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GENERALIZED DERIVATIONS WITH NILPOTENT VALUES ON MULTILINEAR POLYNOMIALS

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Abstract. Let R be a prime ring without nonzero nil one-sided ideals. Suppose that g is a generalized derivation of R and that $f(X_1, \cdots, X_k)$ is a multilinear polynomial not central-valued on R such that $g(f(x_1, \cdots, x_k))$ is nilpotent for all $x_1, \cdots, x_k$ in some nonzero ideal of R. Then $g = 0$.

1. Introduction and Results

The study of derivations having values satisfying certain properties has been investigated in various papers. As to derivations having nilpotent values, Herstein and Giambruno [9] proved that if R is a semiprime ring and d is a derivation of R such that $d(x)^n = 0$ for all x in some nonzero ideal I of R, where $n \geq 1$ is a fixed integer, then $d(I) = 0$. In [7] Felzenszwalb and Lanski proved that if R is a ring with no nonzero nil one-sided ideals and d is a derivation such that $d(x)^n = 0$ for all x in some nonzero ideal I of R, where $n = n(x) \geq 1$ is an integer depending on x, then $d(I) = 0$. The extensions of this theorem to Lie ideals were obtained by Carini and Giambruno [3] in case char$R \neq 2$ and by Lanski [12] in case of arbitrary characteristic. A full generalization in this vein was proved by Wong [19]. She showed that if d is a derivation of a prime ring R such that $d(f(x_1, \cdots, x_k))^n = 0$ for all x_i in some nonzero ideal of R, where $n = n(x_1, \cdots, x_k) \geq 1$ is an integer depending on x_i and $f(X_1, \cdots, X_k)$ is a multilinear polynomial not central-valued on R, then $d = 0$ provided that n is fixed or R contains no nonzero nil one-sided ideals.

Let R be a ring. An additive mapping $g : R \to R$ is called a generalized derivation of R if there exists a derivation d of R such that $g(xy) = g(x)y + xd(y)$

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for all $x, y \in R$. In [10] Hvala proved a result concerning generalized derivations with nilpotent values of bounded index. In fact, he proved that if R is a prime ring of char$R > n$ and g is a generalized derivation of R satisfying $g(x)^n = 0$ for all $x \in R$, then $g = 0$. Later, Lee [15] extended this result to Lie ideals. Recently, [18] Wang showed that if g is a generalized derivation of a prime ring R such that $g(f(x_1, \cdots, x_k))^n = 0$ for all x_i in some nonzero ideal of R, where $n \geq 1$ is a fixed integer and $f(X_1, \cdots, X_k)$ is a multilinear polynomial not central-valued on R, then $g = 0$. In this paper we shall prove the unbounded version of Wang's result. Precisely, we will prove the following

Theorem 1. *Let K be a commutative ring with unity and let R be a prime K-algebra without nonzero nil one-sided ideals. Let $f(X_1, \cdots, X_k)$ be a multilinear polynomial over K with at least one coefficient invertible in K. Suppose that g is a generalized derivation of R and $f(X_1, \cdots, X_k)$ is not central-valued on R such that $g(f(x_1, \cdots, x_k))$ is nilpotent for all $x_1, \cdots, x_k$ in some nonzero ideal I of R. Then $g = 0$.*

Let R be a ring. For $x, y \in R$, we denote $[x, y] = xy - yx$. An additive subgroup L of R is said to be a Lie ideal of R if $[u, r] \subseteq L$ for all $u \in L$ and $r \in R$. A Lie ideal L of R is called noncommutative if $[L, L] \neq 0$. It is well-known that if L is a noncommutative Lie ideal of a prime ring R, then $[x_1, x_2] \subset L$ for all x_1, x_2 in some nonzero ideal I of R (see the proof of [8, Lemma 1.3]). So we immediately obtain the following result from Theorem 1.

