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APPROXIMATION TO OPTIMAL STOPPING RULES FOR WEIBULL RANDOM VARIABLES WITH UNKNOWN SCALE PARAMETER

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Abstract. Let $X_1, X_2, \cdots, X_n, \cdots$ be independent, identically distributed Weibull random variables with an unknown scale parameter α. If we define the reward sequence $Y_n = \max\{X_1, X_2, \cdots, X_n\} - cn$ for $c > 0$, the optimal stopping rule for Y_n depends on the unknown scale parameter α. In this paper we propose an adaptive stopping rule that does not depend on the unknown scale parameter α and show that the difference between the optimal expected reward and the expected reward using the proposed adaptive stopping rule vanishes as c goes to zero.

1. INTRODUCTION

An optimal stopping rule stops the sampling process at a sample size n that maximizes the expected reward. Weibull distribution, one of the prominent probability models, has been widely used in reliability engineering. The purpose of this paper is to find the approximation to optimal stopping rule for Weibull random variables with unknown scale parameters in the hope to maximize the expected reward in the sampling process.

Let $X_1, X_2, \cdots, X_n, \cdots$ be independent, identically distributed Weibull random variables with an unknown scale parameter α. The X_i is observed sequentially and we are allowed to stop observing at any stage. If we stop at the nth observation then we will receive a reward Y_n, where Y_n is a measurable function of $X_1, X_2, \cdots, X_n$. Optimal stopping rule depends on the distribution of the X_i which has the consequence that determination of an optimal stopping rule requires complete knowledge of the underlying distribution for the data. If only partial information is available, e.g., some parameter values are unknown, then it becomes necessary to use an adaptive stopping rule to approximate the optimal rule.

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In this paper, we assume that the X_i is an independent Weibull random variable with common probability density function

$$f(x;\alpha,\beta)=\frac{\beta}{\alpha}x^{\beta-1}exp\left(-\frac{x^{\beta}}{\alpha}\right),\quad x>0,$$

where $\beta>0$ is a known constant and α is an unknown scale parameter. Let $\max\{X_1,X_2,\cdots,X_n\}$ be the reward for the first n trials and let $c>0$ be the cost for each trial. Then we will consider reward or net gain functions of the form $Y_n=\max\{X_1,X_2,\cdots,X_n\}-cn$. Such reward function arises in the context of sampling with recall. Discussion of their motivations and utility can be found in [5] or [6]. The purpose of this paper is to find an adaptive stopping rule in the case of sequential observed Weibull random variable with unknown scale parameter. Using a proposed adaptive stopping rule we prove that the difference between the optimal expected reward and the expected reward using the proposed adaptive stopping rule vanishes as c goes to zero.

The problem of finding an adaptive stopping rule to approximate stopping rule has been studied by [1] that proved that in certain cases involving unknown location parameters, the ratio of the expected reward under an adaptive stopping rule to the optimal expected reward will approach one as c goes to zero. [10] assumed that X_i is an exponential distributed random variable with unknown mean. [12] considered the case where the X_i has common density function $(\alpha-1)x^{-\alpha}I_{[1,\infty]}$ with unknown α, where $I_A(\bullet)$ denotes the indicator function for the set A. [8] considered exponential distributed random variables with unknown location and scale parameters. Under the distribution discussed by [8, 10], and [12] the optimal stopping rules have closed forms. [11] considered the case where the X_i is normal with unknown mean and [7] generalized [11]'s results to include the case where both the mean and variance are unknown. [9] treated the situation in which the X_i is Gamma distribution with unknown scale parameter, while [2] generalized [9]'s results to include the case where both the location and scale parameters are unknown. In the situations of [2, 7, 9], and [11] the optimal stopping rules no longer have a closed form and adaptive stopping rules were used to approximate the optimal stopping rules.

2. Primary Results

In this paper, we define the optimal stopping rule as

$$\tau_c^*=\inf\{n\geq 1: X_n\geq\gamma_c\} \tag{1}$$

where γ_c satisifies $\boldsymbol{E}(X_1-\gamma_c)^+=c$, and $(X_1-\gamma_c)^+=\max\{X_1-\gamma_c,0\}$. The stopping rule τ_c^* maximizes $\boldsymbol{E}(Y_\tau)$ over all stopping rules τ with $\boldsymbol{E}(Y_\tau^-)<\infty$

where $Y_\tau^- = -\min\{Y_\tau, 0\}$, and the expected reward is $\boldsymbol{E}(Y_{\tau_c^*}) = \boldsymbol{E}(X_{\tau_c^*}) - c\boldsymbol{E}(\tau_c^*) = \gamma_c$. For more details see [4, p. 56-58].

