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ON THE RECURSIVE SEQUENCE $x_{n+1} = \dfrac{\alpha + \beta x_{n-k}}{f(x_n, ..., x_{n-k+1})}$

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Abstract. The boundedness, the oscillatory behavior and the global stability of the nonnegative solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{f(x_n, ..., x_{n-k+1})}$$

is investigated, where $k \in \mathbf{N}$, the parameters α and β are nonnegative real numbers and $f : \mathbf{R}_+^k \to \mathbf{R}_+$ is a continuous function nondecreasing in each variable such that $f(0, ..., 0) > 0$.

1. Introduction

In [3] the authors investigate behavior of the nonnegative solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\gamma + x_n},$$

where the parameters α, β and γ are nonnegative real numbers.

Behavior of the nonnegative solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{1 + g(x_n)},$$

where g is a nonnegative increasing function on $[0, \infty)$, was investigated in [14].

In this paper we investigate the oscillatory behavior, the boundedness character and the global stability of the nonnegative solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{f(x_n, ..., x_{n-k+1})}, \tag{1}$$

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where $k \in \mathbf{N}$, the parameters α and β are nonnegative real numbers and $f : \mathbf{R}_+^k \to \mathbf{R}_+$ is a continuous function nondecreasing in each variable such that $f(0, ..., 0) > 0$.

This equation is a natural generalization of the above mentioned equations. Among other results in Section 5 we solve and generalize an open problem posed in [14].

Similar properties were discussed in the literature (see, e.g. [1, 2, 4, 5, 7-13, 15-18]) for several classes of nonlinear difference equations.

In what follows we may assume that the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are positive real numbers.

In our analysis the function

$$g(x) = f(x, ..., x), \quad x \geq 0$$

plays an important role.

Let $\delta = f(0, ..., 0)$. Without loss of generality we may assume that $\delta = 1$, since we can consider the equation

$$y_{n+1} = \frac{\alpha_1 + \beta_1 y_{n-k}}{f_1(y_n, ..., y_{n-k+1})}$$

where $\alpha_1 = \alpha/\delta, \beta_1 = \beta/\delta$ and $f_1(z_1, ..., z_k) = f(z_1, ..., z_k)/\delta$.

2. Semicycle Analysis About a Positive Equilibrium

A *positive semicycle* of a solution (x_n) of Eq. (1) consists of a ”string” of terms $\{x_l, x_{l+1}, ..., x_m\}$, all greater than or equal to $\bar{x}$, with $l \geq -k$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -k, \quad \text{or} \quad l > -k \quad \text{and} \quad x_{l-1} < \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \bar{x}.$$

A *negative semicycle* of a solution (x_n) Eq. (1) consists of a ”string” of terms $\{x_l, x_{l+1}, ..., x_m\}$, all less than $\bar{x}$, with $l \geq -k$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -k, \quad \text{or} \quad l > -k \quad \text{and} \quad x_{l-1} \geq \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \bar{x}.$$

The first semicycle of a solution starts with the term x_{-k} and is positive if $x_{-k} \geq \bar{x}$ and negative if $x_{-k} < \bar{x}$.

We say that a sequence (x_n) oscillates about $\bar{x}$ if for every $n_0 \in \mathbf{N}$ there are $p, q \geq n_0$ such that $(x_p - \bar{x})(x_q - \bar{x}) \leq 0$.

The following theorem is the main result of this section and it generalizes Theorem 3.2 in [3].

Theorem 1. *Let $k \in \mathbf{N}$ be fixed and consider a continuous function $H : (0,\infty)^{k+1} \to (0,\infty)$ having the following properties: There is an index $i_0 \in \{1,2,...,,k\}$ such that $H(z_1,...,z_k,y)$ is nonincreasing in each $z_i, i \in \{1,...,k\} \setminus \{i_0\}$, decreasing in z_{i_0}, and increasing in y. Let $\bar{x}$ be a positive equilibrium of the difference equation*

$$x_{n+1} = H(x_n,...,x_{n-k+1},x_{n-k}), \quad n = 0,1,2,... \tag{2}$$

Then, except possibly for the first semicycle, every oscillatory solution of Eq. (2) *with positive initial values has semicycles of length at most k.*

Proof. Let (x_n) be an oscillatory solution of Eq. (2) with at least two semicycles. If a semicycle has length greater than or equal to k, then there is an $N \geq 0$ such that either

$$x_{N-k} < \bar{x} \leq x_{N-k+1},...,x_N \quad \text{or} \quad x_{N-k} \geq \bar{x} > x_{N-k+1},...,x_N.$$

Using the conditions of the theorem in the first case we obtain

$$x_{N+1} = H(x_N,...,x_{N-k+1},x_{N-k}) < H(\bar{x},...,\bar{x}) = \bar{x},$$

and in the second case we get

$$x_{N+1} = H(x_N,...,x_{N-k+1},x_{N-k}) > H(\bar{x},...,\bar{x}) = \bar{x},$$

as desired.