Theorem 2. *Let R be a prime ring without nonzero nil one-sided ideals and let L be a noncommutative Lie ideal of R. Suppose that g is a generalized derivation of R such that $g(u)$ is nilpotent for each $u \in L$. Then $g = 0$.*

Finally, we extend Wang's result to the case of semiprime rings.

Theorem 3. *Let R be a semiprime K-algebra, where K is a commutative ring with unity. Let $f(X_1, \cdots, X_k)$ be a multilinear polynomial over K with at least one coefficient invertible in K. Suppose that g is a generalized derivation of R such that $g(f(x_1, \cdots, x_k))^n = 0$ for all $x_1, \cdots, x_k \in R$, where $n \geq 1$ a fixed integer. Then $[f(x_1, \cdots, x_k), x]g(y) = 0$ for all $x_1, \cdots, x_k, x, y \in R$.*

2. Preliminaries

Throughout, unless specially stated, let R be a prime K-algebra, where K is a commutative ring with unity and $f(X_1, \cdots, X_k)$ abbreviated by f or $f(X_i)$, will be a multilinear polynomial over K with at least one coefficient invertible in K.

An additive mapping $g : R \to R$ is called a generalized derivation of R if there exists a derivation d of R such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in R$.

We let U be the maximal right ring of quotients of R and let Q stand for the two sided Martindale quotient ring of R. The center C of U (and Q) is called the extended centroid of R (see [1] for details). It is well-known that any derivation of R can be uniquely extended to a derivation of Q. Without loss of generality, we may write

$$f(X_1, \cdots, X_k) = \alpha_1 X_1 \cdots X_k + \sum_{\sigma \neq id} \alpha_\sigma X_{\sigma(1)} \cdots X_{\sigma(k)},$$

where α_1 is invertible in K and the sum is taken over all permutations σ except the identity id in the symmetric group S_k.

We include two preliminary lemmas.

Lemma 1.1. *Let R be a prime ring with nonzero socle H. Suppose that R is not a domain and d is a derivation of R such that $d(e)e = 0$ for all $e = e^2 \in H$. Then $d = 0$. By symmetry, if $ed(e) = 0$ for all $e = e^2 \in H$, then $d = 0$.*

Proof. Let $x \in R$. For $e = e^2 \in H$, $e + (1 - e)xe$ is still an idempotent in H. Assume first that d is X-inner, that is, $d(x) = ax - xa$ for some $a \in Q$. Then $(ae - ea)e = 0$ for $e = e^2 \in H$. Hence $ae = eae$ for $e = e^2 \in H$. Let $y \in H$ and $e = e^2 \in H$. Then $(1 - e)y \in H$. Note that H is a regular ring [6, Lemma 1]. So $(1 - e)yH = hH$ for some $h = h^2 \in H$. Hence $eh = 0$. Since $ah = hah$, we have $eah = 0$. Therefore $ea(1 - e)y = 0$ and then $ea(1 - e)H = 0$ implies that $ea(1 - e) = 0$. Thus $ae - ea = 0$ for all $e^2 = e \in H$. In particular, $a(e + (1 - e)xe) = (e + (1 - e)xe)a$. Then $a(1 - e)xe = (1 - e)xea$ for all $x \in R$. Since R is not a domain, there exists $e = e^2 \in H$ and $e \neq 0, 1$. By Martindale's Lemma [17, Theorem 2 (a)], $a(1 - e) = \lambda(1 - e)$ and $ea = \lambda e$ for some $\lambda \in C$. So $a = \lambda$ and then $d = 0$, as desired. Assume next that d is not X-inner. Let $x \in R$. Expanding $d(e + (1 - e)xe)(e + (1 - e)xe) = 0$ and using $d(e)e = 0$ to yield that

$$d(e)(1 - e)xe + d(1 - e)xe + (1 - e)d(x)e + (1 - e)xd(e)(1 - e)xe = 0$$

for all $x \in R$. Thus $(1 - e)d(x)e + (1 - e)xd(e)xe = 0$. Applying Kharchenko's Theorem [11] by replacing $d(x), x$ with $y, 0$ respectively, we have that $(1-e)ye = 0$ for all $y \in R$. Thus $e = 0$ or 1 for $e = e^2 \in H$, a contradiction. This proves the lemma.