However, in order to use the optimal stopping rule τ_c^* it is necessary to know γ_c, which in turn requires knowledge of distribution of X_i. If only partial information about the distribution is available, it would be desirable to find an adaptive stopping rule to approximate the optimal rule τ_c^* and the optimal reward $\boldsymbol{E}(Y_{\tau_c^*})$ as well. Throughout the rest of this paper we assume that the X_i is an independent Weibull random variable with the common probability density function

$$f(x;\alpha,\beta) = \frac{\beta}{\alpha} x^{\beta-1} exp\left(-\frac{x^\beta}{\alpha}\right), \quad x > 0.$$

We define the function $\boldsymbol{E}(X_1 - x)^+ = f(x, \alpha)$, and

$$f(x,\alpha) = \alpha^{\frac{1}{\beta}}\left[1 - G(\frac{x^\beta}{\alpha})\right] - x\exp\left(-\frac{x^\beta}{\alpha}\right), \tag{2}$$

where $G(\frac{x^\beta}{\alpha}) = \int_0^{\frac{x^\beta}{\alpha}} t^{\frac{1}{\beta}} \exp(-t)dt$. Let γ_c satisfy

$$f(\gamma_c, \alpha) = c. \tag{3}$$

In this case the optimal stopping rule τ_c^* will depend on the unknown parameter α. Therefore, while α is replaced by its estimator $\hat{\alpha}$, we obtain an adaptive stopping rule $\hat{\tau}_c$ which is

$$\hat{\tau}_c = \inf\{n \geq n_c : X_n \geq \hat{\gamma}_{c,n}\}, \tag{4}$$

where $\hat{\gamma}_{c,n}$ satisfies

$$f(\hat{\gamma}_{c,n}, \hat{\alpha}_n) = c, \tag{5}$$

$\hat{\alpha}_n = \left\{\dfrac{\overline{X}}{\Gamma(1+1/\beta)}\right\}^\beta$ and n_c is a function of c.

First we state some properties of $f(x,\alpha)$, which will be needed later.

Lemma 2.1. *For fixed α, $f(x,\alpha)$ is a strictly decreasing function in x, and $x \in (0,\infty)$.*

Proof.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \alpha^{\frac{1}{\beta}}(-\frac{\beta}{\alpha}x^{\beta-1})\frac{x}{\alpha^{\frac{1}{\beta}}}\exp(-\frac{x^\beta}{\alpha}) - \exp(-\frac{x^\beta}{\alpha}) + \frac{\beta}{\alpha}x^\beta\exp(-\frac{x^\beta}{\alpha}) \\ &= -\exp(-\tfrac{x^\beta}{\alpha}) < 0.\end{aligned}$$

We have proved that $f(x,\alpha)$ is a strictly decreasing function in x.

Lemma 2.2. *For fixed x, $f(x,\alpha)$ is a strictly increasing function in α for all $\alpha > 0$.*

Proof.

$$\begin{aligned}\frac{\partial f}{\partial \alpha} &= \frac{1}{\beta}\alpha^{\frac{1}{\beta}-1}\left[1 - G(\frac{x^\beta}{\alpha})\right] + \alpha^{\frac{1}{\beta}}\left[\frac{x^\beta}{\alpha^2}\frac{x}{\alpha^{\frac{1}{\beta}}}\exp(-\frac{x^\beta}{\alpha})\right] - \frac{x^{\beta+1}}{\alpha^2}\exp(-\frac{x^\beta}{\alpha})\\ &= \frac{1}{\beta}\alpha^{\frac{1}{\beta}-1}\left[1 - G(\frac{x^\beta}{\alpha})\right] > 0.\end{aligned}$$

We have proved that $f(x,\alpha)$ is a strictly increasing function in α for all $\alpha > 0$.

Using Lemma 2.1 and Lemma 2.2, it is easy to obtain Lemma 2.3

Lemma 2.3. *If $0 < \alpha_1 < \alpha_2$ and $f(x,\alpha_1) = f(y,\alpha_2)$ then $y > x$.*

Let γ_c satisfy $f(\gamma_c,\alpha) = c$, this implies $\boldsymbol{E}(X_1 - \gamma_c)^+ = c$ in this case. For fixed α, by Lemma 2.2, we see that γ_c is a decreasing function of c.
Therefore, we can choose c_0 small enough such that for all $c \in (0, c_0)$, $\gamma_c^\beta > \alpha$.

Lemma 2.4. *For $\beta > 1$, we have $\boldsymbol{P}(X_1 \geq \gamma_c) \geq c\left\{\beta\alpha^{-1/\beta}\right\}$*

Proof. From the equality

$$c = f(\gamma_c,\alpha) = \frac{1}{\beta}\alpha^{1/\beta}\int_{\frac{\gamma_c^\beta}{\alpha}}^{\infty} z^{1/\beta-1}exp(-z)dz$$

and $\beta > 1$, we have

$$\begin{aligned}\frac{c}{\boldsymbol{P}(X_1 \geq \gamma_c)} &= \frac{1}{\beta}\alpha^{1/\beta}\int_{\frac{\gamma_c^\beta}{\alpha}}^{\infty} z^{1/\beta-1}exp(-z+\frac{\gamma_c^\beta}{\alpha})dz\\ &\leq \frac{1}{\beta}\alpha^{1/\beta}\int_{\frac{\gamma_c^\beta}{\alpha}}^{\infty} exp(-z+\frac{\gamma_c^\beta}{\alpha})dz\\ &= \frac{1}{\beta}\alpha^{1/\beta}.\end{aligned}$$

Therefore, we obtain $\boldsymbol{P}(X_1 \geq \gamma_c) \geq c\left\{\beta\alpha^{-1/\beta}\right\}$ for all $\beta > 1$.

