempotent. Suppose that B_1 and B_2 are any right R-modules such that $eR \oplus B_1 \cong eR \oplus B_2$. By virtue of Lemma 3, we get a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} eR & B_1 \end{array} \\ \begin{array}{c} eR \\ B_2 \end{array} & \left(\begin{array}{cc} A' & B_1' \\ B_2' & C' \end{array} \right). \end{array}$$

Hence $eR \cong A' \oplus B_1' \cong A' \oplus B_2'$ with $A', B_1', B_2' \in \mathcal{V}(R)$. Clearly, we have an idempotent $g \in eRe \subseteq I$ such that $A' \cong gR$. So $gR \oplus B_1' \cong gR \oplus B_2'$ in $\mathcal{V}(R)$, and then we have orthogonal order-ideals S_1 and S_2 in $\mathcal{V}(R)$ and elements $c_1' = [C_1'], c_2' = [C_2']$, such that $c_1' \in S_1, c_2' \in S_2$ and $[B_1'] + c_1' = [B_2'] + c_2'$. That is, $B_1' \oplus C_1' \cong B_2' \oplus C_2'$. This infers that $eR \oplus C_1' \cong A' \oplus (B_1' \oplus C_1') \cong A' \oplus (B_2' \oplus C_2') \cong eR \oplus C_2'$. By Lemma 3 again , we have a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} eR & C_1' \end{array} \\ \begin{array}{c} eR \\ C_2' \end{array} & \left(\begin{array}{cc} A'' & C_1 \\ C_2 & C'' \end{array} \right). \end{array}$$

Since $[C_1'] \in S_1$ and $[C''] \leq [C_1']$, we have $[C''] \in S_1$. Likewise, we have $[C''] \in S_2$. It follows from $S_1 \cap S_2 = 0$ that $C'' = 0$. Therefore there exist some idempotents $h_1, h_2, k_1, k_2 \in eRe$ such that $h_1 + k_1 = e = h_2 + k_2, h_1R \cong h_2R, k_1R \cong C_1$ and $k_2R \cong C_2$. For $i = 1, 2$ let $I_i = \{\sum RpR \mid p = p^2, pR \in S_i\}$, respectively. Then $I_1 \cap I_2 = 0$; hence, $I_1 \perp I_2 = 0$. Furthermore, we have $k_1 \in I_1$ and $k_2 \in I_2$. Clearly, $C_1I_1 = C_1$ and $C_2I_1 = C_2$. Moreover, we get $B_1' \oplus C_1 \cong B_1' \oplus C_1' \cong B_2' \oplus C_2' \cong B_2' \oplus C_2$, and then $B_1 \oplus C_1 \cong (B_1' \oplus C') \oplus C_1 \cong C' \oplus (B_2' \oplus C_2) \cong B_2 \oplus C_2$, as required.

$(3) \Rightarrow (1)$ Let $e \in I$ be an idempotent. Then eRe is a regular ring. Let $a \in eRe$. There exists $b \in eRe$ such that $a = aba$ and $b = bab$. Set $p = ab$ and $q = ba$. As in the proof of [2, Theorem 8.7], we see that $eR \oplus (e-p)R \cong qR \oplus (e-q)R \oplus (e-p)R \cong pR \oplus (e-q)R \oplus (e-p)R \cong eR \oplus (e-q)R$. So we have right R-modules C_1 and C_2 such that $(e-p)R \oplus C_1 \cong (e-q)R \oplus C_2$ and a pair of orthogonal ideals I_1 and I_2 such that $C_1I_1 = C_1$ and $C_2I_2 = C_2$. As $e - p = (e-p)^2 \in I$, by Lemma 3, we get a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} (e-p)R & C_1 \end{array} \\ \begin{array}{c} (e-q)R \\ C_2 \end{array} & \left(\begin{array}{cc} A' & C_1' \\ C_2' & C' \end{array} \right). \end{array}$$

Inasmuch as $C'I_1 = C' = C'I_2, C' = C'I_1 = (C'I_1)I_1 = (C'I_2)I_1 = 0$. Hence we have idempotents $e_1, f_1 \in (e-p)R(e-p), e_2, f_2 \in (e-q)R(e-q)$ such that $e - p = e_1 + f_1, e - q = e_2 + f_2, e_1R \cong A' \cong e_2R$, and $f_1R \cong C_1'$ and $f_2R \cong C_2'$. As $e_1R \cong e_2R$, we can find $c \in e_1Re_2$ and $d \in e_2Re_1$ such that $e_1 = cd$ and $e_2 = dc$. Clearly, $a \in p(eRe)q$ and $c \in (e-p)(eRe)(e-q)$ are both regular in eRe. By [2, Lemma 2.7], $a \leq a + c$. Furthermore, it follows from $b \in qRp, d \in (e-q)R(e-p)$ that $e-(a+c)(b+d) = e-ab-cd = (e-p)-e_1 = f_1$. Likewise, $e - (b+d)(a+c) = f_2$. Obviously, $C_1'I_1 = C_1'$ and $C_2'I_2 = C_2$. From this, we deduce that $f_1RI_1 = f_1R$ and $f_2RI_2 = f_2R$; hence, $f_1 \in I_1$ and $f_2 \in I_2$. From $I_1 \perp I_2$, it follows that $\big(e-(a+c)(b+d)\big)R\big(e-(b+d)(a+c)\big)=0$ and $\big(e-(b+d)(a+c)\big)R\big(e-(a+c)(b+d)\big)=0$. So $a+c \in (eRe)_q^{-1}$. Therefore we complete the proof by [2, Theorem 8.4] and Lemma 1. ■