The second lemma is implicit in the proof of [7, Theorem 5].

Lemma 1.2. *Let R be a ring and $v \in R, v^2 = 0$. Suppose that for each $x \in R$ with $x^2 = 0$ we have either $xv = 0$ or $vx = 0$. Then $vhv = 0$ for all nilpotent elements h in R.*

Proof. Assume on the contrary that $vhv \neq 0$ for some nilpotent element h. Since h is nilpotent, there exists some $\ell \geq 1$ such that $vh^k v = 0$ and $vh^\ell v \neq 0$ for all $k > \ell$. Note that $((1+h^\ell)v(1+h^\ell)^{-1})^2 = 0$. By assumption, either $v(1+h^\ell)v(1+h^\ell)^{-1} = 0$ or $(1+h^\ell)v(1+h^\ell)^{-1}v = 0$. Thus either $v(1+h^\ell)v = 0$ or $v(1+h^\ell)^{-1}v = 0$. So $0 = v(1+h^\ell)^{-1}v = v(1-h^\ell+h^{2\ell}-h^{3\ell}+\cdots)v$. This implies that $vh^\ell v = 0$, a contradiction.

2. Proof of Theorem 1 and Theorem 3

Before proving Theorem 1, we make the following remark. For each coefficient α of f, since α and $d(\alpha)$ are all contained in C, we may choose a nonzero ideal I_α of R such that $\alpha I_\alpha \cup d(\alpha)I_\alpha \subseteq R$. Replacing I by $I \cdot (\cap_\alpha I_\alpha)$, where the intersection runs over all coefficients of f, we may assume that $\alpha I \cup d(\alpha)I \subseteq R$ for each coefficient α of f. If $k = 1$, then $f(X_1) = \alpha_1 X_1$, where $\alpha_1^{-1} \in K$. Observe that $f(X_1)X_2 = \alpha_1 X_1 X_2$ is not central-valued on R; otherwise R is commutative and then f is central-valued on R. Replacing f by fX_2, we may always assume that $k \geq 2$.

We divide the proof of Theorem 1 into several lemmas.

Lemma 2.1. *Theorem 1 holds if R is a semisimple algebra.*

Proof. Let ${}_RM$ be an irreducible left R-module and $\mathrm{Ann}_R(M)$= $\{r \in R \mid rm = 0 \text{ for all } m \in M\}$. Let $J = \alpha_1 I^2$. Since $\alpha_1^{-1} \in K$, J is a nonzero ideal of R contained in I. We claim that either $g(J^2) \subseteq \mathrm{Ann}_R(M)$ or $g(f(x_i))^{k+1} \subseteq \mathrm{Ann}_R(M)$ for $x_i \in I$. If $J \subseteq \mathrm{Ann}_R(M)$, then $g(J^2) \subseteq \mathrm{Ann}_R(M)$. So we may assume that $JM \neq 0$ and then M is also an irreducible left J-module. Let $D = \mathrm{End}({}_RM) = \mathrm{End}({}_JM)$. Suppose first that $\dim M_D \leq k+1$. Then $\overline{R} = R/\mathrm{Ann}_R(M) \cong M_m(D)$, where $m \leq k+1$. Since $\overline{g(f(x_i))} = g(f(x_i))+\mathrm{Ann}_R(M)$ is nilpotent in $\overline{R}$, we must have $\overline{g(f(x_i))}^m = \overline{0}$, that is, $g(f(x_i))^m \in \mathrm{Ann}_R(M)$ for all $x_i \in I$.