From Lemma 2.4, and using $\boldsymbol{P}(X_1 \geq \gamma_c) = exp(-\frac{\gamma_c^\beta}{\alpha})$, we can obtain the result of Lemma 2.5.

Lemma 2.5. *When $\beta > 1$, $\gamma_c^\beta = o(c^{-b})$, for all $b > 0$ as $c \longrightarrow 0$.*

Lemma 2.6. *Let τ_c^* be as defined in (1) and (3). Then $\{(c\tau_c^*)^p : 0 \le c \le c_0\}$ is uniformly integrable for all $p > 0$.*

Proof. Since τ_c^* is a geometric random variable, we have $c\boldsymbol{E}(\tau_c^*) = c[\boldsymbol{P}(X_1 \ge \gamma_c)]^{-1}$. Using Lemma 2.4 for all $c \in (0, c_0)$, we obtain $\sup_{0 \le c \le c_0} c\boldsymbol{E}(\tau_c^*) \le \dfrac{1}{\beta}\alpha^{1/\beta}$. This implies $\sup_{0 \le c \le c_0} \boldsymbol{E}(c\tau_c^*)^p \le M_p\{\dfrac{1}{\beta}\alpha^{1/\beta}\}^p$, where M_p only depends on p.

3. Performance of $\hat{\tau}_c$

Unlike τ_c^* the adaptive stopping rule $\hat{\tau}_c$ defined by (4) and (5) is not a geometric random variable. The key to studying the behavior of $\hat{\tau}_c$ is to approximate $\hat{\tau}_c$ by $\tau_{c,b}^+$ and $\tau_{c,b}^-$ which are defined as follows:

$$\tau_{c,b}^+ = \inf\{n \ge 1 : X_n \ge \gamma_{c,b}^+\} \tag{6}$$

and

$$\tau_{c,b}^- = \inf\{n \ge 1 : X_n \ge \gamma_{c,b}^-\} \tag{7}$$

where $\gamma_{c,b}^+$ and $\gamma_{c,b}^-$ satisfy

$$f(\gamma_{c,b}^+, (1 + c^b)^\beta \alpha) = c, \tag{8}$$

and

$$f(\gamma_{c,b}^-, (1 - c^b)^\beta \alpha) = c, \tag{9}$$

respectively. By Lemma 2.3, we have $\gamma_{c,b}^- \le \gamma_c \le \gamma_{c,b}^+$. For fixed positive α, the function $f(x, \alpha)$ is a function of x only and let $f(x, \alpha) = h(x)$. From (2), (3), (6), and (7) it is easy to obtain Lemma 3.1.

Lemma 3.1. *For fixed positive α, and $\beta > 1$, we have*

(*a*) $\gamma_c = h^{-1}(c)$;

(*b*) $\gamma_{c,b}^+ = (1 + c^b)h^{-1}(\frac{c}{1+c^b})$;

(*c*) $\gamma_{c,b}^- = (1 - c^b)h^{-1}(\frac{c}{1-c^b})$.

Proof. From (2) and integration by parts, we have $f(\gamma_c, \alpha) = \frac{1}{\beta}\alpha^{\frac{1}{\beta}} \int_{\frac{\gamma_c^\beta}{\alpha}}^{\infty} z^{\frac{1}{\beta}-1} \exp(-z)dz = c$. For fixed α,

$$f(\gamma_c,\alpha)=h(\gamma_c)=c,$$

hence $\gamma_c=h^{-1}(c)$.
For (b), from the definition we have

$$\begin{aligned} f(\gamma^+_{c,b},(1+c^b)^\beta\alpha) &= \frac{1}{\beta}(1+c^b)\alpha^{\frac{1}{\beta}}\int_{(\frac{\gamma^+_{c,b}}{1+c^b})^\beta\frac{1}{\alpha}}^{\infty} z^{\frac{1}{\beta}-1}\exp(-z)dz \\ &= (1+c^b)h(\frac{\gamma^+_{c,b}}{1+c^b})=c. \end{aligned}$$

We obtain $\gamma^+_{c,b}=(1+c^b)h^{-1}(\frac{c}{1+c^b})$. Similarly we can obtain $\gamma^-_{c,b}=(1-c^b)h^{-1}(\frac{c}{1-c^b})$.