Theorem 5. *Let I be a regular ideal of R. Then the following are equivalent:*

(1) *I is a QB-ideal.*

(2) *For all finitely generated projective right R-module A with $AI = A$, if B_1 and B_2 are any right R-modules such that $A \oplus B_1 \cong A \oplus B_2$, then there exists a pair of orthogonal ideals I_1 and I_2 and $B_1 \oplus C_1 \cong B_2 \oplus C_2$ such that $C_1I_1 = C_1$ and $CI_2 = C_2$.*

Proof. $(2) \Rightarrow (1)$ Given any idempotent $e \in I$, then we have a finitely generated projective right R-module eR such that $eRI = eR$. In view of Theorem 4, I is a QB-ideal of R.

$(1) \Rightarrow (2)$ Let A be a finitely generated projective right R-module A with $AI = A$. Suppose that $A \oplus B_1 \cong A \oplus B_2$. By virtue of [9, Lemma 6], there exist idempotents $e_1, \cdots, e_n \in I$ such that $A \cong e_1R \oplus \cdots \oplus e_nR$. So $diag(e_1, \cdots, e_n)R^{n\times 1} \oplus B_1 \cong diag(e_1, \cdots, e_n)R^{n\times 1} \oplus B_2$, and then $diag(e_1, \cdots, e_n)M_n(R) \oplus B_1 \bigotimes_R R^{1\times n} \cong diag(e_1, \cdots, e_n)M_n(R) \oplus B_2 \bigotimes_R R^{1\times n}$. Clearly, $diag(e_1, \cdots, e_n) \in M_n(I)$. By [5, Lemma 2] and [2, Remark 6.5], $M_n(I)$ is a regular QB-ideal of $M_n(R)$. It follows from Theorem 4 that there exists a pair of orthogonal ideals $M_n(I_1)$ and $M_n(I_2)$ of $M_n(R)$ and $B_1 \bigotimes_R R^{1\times n} \oplus C_1' \cong B_2 \bigotimes_R R^{1\times n} \oplus C_2'$ such that $C_1'M_n(I_1) = C_1'$ and $C_2'M_n(I_2) = C_2'$. Obviously, $I_1 \perp I_2$. Set $C_i = C_i' \bigotimes_{M_n(R)} R^{n\times 1} (i = 1, 2)$. Then $B_1 \oplus C_1 \cong B_2 \oplus C_2$. As $R^{n\times 1}I \cong M_n(I)R^{n\times 1}$, we deduce that $C_1I = C_1$ and $C_2I = C_2$, as required. ■

As a result, we prove that a regular ring R is a QB-ring if and only for all finitely generated projective right R-module A, if B_1 and B_2 are any right R-modules such that $A \oplus B_1 \cong A \oplus B_2$, then there exists a pair of orthogonal ideals I_1 and I_2 and $B_1 \oplus C_1 \cong B_2 \oplus C_2$ such that $C_1I_1 = C_1$ and $CI_2 = C_2$, which extend [2, Theorem 8.7] and gives a new characterization of regular QB-rings.

Corollary 6. *Let I be a purely infinite simple regular ideal of a ring R, and let A be a finitely generated projective right R-module such that $A = AI$. If B_1 and B_2 are any right R-modules such that $A \oplus B_1 \cong A \oplus B_2$, then there exists a pair of orthogonal ideals I_1 and I_2 and $B_1 \oplus C_1 \cong B_2 \oplus C_2$ such that $C_1I_1 = C_1$ and $CI_2 = C_2$.*

Proof. According to [4, Corollary 1.11], I is a QB-ideal of R. So the result follows by Theorem 5. ■

Following P. Ara et al. (cf. [3] and [11]), we say that R is a separative ring if the following condition holds for all finitely generated projective right R-modules $A, B : A \oplus A \cong A \oplus B \cong B \oplus B \Longrightarrow A \cong B$. A ring R is said to be one-sided unit-regular in case for any $x \in R$ there exists a right or left invertible $u \in R$ such that $x = xux$ (cf. [7-8]). A simple ring R is said to be purely infinite if R is not a division ring, but for any non-zero element $x \in R$ there are $s, t \in R$ such that $sxt = 1$(see [3]). The class of purely infinite simple regular rings is rather large (cf.[1]). We claim that every purely infinite simple regular ring is separative.