Suppose now that $\dim M_D > k+1$. By [15, Theorem 4], we may write $g(x) = ax + d(x)$ for all $x \in R$, where $a \in U$ and d a derivation of R. Notice that $aR \subseteq g(R) - d(R) \subseteq R$. Define an additive map $\overline{d} : J \to \mathrm{End}(M_D)$ given by $\overline{d}(r) = L_{d(r)}$, where $L_{d(r)}(v) = d(r) \cdot v$ for $v \in M$ (see [2, p.326]). We divide the proof into two cases.

Case 1. Assume that $\overline{d}$ is M-inner [2, Definition 4.1]. That is, there exists an additive endomorphism T of M such that $d(r)v = T(rv) - rT(v)$ for all $r \in J$ and $v \in M$. Suppose first that v and $T(v)$ are linear dependent over D for all $v \in M$. Then by [2, Lemma 7.1] there exists $\lambda \in D$ such that $T(v) = v\lambda$ for all $v \in M$. Hence $d(r)v = (rv)\lambda - r(v\lambda) = 0$ for $r \in J, v \in M$, that is, $d(J)M = 0$

and so $d(J) \subseteq \text{Ann}_R(M)$. If $(aJ)M = 0$, then $g(J^2) \subseteq \text{Ann}_R(M)$, as claimed. Hence we may assume that $(a(\alpha_1 y))v \neq 0$ for some $y \in I^2$ and $v \in M$. Let $w = (a(\alpha_1 y))v$ and $w = u_1, \cdots, u_k$ be k D-independent vectors in M. Since M is an irreducible left J-module, by the Jacobson Density Theorem, there exist $r_1, \cdots, r_k \in J$ such that $r_k u_1 = u_2, r_{k-1}u_2 = u_3, \cdots, r_2 u_{k-1} = u_k, r_1 u_k = v$ and $r_i u_j = 0$ for all other possible choices of i and j. Then $af(yr_1, \cdots, r_k)w = w$ and $d(f(yr_1, \cdots, r_k)) \in d(J)$. Hence $g(f(yr_1, \cdots, r_k))w = (af(yr_1, ..., r_k))w = w$. In particular, $g(f(yr_1, ..., r_k))^n w = w$ for all $n \geq 1$, a contradiction.

So we may assume that there exists $v \in M$ such that v and $T(v)$ are linear independent over D. Let $v = u_0, T(v) = u_1, \cdots, u_k$ be $k+1$ D-independent vectors in M. By the Jacobson Density Theorem, there exist $y \in I^2$ and $r_1, \cdots, r_k \in J$ such that $(\alpha_1 y)v = v$, $r_k u_1 = u_2, \cdots, r_2 u_{k-1} = u_k, r_1 u_k = -v$ and $r_i u_j = 0$ for all other possible choices of i and j. Hence we have

$$g(f(yr_1, \cdots, r_k))^n v = (af(yr_1, \cdots, r_k) + Tf(yr_1, \cdots, r_k) - f(yr_1, \cdots, r_k)T)^n v = v$$

for all $n \geq 1$, a contradiction.

Case 2. Assume that $\overline{d}$ is not M-inner. We denote by $f^d(X_1, \cdots, X_k)$ the polynomial obtained from $f(X_1, \cdots, X_k)$ by replacing each coefficient α with $d(\alpha \cdot 1)$. Let $v_1, \cdots, v_k$ be k D-independent vectors in M. By the Extended Jacobson Density Theorem [2, Theorem 4.6], there exist $r_1, \cdots, r_k \in J$ such that

$$d(r_k)v_k = v_{k-1}, r_{k-1}v_{k-1} = v_{k-2}, \cdots, r_2 v_2 = v_1, r_1 v_1 = v_k$$

and

$$r_i v_j = 0, d(r_i)v_j = 0 \text{ for all other possible choices of } i \text{ and } j.$$