Lemma 3.2. *For any* $b>0$, $0\le\gamma^+_{c,b}-\gamma^-_{c,b}=o(c^{b/4})$.

Proof. Since

$$h^{-1}(\frac{c}{1+c^b})\ge h^{-1}(\frac{c}{1-c^b}),$$

and by Lemma 3.1, we have

$$\begin{aligned} 0\le\gamma^+_{c,b}-\gamma^-_{c,b} &= h^{-1}(\frac{c}{1+c^b})-h^{-1}(\frac{c}{1-c^b})+c^b\left[h^{-1}(\frac{c}{1+c^b})+h^{-1}(\frac{c}{1-c^b})\right] \\ &\le h^{-1}(\frac{c}{1+c^b})-h^{-1}(\frac{c}{1-c^b})+2c^bh^{-1}(\frac{c}{1+c^b}). \end{aligned}$$

Using the Mean-Value theorem, we get

$$\begin{aligned} h^{-1}(\frac{c}{1+c^b})-h^{-1}(\frac{c}{1-c^b}) &= (h^{-1})'(cx^*)\left(\frac{-2c^{b+1}}{1-c^{2b}}\right) \\ &= \left(\frac{-2c^{b+1}}{1-c^{2b}}\right)\left\{-\exp\left(-\frac{\left[h^{-1}(cx^*)\right]^\beta}{\alpha}\right)\right\}^{-1}, \end{aligned}$$

where $x^*\in(\frac{1}{1+c^b},\frac{1}{1-c^b})$ and $(h^{-1})'$ is the first derivative of h^{-1}. Using

$$h^{-1}(cx^*)\le h^{-1}(\frac{c}{1+c^b})=\frac{\gamma^+_{c,b}}{1+c^b},$$

and letting $c'=\frac{c}{1+c^b}$ in lemma 3.1,we have $\gamma_{c'}=h^{-1}(c')=\frac{\gamma^+_{c,b}}{1+c^b}$. Replacing c by c', we get

$$\begin{aligned} \gamma^+_{c,b}-\gamma^-_{c,b} &\le \frac{2c^{1+b}}{\boldsymbol{P}(X_1\ge\ \gamma_{c'})(1-c^{2b})}+2c^b\gamma_{c'} \\ &= \frac{2c^bc'}{\boldsymbol{P}(X_1\ge\ \gamma_{c'})(1-c^b)}+2c^b\gamma_{c'} \\ &= \frac{2c^b}{1-c^b}O(1)+2c^bo(c^{-\frac{b}{4}})\le o(c^{-\frac{b}{4}}). \end{aligned}$$

Since $\tau^+_{c,b}$ and $\tau^-_{c,b}$ are geometric distributed, it is easy to obtain Lemma 3.3.

Lemma 3.3.

(i) $\{(c\tau^+_{c,b})^p : 0 < c \le c_0\}$ *is uniformly integrable for all* $p > 0$.

(ii) $\{(c\tau^-_{c,b})^p : 0 < c \le c_0\}$ *is uniformly integrable for all* $p > 0$.

Lemma 3.4. *Let* $\hat{\tau}_c$ *be as defined in (4) and (5) with* $n_c = \delta c^{-\theta}$ *and* $0 \le \theta \le 1$. *For* $0 < b < \frac{\theta}{2}$ *and as* $c \longrightarrow 0$, *we have*

$$\boldsymbol{E}(\hat{\tau}_c) \le o(1) + (n_c - 1) + \boldsymbol{E}(\tau^+_{c,b}).$$

Proof. We define

$$L_{c,b} = \sup\{n \ge 1 : |\overline{X}_n - \alpha^{\frac{1}{\beta}}\Gamma(1+\frac{1}{\beta})| \ge c^b\alpha^{\frac{1}{\beta}}\Gamma(1+\frac{1}{\beta})\}. \tag{10}$$

By [3,Theorem 7],

$$\{(c^{2b}L_{c,b})^p : 0 \le c \le c_0\} \text{ is uniformly integrable for all } p > 0. \tag{11}$$

For K sufficiently large, $Kc^{-1} > 2n_c$ for $c < c_0$. Treating $Kc^{-1}/2$ as an integer, we have

$$\boldsymbol{P}(c\hat{\tau}_c > K) \le \boldsymbol{P}(L_{c,b} > Kc^{-1}/2) + \boldsymbol{P}(c\hat{\tau}_c > K, L_{c,b} \le Kc^{-1}/2) \tag{12}$$