Corollary 7. *Let R be a simple regular ring. Then the following are equivalent:*

(1) *R is a QB-ring.*

(2) *R is a separative ring.*

(3) *R is one-sided unit-regular.*

(4) *R either has stable rank 1 or is purely infinite.*

Proof. $(1) \Rightarrow (2)$ Suppose that A, B_1 and B_2 are finitely generated projective right R-modules such that $A \oplus B_1 \cong A \oplus B_2$. In view of Theorem 5, there exists a pair of orthogonal ideals I_1 and I_2 and $B_1 \oplus C_1 \cong B_2 \oplus C_2$ such that $C_1 I_1 = C_1$ and $CI_2 = C_2$. Since R is a simple ring, either I_1 or I_2 is zero. This infers that $C_1 = 0$ or $C_2 = 0$. So $B_1 \lesssim^{\oplus} B_2$ or $B_2 \lesssim^{\oplus} B_1$. By [8,Theorem 8], R is one-sided unit-regular.

$(2) \Rightarrow (3)$ Let R be a simple regular separative ring. If R is directly finite, R has stable range one from [3, Theorem 3.4]. If R is directly infinite, then $R \oplus D \cong R$ for some nonzero right R-module D. Given any right R-modules P and Q. If either P or Q is zero, then $P \lesssim^{\oplus} Q$ or $Q \lesssim^{\oplus} P$. Now we assume that P and Q are both nonzero. Since R is simple, there exists a positive integer n such that $P \lesssim^{\oplus} nD$. Thus $P \oplus R \lesssim^{\oplus} nD \oplus R \cong R$. So $P \oplus R \lesssim^{\oplus} R \lesssim^{\oplus} Q \oplus R$, and then $R \oplus (P \oplus E) \cong R \oplus Q$ for a right R-module E. Inasmuch as $P \oplus E$ and Q are nonzero, we have $R \lesssim^{\oplus} s(P \oplus E)$ and $R \lesssim^{\oplus} tQ$ for positive integers s and t. Applying [3,Lemma 2.1], $P \lesssim^{\oplus} P \oplus E \cong Q$. Therefore R is one-sided unit-regular.

$(3) \Rightarrow (1)$ According to [2, Example 8.8], R is a QB-ring.

$(1) \Leftrightarrow (4)$ is clear by [1, Remark 1.8] and [2, Proposition 3.10]. ■

References

1. P. Ara, K. R. Goodearl and E. Pardo, K_0 of purely infinite simple regular rings, *K-Theory*, **26** (2002), 69-100.

2. P. Ara, G. K. Pedersen and F. Perera, An infinite analogue of rings with stable range one, *J. Algebra* **230** (2000), 608-655.

3. P. Ara, K. R. Goodearl, K. C. O'Meara and E. Pardo, Separative cancellation for projective modules over exchange rings, *Israel J. Math.*, **105** (1998), 105-137.

4. P. Ara, G. K. Pedersen and F. Perera, Extensions and pullbacks in QB-rings, *Algebra Represent Theory*, **8** (2005), 75-97.

5. B. Brown, N. H. Mccoy, The maximal regular ideal of a ring, *Proc. Amer. Math. Soc.*, **1** (1950), 165-171.

6. C. Y. Hong, N. K. Kim, Nam and Y. Lee, Exchange rings and their extensions, *J. Pure Appl. Algebra*, **179** (2003), 117-126.

7. H. Chen, Elements in one-sided unit-regular rings, *Comm. Algebra*, **25** (1997), 2517-2529.

8. H. Chen, On exchange QB-rings, *Comm. Algebra*, **31** (2003), 831-841.

9. H. Chen and M. Chen, On unit-regular ideals, *New York J. Math.*, **9** (2003), 295-302.

10. K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London-San Francisco-Melbourne, 1979; 2nd ed., Krieger, Malabar, Fl., 1991.

11 K. R. Goodearl, von Neumann regular rings and direct sum decomposition problems, Facchini Alberto (ed.) et al., Abelian groups and modules, *Proc. Padaova Conference*, Padova, Italy, 1994. Dordrecht: Kluwer Academic Publishers, *Math. Appl. Dordr.*, **343** (1995), 249-255.

12. F. Perera, Lifting units modulo exchange ideals and C^*-algebras with real rank zero, *J. reine. Math.*, **522** (2000), 51-62.

13. H.P. Yu, Stable range one for exchange rings, *J. Pure Appl. Algebra*, **98** (1995), 105-108.

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