Let $y \in I^2$ such that $(\alpha_1 y)v_k = v_k$. Then $af(yr_1, \cdots, r_k)v_k = 0$, $f^d(yr_1, \cdots, r_k)v_k = 0$,

$$f(d(yr_1), r_2, \cdots, r_k)v_k = f(d(y)r_1 + yd(r_1), r_2, \cdots, r_k)v_k = 0$$

and $f(yr_1, \cdots, d(r_i), \cdots, r_k)v_k = 0$. But $f(yr_1, \cdots, r_{k-1}, d(r_k))v_k = v_k$. So we have $g(f(yr_1, \cdots, r_k))v_k = \big(af(yr_1, \cdots, r_k) + d(f(yr_1, \cdots, r_k))\big)v_k = v_k$. Hence $g(f(yr_1, \cdots, r_k))^n v_k = v_k$ for all $n \geq 1$, a contradiction.

So now we have $g(J^2)Rg(f(x_i))^{k+1} \subseteq \cap_M \text{Ann}_R(M) = 0$, where the intersection runs over all irreducible left R-modules M. If $g(J^2) = 0$, then $g = 0$ by [15, Theorem 6]. Otherwise, by primeness of R, $g(f(x_i))^{k+1} = 0$ for all $x_i \in I$. Thus $g = 0$ follows from [18, Theorem 1].

From now on we may assume that R is not a semisimple algebra, that is, $J(R)$, the Jacobson radical of R, is nonzero.

Lemma 2.2. *Theorem 1 holds if there exist* $b, c \in Q$ *with* $bc = 0$ *but* $bd(c) \neq 0$.

Proof. We first claim that if $u, v \in Q$ with $uv = 0$ but $ud(v) \neq 0$, then f vanishes on Qu. Let I' be a nonzero ideal of R such that $vI', I'v$ and $I'u$ are all contained in I. Rewrite f in a form that

$$f = X_1 f_1(X_2, \cdots, X_k) + X_2 f_2(X_1, X_3, \cdots, X_k) + \cdots + X_k f(X_1, \cdots, X_{k-1}).$$

For all $x_1, \cdots, x_k \in I'$, we have

$$f(vx_1, x_2u, \cdots, x_ku) = vx_1 f_1(x_2u, \cdots, x_ku)$$

and

$$g(f(vx_1, x_2u, \cdots, x_ku))v = vx_1 d(f_1(x_2u, \cdots, x_ku))v.$$

Thus

$$(g(f(vx_1, x_2u, \cdots, x_ku)))^n v = v(x_1 d(f_1(x_2u, \cdots, x_ku))v)^n = 0$$

for some $n = n(x_i) \geq 1$. Hence $I'd(f_1(x_2u, \cdots, x_ku))v$ is a nil left ideal of R. So $d(f_1(x_2u, \cdots, x_ku))v = 0$. And then

$$f_1(x_2u, \cdots, x_ku)d(v) = d(f_1(x_2u, \cdots, x_ku)v) - d(f_1(x_2u, \cdots, x_ku))v = 0$$

for all $x_i \in I'$ and hence for all $x_i \in Q$ by [5, Theorem 2]. By [19, Lemma 4], $f_1(x_2u, \cdots, x_ku) = 0$ for all $x_i \in Q$. In a similar way, we have $f_i(x_ju) = 0$ for all $x_j \in Q$ and $i = 2, \cdots, k$. Therefore, $f(x_1u, \cdots, x_ku)$ is a GPI of Q. Since $bc = 0$ and $bd(c) \neq 0$, Q satisfies the nontrivial GPI $f(x_1b, \cdots, x_kb)$. By Martindale's Theorem [17], Q is a primitive ring with nonzero socle H and its associated division ring D is finite-dimensional over C. Moreover, Q is isomorphic to a dense subring of the ring of linear transformations of a vector space M over D and H consists of linear transformations of finite rank. If dim$M_D = m$, then $Q \cong M_m(D)$. Then $g(f(x_i))^m = 0$ for all $x_i \in I$. By [18, Theorem 1], we are done. So we assume that dim$_D M = \infty$. Note that f is not a PI of $Q(1-e)$ for $e^2 = e \in H$. Otherwise, $Q(1-e) = Qh$ for some $h^2 = h \in H$ by [13, Proposition]. Thus $(1-e)(1-h) = 0$. This implies that $1 = e + (1-e)h \in H$, contrary to the infinite-dimensional of ${}_DM$. Since $e(1-e) = 0$, we have $0 = ed(1-e) = -ed(e)$ for all $e^2 = e \in H$. By Lemma 1.1, $d = 0$. This contradicts that $bd(c) \neq 0$.