From the definition of $L_{c,b}$, we have

$$\{L_{c,b} \le Kc^{-1}/2\} = \{\alpha(1-c^b)^\beta \le \hat{\alpha}_n \le \alpha(1+c^b)^\beta \text{ for all } Kc^{-1}/2 < n\},$$

where $\hat{\alpha}_n = \left(\frac{\overline{X}_n}{\Gamma(1+\frac{1}{\beta})}\right)^\beta$ and $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Therefore we obtain

$$\begin{aligned} &\{c\hat{\tau}_c > K, L_{c,b} \le Kc^{-1}/2\} \\ &\subseteq \{\alpha(1-c^b)^\beta \le \hat{\alpha}_n \le \alpha(1+c^b)^\beta, X_n < \hat{\gamma}_{c,n} \text{ for all } Kc^{-1}/2 < n < Kc^{-1}\} \\ &\subseteq \{X_n < \gamma^+_{c,b} \text{ for all } Kc^{-1}/2 < n < Kc^{-1}\} \\ &\subseteq \{\tilde{\tau}^+_{c,b} > Kc^{-1}/2\} \end{aligned} \tag{13}$$

where $\tilde{\tau}^+_{c,b} \equiv \inf\{m \ge 1 : X_{m+Kc^{-1}/2} \ge \gamma^+_{c,b}\}$. By (12) and (13), we get

$$\boldsymbol{P}(c\hat{\tau}_c > K) \le \boldsymbol{P}(c^{2b}L_{c,b} > K/2) + \boldsymbol{P}(c\tilde{\tau}^+_{c,b} > K/2).$$

From (11) and Lemma 3.3, it is easy to see that

(14) $\{(c\hat{\tau}_c)^p : 0 \le c \le c_0\}$ is uniformly integrable for all $p > 0$.

Taking $p = 2$ in (14) and $p > (\theta/2 - b)^{-1}$ in (11), we get

$$\begin{aligned}\boldsymbol{E}(\hat{\tau}_c) &= \boldsymbol{E}(\hat{\tau}_c \boldsymbol{I}_{[L_{c,b} \ge n_c]}) + \boldsymbol{E}(\hat{\tau}_c \boldsymbol{I}_{[L_{c,b} < n_c]}) \\ &\le \left[\boldsymbol{E}(\hat{\tau}_c^2)\right]^{1/2} \left[\boldsymbol{P}(L_{c,b} \ge n_c)\right]^{1/2} + \boldsymbol{E}(\inf\{n \ge n_c : X_n \ge \ \gamma_{c,b}^{+}\}) \\ &\le \left[\boldsymbol{E}(\hat{\tau}_c^2)\right]^{1/2} n_c^{-p/2} \left[\boldsymbol{E}(L_{c,b})^p\right]^{1/2} + (n_c - 1) + \boldsymbol{E}(\tau_{c,b}^{+}) \\ &= o(1) + (n_c - 1) + \boldsymbol{E}(\tau_{c,b}^{+}).\end{aligned}$$

The proof is completed.

Lemma 3.5. *Let $\hat{\tau}_c$ be as defined in (4) and (5) with $n_c = \delta c^{-\theta}$ and $0 < \theta < 1$. Then for $0 < b < \theta/2$, as $c \longrightarrow 0$,*

$$\boldsymbol{E}(\hat{\tau}_c) \ge \boldsymbol{E}(\tau_{c,b}^{-}) + (n_c - 1) - o(1).$$

Proof. Let $L_{c,b}$ be as defined in (10)

$$\begin{aligned}\boldsymbol{E}(\hat{\tau}_c) &\ge \boldsymbol{E}(\hat{\tau}_c \boldsymbol{I}_{[L_{c,b} < n_c]}) \\ &\ge \boldsymbol{E}(\left[\inf\{n \ge n_c : X_n \ge \gamma_{c,b}^{-}\}\right] \boldsymbol{I}_{[L_{c,b} < n_c]}) \\ &\ge \boldsymbol{E}(\inf\{n \ge n_c : X_n \ge \gamma_{c,b}^{-}\}) - \boldsymbol{E}(\left[\inf\{n \ge n_c : X_n \ge \gamma_{c,b}^{-}\}\right] \boldsymbol{I}_{[L_{c,b} \ge n_c]}).\end{aligned}$$

Taking $p = 2$ in Lemma 3.3 and $p > (\theta/2 - b)^{-1}$ in (11), we have

$$\begin{aligned}\boldsymbol{E}(\hat{\tau}_c) &\ge (n_c - 1) + \boldsymbol{E}(\tau_{c,b}^{-}) - \{\boldsymbol{E}\left[(n_c - 1) + \tau_{c,b}^{-}\right]^2\}^{1/2}\{\left[n_c^{-p}\boldsymbol{E}(L_{c,b})^p\right]^{1/2}\} \\ &\ge (n_c - 1) + \boldsymbol{E}(\tau_{c,b}^{-}) - \{O(c^{-2\theta}) + O(c^{-\theta-1}) + O(c^{-2})\}^{1/2} O(c^{(\theta/2-b)p}) \\ &\ge (n_c - 1) + \boldsymbol{E}(\tau_{c,b}^{-}) - O(c^{(\theta/2-b)p-1}) \\ &\ge (n_c - 1) + \boldsymbol{E}(\tau_{c,b}^{-}) - o(c^q), \quad \text{for some } q > 0.\end{aligned}$$

From (11), it is easy to obtain Lemma 3.6.