By Lemma 2.2, now we may assume that $xy = 0$ implies that $xd(y) = 0$ for $x, y \in Q$.

Lemma 2.3. *Let* R *be a non-GPI ring. Then Theorem 1 holds.*

Proof. Let

$$S = \{s \in R \mid s^2 = 0\}.$$

If $S = 0$, then R is a prime reduced ring and hence is a domain. So $g(f(x_i)) = 0$ for all $x_i \in I$. By [18, Theorem 1], we are done. Now we assume that $S \neq 0$. We first to show that $d(S) = 0$.

Now let

$$T = \{t \in R \mid xty = 0 \text{ whenever } xy = 0 \text{ for } x, y \in Q\}.$$

Note that T is a subring of R. We also remark that S and T are invariant under inner automorphisms of R. For $x, y \in Q$ with $xy = 0$ and $s \in S$, we have $xd(y) = 0 = sd(s)$ and $x(1-s)(1+s)y = 0$. Thus

$$0 = x(1-s)d((1+s)y) = x(1-s)(1+s)d(y) + x(1-s)d(1+s)y = xd(s)y.$$

So $d(S) \subseteq T$. Also $d(s)s = d(s^2) - sd(s) = 0$ implies that $d(s)^2 = 0$ for $s \in S$, that is, $d(S) \subseteq S$.

Suppose first that $T \cap S = 0$. Then $d(S) = 0$. We are done. So suppose now that $W = T \cap S \neq 0$. Note that $(1+z)W(1+z)^{-1} \subseteq W$ for $z \in J(R)$. We claim that there exists some $0 \neq v \in R$ such that $v \in W$ and $vRv \subseteq T$. Fix $0 \neq w \in W$. If $wW = 0$, then $w(1+z)W(1+z)^{-1} = 0$ for $z \in J(R)$. This implies $wJ(R)W = 0$ and so $w = 0$, a contradiction. Choose $t \in W$ such that $wt \neq 0$. Recall that $w^2 = t^2 = wtw = 0$ and $(trwt)^2 = 0$ for $r \in R$. Hence

$$(1 + trwt)w(1 - trwt) - w = w - wtrwt \in T.$$

Let $v = wt$. Then $0 \neq v \in W$ and $vRv \subseteq T$. Let

$$V = \{v \in W \mid vRv \subseteq T\}.$$

Obviously, $(1+z)V(1+z)^{-1} \subseteq V$ for $z \in J(R)$. And for $v \in V$ and $s^2 = 0$, $svRvs \subseteq sTs = 0$ yields that either $vs = 0$ or $sv = 0$. Since $g(f(x_i))$ is nilpotent, by Lemma 1.2, $vg(f(x_i))v = 0$ for all $v \in V$. Let L be the additive subgroup of R generated by $\{f(x_i) : x_i \in I\}$. Let $y \in R$. Using multilinearity of $f(X_i)$, we have $[y, f(x_1, \cdots, x_k)] = \sum_{i=1}^{k} f(x_1, \cdots, [y, x_i], \cdots, x_k)$. Hence $[R, L] \subseteq L$ and then L is a Lie ideal of R. Obviously, $vg(L)v = 0$. Since R is a non-GPI ring, L must be noncommutative. Moreover, we have $vg(R)v = 0$ by [14, Theorem 2]. From the definition of T we see that $vg(r)tv = 0$ for $t \in T$. Hence