Lemma 3.6. *Let $L_{c,b}$ be as defined in (10). If $n_c = \delta c^{-\theta}$ for some $\delta > 0$ and $0 < \theta < 1$, then*

$$\sum_{j=n_c}^{\infty} \boldsymbol{E}(|X_j| \boldsymbol{I}_{[L_{c,b} \ge j]}) \longrightarrow 0, \quad \text{as } c \longrightarrow 0;$$

for $b \in (0, \theta/2)$.

Proof. Since

$$\begin{aligned}
&\sum_{j=n_c}^{\infty} \boldsymbol{E}(|X_j| \boldsymbol{I}_{\left[L_{c,b} \geq j\right]}) \\
&\leq \sum_{j=n_c}^{\infty} \{\boldsymbol{E}(X_j^2) \boldsymbol{P}(L_{c,b} \geq j)\}^{1/2} \\
&\leq \sum_{j=n_c}^{\infty} \{\boldsymbol{E}(X_j^2) \boldsymbol{E}(L_{c,b})^{2p_1} / j^{2p_1}\}^{1/2} \\
&\leq \{\boldsymbol{E}(X_1^2)\}^{1/2} \left[\boldsymbol{E}(c^{2b} L_{c,b})^{2p_1}\right]^{1/2} \sum_{j=n_c}^{\infty} c^{-2bp_1} j^{-p_1} \\
&\leq \{\boldsymbol{E}(X_1^2)\}^{1/2} \left[\boldsymbol{E}(c^{2b} L_{c,b})^{2p_1}\right]^{1/2} c^{-2bp_1} O(n_c^{-p_1+1}) \\
&\leq O(c^{-2bp_1 - \theta + p_1 \theta}) \\
&= O(c^{p_1(\theta - 2b) - \theta}).
\end{aligned}$$

Therefore taking p_1 such that $p_1(\theta - 2b) - \theta > 0$, we have

$$\sum_{j=n_c}^{\infty} \boldsymbol{E}(|X_j| \boldsymbol{I}_{\left[L_{c,b} \geq j\right]}) \longrightarrow 0, \text{ as } c \to 0.$$

Lemma 3.7. $\boldsymbol{E}(X_{\hat{\tau}_c}) \geq \boldsymbol{E}(X_1 \boldsymbol{I}_{\left[X_1 \geq \gamma_{c,b}^+\right]}) \boldsymbol{E}(\tau_{c,b}^-) + o(1)$ *as* $c \longrightarrow 0$.

Proof. Let $L_{c,b}$ be as defined in (10)

$$\begin{aligned}
\boldsymbol{E}(X_{\hat{\tau}_c}) &= \sum_{j=n_c}^{\infty} \boldsymbol{E}(X_j \boldsymbol{I}_{[\hat{\tau}_c = j]}) \\
&\geq \sum_{j=n_c}^{\infty} \boldsymbol{E}(X_j \boldsymbol{I}_{\left[\hat{\tau}_c = j, L_{c,b} < j\right]}) + o(1) \\
&\geq \sum_{j=n_c}^{\infty} \boldsymbol{E}(X_j \boldsymbol{I}_{\left[\hat{\tau}_c = j, \alpha(1-c^b)^\beta \leq \hat{\alpha}_n \leq \alpha(1+c^b)^\beta \text{ for all } n \geq j\right]}) + o(1) \\
&\geq \sum_{j=n_c}^{\infty} \boldsymbol{E}(X_j \boldsymbol{I}_{\left[\hat{\tau}_c = j, \alpha(1-c^b)^\beta \leq \hat{\alpha}_n \leq \alpha(1+c^b)^\beta \text{ for all } n \geq j, X_j \geq \gamma_{c,b}^+\right]}) + o(1) \\
&\geq \sum_{j=n_c}^{\infty} \boldsymbol{E}(X_j \boldsymbol{I}_{\left[\hat{\tau}_c \geq j, \alpha(1-c^b)^\beta \leq \hat{\alpha}_n \leq \alpha(1+c^b)^\beta \text{ for all } n \geq j, X_j \geq \gamma_{c,b}^+\right]}) + o(1) \\
&= \sum_{j=n_c}^{\infty} \boldsymbol{E}(X_j \boldsymbol{I}_{\left[X_j \geq \gamma_{c,b}^+\right]} \boldsymbol{I}_{\left[\hat{\tau}_c \geq j, L_{c,b} < j\right]})
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{j=n_c}^{\infty} \boldsymbol{E}(X_j \boldsymbol{I}_{[X_j \geq \gamma_{c,b}^{+}]} \{\boldsymbol{I}_{[\hat{\tau}_c \geq j]} - \boldsymbol{I}_{[\hat{\tau}_c \geq j, L_{c,b} \geq j]}\}) \\
&\geq \sum_{j=n_c}^{\infty} \boldsymbol{E}(X_j \boldsymbol{I}_{[X_j \geq \gamma_{c,b}^{+}]} \boldsymbol{I}_{[\hat{\tau}_c \geq j]}) - \sum_{j=n_c}^{\infty} \boldsymbol{E}(X_j \boldsymbol{I}_{[X_j \geq \gamma_{c,b}^{+}]} \boldsymbol{I}_{[\hat{\tau}_c \geq j, L_{c,b} \geq j]}).
\end{aligned}
$$