$$vrd(t)v = vg(rt)v - vg(r)tv = 0$$

for all $r \in R$. This implies that $d(t)v = 0$ for all $t \in T$ and $v \in V$. So it follows that $d(t)J(R)v = 0$ from $d(t)(1+z)v(1+z)^{-1} = 0$ for $z \in J(R)$. Thus $d(T) = 0$.

In particular, $d(V)=0$. Let $0\neq v\in V$ and $s^2=0$. Then either $sv=0$ or $vs=0$. If $vs=0$, then $vd(s)=0$. If $sv=0$, then $vs=(1-s)v(1+s)-v\in T$ and so $0=d(vs)=d(v)s+vd(s)=vd(s)$. Using $(1+z)^{-1}v(1+z)d(s)=0$ for $z\in J(R)$, we obtain that $d(S)=0$.

Next we claim that $d=0$. For $0\neq s\in S$, obviously we have $sRs\subseteq S$. So $0=d(sRs)=d(sR)s=sd(R)s$. This yields that $sd(R)\subseteq S$. Thus $0=d(sd(R))=sd^2(R)$ for all $s\in S$. Therefore $(1+z)^{-1}s(1+z)d^2(R)=0$ for $z\in J(R)$, implying that $d^2(R)=0$. By [4, Theorem 2], we may assume that the characteristic of R is equal to 2. Using $0=d(sR)s$ and in view of [4, Lemma 4], there exists some $p_s\in Q$ depending on s such that $d(x)=p_sx-xp_s$ and $p_ssR=0$. So $p_ss=0$. Since $0=d^2(x)=p_s^2x-xp_s^2$, we see that $p_s^2\in C$ for all $0\neq s\in S$. Thus it follows that $p_s^2=0$ from $p_ss=0$. Suppose that $p_s\neq p_{s'}$ for some $0\neq s,s'\in S$. Then $p_s-\alpha=p_{s'}$ for some $\alpha\in C$ and $(p_s-\alpha)^2=0=p_s^2$. This implies that $\alpha=0$, a contradiction. So we may assume that $d(x)=px-xp$ for some $p\in Q$ and $ps=0$ for all $s\in S$. Using $p(1+z)S(1+z)^{-1}=0$ for $z\in J(R)$, we have $p=0$. Hence $d=0$, as claimed.

So now $g(x)=ax$ for some $a\in U$ [15, Theorem 4]. For $0\neq s\in S$, we have

$$sg(f(sx_1,\cdots sx_{k-1},sx_ks))=saf(sx_1,\cdots sx_{k-1},sx_ks)=sah(sx_1,\cdots,sx_{k-1})sx_ks$$

for some multilinear polynomial $h(x_1,\cdots,x_{k-1})$. Thus

$$0=sg(f(sx_1,\cdots sx_{k-1},sx_ks))^m=(sah(sx_1,\cdots,sx_{k-1})sx_k)^ms$$

for m large enough. Hence $sah(sx_1,\cdots,sx_{k-1})sI$ is a nil right ideal of R. So $sah(sx_1,\cdots,sx_{k-1})sx_k=0$ for all $x_i\in I$. Since R is a non-GPI ring, we have $sas=0$ for all $s\in S$. Also we have

$$sg(f(x_1,sx_2,\cdots sx_{k-1},sx_ks))=sax_1h'(sx_2,\cdots,sx_{k-1})sx_ks$$

for some multilinear polynomial $h'(x_2,\cdots,x_{k-1})$. Thus

$$0=sg(f(x_1,sx_2,\cdots sx_{k-1},sx_ks))^m=(sax_1h'(sx_2,\cdots,sx_{k-1})sx_k)^ms$$

for m large enough. Hence $sax_1h'(sx_2,\cdots,sx_{k-1})sI$ is a nil right ideal of R. So $sax_1h'(sx_2,\cdots,sx_{k-1})sx_k=0$ for all $x_i\in I$. Since R is a non-GPI ring, it follows that $sa=0$ for all $s\in S$. Using $(1+z)^{-1}S(1+z)\subseteq S$, we may easily get $a=0$. So $g=0$. This proves the lemma.