From Lemma 3.6, it is easy to obtain

$$
\begin{aligned}
\boldsymbol{E}(X_{\hat{\tau}_c}) &\geq \sum_{j=n_c}^{\infty} \boldsymbol{P}\{\hat{\tau}_c \geq j\} \boldsymbol{E}(X_j \boldsymbol{I}_{[X_j \geq \gamma_{c,b}^{+}]}) + o(1) \\
&= \boldsymbol{E}(X_1 \boldsymbol{I}_{[X_1 \geq \gamma_{c,b}^{+}]}) \left[\boldsymbol{E}(\hat{\tau}_c) - (n_c - 1)\right] + o(1).
\end{aligned}
$$

By Lemma 3.5, we have

$$
\boldsymbol{E}(X_{\hat{\tau}_c}) \geq \boldsymbol{E}(X_1 \boldsymbol{I}_{[X_1 \geq \gamma_{c,b}^{+}]}) \boldsymbol{E}(\tau_{c,b}^{-}) + o(1).
$$

Lemma 3.8. *For all* $b > 0$, $\gamma_c(1 - \boldsymbol{P}(X_1 \geq \gamma_{c,b}^{+})/\boldsymbol{P}(X_1 \geq \gamma_{c,b}^{-})) \longrightarrow 0$ *as* $c \longrightarrow 0$.

Proof. Note that $\boldsymbol{P}(X_1 \geq \gamma_{c,b}^{-}) = exp(-\frac{(\gamma_{c,b}^{-})^{\beta}}{\alpha})$ and $\gamma_c\{1 - \boldsymbol{P}(X_1 \geq \gamma_{c,b}^{+})/\boldsymbol{P}(X_1 \geq \gamma_{c,b}^{-})\} = \gamma_c\{\boldsymbol{P}(\gamma_{c,b}^{-} \leq X_1 \leq \gamma_{c,b}^{+})\}/\boldsymbol{P}(X_1 \geq \gamma_{c,b}^{-})$. Using the Mean Value theorem to compute $\boldsymbol{P}(\gamma_{c,b}^{-} \leq X_1 \leq \gamma_{c,b}^{+})$, we have

$$
\begin{aligned}
&\gamma_c(1 - \boldsymbol{P}(X_1 \geq \gamma_{c,b}^{+})/\boldsymbol{P}(X_1 \geq \gamma_{c,b}^{-})) \\
\leq\ & \gamma_c(\gamma_{c,b}^{+} - \gamma_{c,b}^{-}) \frac{\beta}{\alpha} \left[\gamma_{c,b}^{+}\right]^{\beta - 1} \\
\leq\ & o(c^{b/4}) O(c^{-\eta}), \quad \text{for all } \eta > 0
\end{aligned}
$$

Therefore, we can choose η such that $b/4 > \eta$, then

$$
\gamma_c(1 - \boldsymbol{P}(X_1 \geq \gamma_{c,b}^{+})/\boldsymbol{P}(X_1 \geq \gamma_{c,b}^{-})) \longrightarrow 0 \quad \text{as} \quad c \longrightarrow 0.
$$

Lemma 3.9. $\boldsymbol{E}(X_1 \boldsymbol{I}_{[\gamma_c \leq X_1 \leq \gamma_{c,b}^{+}]}) \boldsymbol{E}(\tau_{c,b}^{-}) \longrightarrow 0$ *as* $c \longrightarrow 0$.

Proof.

$$
\begin{aligned}
\boldsymbol{E}(X_1 \boldsymbol{I}_{[\gamma_c \leq X_1 \leq \gamma_{c,b}^{+}]}) \boldsymbol{E}(\tau_{c,b}^{-}) &\leq \gamma_{c,b}^{+} \boldsymbol{P}(\gamma_{c,b}^{-} \leq X_1 \leq \gamma_{c,b}^{+})/\boldsymbol{P}(X_1 \geq \gamma_{c,b}^{-}) \\
&\leq (\gamma_c + o(c^{b/4}))(1 - \boldsymbol{P}(X_1 \geq \gamma_{c,b}^{+})/\boldsymbol{P}(X_1 \geq \gamma_{c,b}^{-})).
\end{aligned}
$$

Using lemma 3.8 it is easy to obtain the result.

Lemma 3.10. $c\{\boldsymbol{E}(\hat{\tau}_c) - \boldsymbol{E}(\tau^-_{c,b})\} \longrightarrow 0$ *as* $c \longrightarrow 0$.