Proof of Theorem 1. In view of Lemma 2.3, R can be assumed to be a prime GPI-ring. Then by Martindale's Theorem [17], Q is a primitive ring with nonzero socle H and its associated division ring D is finite-dimensional over C. Moreover, Q is isomorphic to a dense subring of the ring of linear transformations of a vector space M over D and H consists of linear transformations of finite rank. If dim$M_D = m$, then $Q \cong M_m(D)$. Hence $g(f(x_i))^m = 0$ for all $x_i \in I$. By [18, Theorem 1], we are done. So we assume that dim$M_D = \infty$. Since $e(1-e) = 0$ for $e^2 = e \in H$, in view of Lemma 2.2 we have $0 = ed(1-e) = -ed(e)$. By Lemma 1.1, $d = 0$. So now $g(x) = ax$. For each $e^2 = e \in H$, it follows from Litöff's Theorem [6] that $eQe \cong M_m(D)$, where dim$(eM)_D = m$. Choose a nonzero ideal I' of R such that $eI'e \subseteq I$. Thus

$$(eaef(ex_1e, \cdots, ex_ke))^m = 0$$

for all $x_i \in I'$ and hence for $x_i \in Q$ by [5, Theorem 2]. Moreover, if $2m - 1 > k$, then f is not cental-valued on eQe and then $eae = 0$ by [18, Theorem 1]. Given $r \in R$ and $h \in H$, there exists $e^2 = e \in H$ such that $arh, rh \in eQe$ and $eQe \cong M_m(D)$, $2m - 1 > k$. Then $arh = earh = eaerh = 0$. This implies that $aRH = 0$. Thus $a = 0$ and so $g = 0$. The proof is now complete.

Proof of Theorem 3. By [15, Theorem 4], we may write $g(x) = ax + d(x)$ for all $x \in R$, where $a \in U$ and d a derivation of R. Since U and R satisfy the same differential identities [16, Theorem 3], $g(f(x_1, \cdots, x_k))^n = 0$ for all $x_1, \cdots, x_k \in U$. Denote by $C = Z(U)$ the center of U. Let P be a maximal ideal of C. Then PU is a prime ideal of U invariant under all derivations of U and $\cap_P PU = 0$, where P's run over all maximal ideals of C (see [16, p.32 (iii)]).

Fix a maximal ideal P of C. Let $\overline{d}$ be the canonical derivation of $\overline{U} = U/PU$ induced by d. Set $\overline{g}(\overline{x}) = \overline{a} \cdot \overline{x} + \overline{d}(\overline{x})$. Note that $\overline{g}$ is a generalized derivation of the prime ring $\overline{U}$. Moreover, $\overline{g}(f(\overline{x_1}, \cdots, \overline{x_k}))^n = 0$. It follows from [18, Theorem 1] that either $\overline{g}(\overline{U}) = 0$ or $f(X_1, \cdots, X_k)$ is central-valued on $\overline{U}$, that is either $g(U) \subset PU$ or $[f(x_1, \cdots, x_k), x] \subset PU$ for $x_1, \cdots, x_k, x \in U$. Hence $[f(x_1, \cdots, x_k), x]g(U) \subset PU$. But since $\cap_P PU = 0$, we obtain $[f(x_1, \cdots, x_k), x]g(y) = 0$ for $x_1, \cdots, x_k, x, y \in U$.

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