Proof. From Lemma 3.4., we have

$$\begin{aligned} c\{\boldsymbol{E}(\hat{\tau}_c) - \boldsymbol{E}(\tau^-_{c,b})\} &\le c\{\boldsymbol{E}(\tau^+_{c,b}) - \boldsymbol{E}(\tau^-_{c,b})\} + c(n_c - 1) + o(1) \\ &= c\{\boldsymbol{E}(\tau^+_{c,b}) - \boldsymbol{E}(\tau^-_{c,b})\} + o(1) \\ &= \frac{c\boldsymbol{P}(\gamma^-_{c,b} \le X_1 \le \gamma^+_{c,b})}{\boldsymbol{P}(X_1 \ge \gamma^+_{c,b})\boldsymbol{P}(X_1 \ge \gamma^-_{c,b})} + o(1) \\ &\le \frac{c\boldsymbol{P}(\gamma^-_{c,b} \le X_1 \le \gamma^+_{c,b})}{\boldsymbol{P}(X_1 \ge \gamma^+_{c,b})\boldsymbol{P}(X_1 \ge \gamma_c)} + o(1) \\ &\le \frac{c}{\boldsymbol{P}(X_1 \ge \gamma_c)}(\gamma^+_{c,b} - \gamma^-_{c,b})\left[\gamma^+_{c,b}\right]^{\beta-1} + o(1). \end{aligned}$$

Using Lemma 2.4. and Lemma 3.2., we have

$$c\{\boldsymbol{E}(\hat{\tau}_c) - \boldsymbol{E}(\tau^-_{c,b})\} \le O(1)o(c^{b/4})o(c^{-\eta\beta}) \quad \text{for all } \eta > 0.$$

We choose η such that $\eta\beta \le b/4$ and the result of this lemma follows.

Theorem. *Let* $\hat{\tau}_c$ *be as defined in (4) and (5) with* $n_c = \delta c^{-\theta}$ *for some* $\delta > 0$ *and* $0 < \theta < 1$. *Then*

$$\boldsymbol{E}(Y_{\tau^*_c}) - \boldsymbol{E}(Y_{\hat{\tau}_c}) \longrightarrow 0 \quad \text{as } c \longrightarrow 0.$$

That is, the expected loss due to not knowing α *vanishes when we use the approximating rule* $\hat{\tau}_c$ *as* $c \longrightarrow 0$.

Proof.

$$\begin{aligned} 0 &\le \boldsymbol{E}(Y_{\tau^*_c}) - \boldsymbol{E}(Y_{\hat{\tau}_c}) = \gamma_c - \boldsymbol{E}(X_{\hat{\tau}_c}) + c\boldsymbol{E}(\hat{\tau}_c) \\ &\le \gamma_c - \boldsymbol{E}(X_1\boldsymbol{I}_{\left[X_1 \ge \gamma^+_{c,b}\right]})\boldsymbol{E}(\tau^-_{c,b}) + c\boldsymbol{E}(\hat{\tau}_c) \\ &\le \gamma_c - \{\boldsymbol{E}(X_1\boldsymbol{I}_{[X_1 \ge \gamma_c]}) - \boldsymbol{E}(X_1\boldsymbol{I}_{\left[\gamma_c \le X_1 \le \gamma^+_{c,b}\right]})\}\boldsymbol{E}(\tau^-_{c,b}) + c\boldsymbol{E}(\hat{\tau}_c) + o(1). \end{aligned}$$

From Lemma 3.7., the second inequality holds. Using Lemma 3.9. and the equality

$$\boldsymbol{E}(X_1\boldsymbol{I}_{[X_1 \ge \gamma_c]}) = c + \gamma_c\boldsymbol{P}(X_1 \ge \gamma_c),$$

we have

$$\begin{aligned} 0 &\le \boldsymbol{E}(Y_{\tau^*_c}) - \boldsymbol{E}(Y_{\hat{\tau}_c}) \\ &\le \gamma_c\{1 - \boldsymbol{P}(X_1 \ge \gamma_c)/\boldsymbol{P}(X_1 \ge \gamma^-_{c,b})\} + c\{\boldsymbol{E}(\hat{\tau}_c) - \boldsymbol{E}(\tau^-_{c,b})\} + o(1) \\ &\le \gamma_c\{1 - \boldsymbol{P}(X_1 \ge \gamma^+_{c,b})/\boldsymbol{P}(X_1 \ge \gamma^-_{c,b})\} + c\{\boldsymbol{E}(\hat{\tau}_c) - \boldsymbol{E}(\tau^-_{c,b})\} + o(1). \end{aligned}$$

By Lemma 3.8 and Lemma 3.10, we obtain

$$0 \quad \leq \quad \boldsymbol{E}(Y_{\tau_c^*}) - \boldsymbol{E}(Y_{\hat{\tau}_c}) \longrightarrow 0 \text{ as } c \longrightarrow 0.$$

The main result is proven.

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