Corollary 1. *Consider Eq.* (1) *where $\alpha,\beta > 0$, $k \in \mathbf{N}$, the initial conditions $x_{-k},x_{-k+1},...,x_{-1}$ and x_0 of Eq.* (1) *are arbitrary positive numbers and $f : \mathbf{R}_+^k \to \mathbf{R}_+$ is a continuous function nondecreasing in each variable and increasing in at least one.*

Then except possibly for the first semicycle, every oscillatory solution of Eq. (1) *has semicycles of length at most k.*

3. The Case $\beta < 1$

In this section we consider the case $\beta < 1$.

Theorem 2. *Consider Eq.* (1)*, where $\alpha > 0$, $\beta \in [0,1)$, $k \in \mathbf{N}$, the initial conditions $x_{-k},x_{-k+1},...,x_{-1}$ and x_0 of Eq.* (1) *are arbitrary positive numbers,*

$f : \mathbf{R}_+^k \to \mathbf{R}_+$ is a continuous function nondecreasing in each variable and where the function $g(x)$ satisfies the following conditions:

(a) $g(0) = 1$;

(b) $g(x)$ *is increasing on* $[0, \infty)$;

(c) $x/(g(x) - 1)$ *is nondecreasing on* $[0, \infty)$.

Then every positive solution of Eq. (1) *converges.*

Proof. First we prove that Eq. (1) has the unique positive equilibrium. The equilibrium points $\bar{x}$ of Eq. (1) satisfy the equation

$$\bar{x} = \frac{\alpha + \beta\bar{x}}{g(\bar{x})}.$$

Let $G(x) = x - \frac{\alpha+\beta x}{g(x)}$. It is clear that G is a continuous function on $[0, \infty)$ such that $G(0) = -\frac{\alpha}{g(0)} < 0$ and $\lim_{x\to+\infty} G(x) = +\infty$. By a well known theorem it follows that there is an $x^* \in (0, \infty)$ such that $G(x^*) = 0$. On the other hand

$$\begin{aligned} G(x) - G(y) &= x - y + \frac{\alpha + \beta y}{g(y)} - \frac{\alpha + \beta x}{g(x)} \\ &= \frac{(x-y)g(y)(g(x) - \beta) + (\alpha + \beta y)(g(x) - g(y))}{g(x)g(y)} > 0, \end{aligned}$$

if $x > y$. So $G(x)$ is an increasing function and consequently x^* is the unique positive equilibrium of Eq. (1).

Further, we prove that every positive solution of Eq. (1) is bounded. We have

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{f(x_n, ..., x_{n-k+1})} < \alpha + \beta x_{n-k} \qquad n = 0, 1, 2, ... \ . \tag{3}$$

From (3) using induction we obtain

$$\begin{aligned} x_{(k+1)m+r+1} &< x_{r-k}\beta^{m+1} + \alpha\left(1 + \beta + \cdots + \beta^m\right) \\ &< x_{r-k} + \frac{\alpha}{1-\beta}, \quad \text{for all } m \in \mathbf{N} \cup \{0\} \text{ and } r \in \{0, 1, ..., k\}, \end{aligned}$$

from which the boundedness follows.

Thus there are the finite $\liminf_{n\to\infty} x_n = l$ and $\limsup_{n\to\infty} x_n = L$. Letting $\liminf_{n\to\infty}$ and $\limsup_{n\to\infty}$ in (1) we obtain

$$l \geq \frac{\alpha + \beta l}{g(L)} \quad \text{and} \quad L \leq \frac{\alpha + \beta L}{g(l)}.$$

From this and by *(c)* we obtain

$$\alpha + (\beta - 1)L \geq L(g(l) - 1) \geq l(g(L) - 1) \geq \alpha + (\beta - 1)l.$$

Since $\beta \in [0, 1)$ we obtain $l = L = x^*$, as desired.

Example 1. Consider the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{1 + x_n^{\gamma}}, \quad n = 0, 1, \ldots \ ,$$

where $k \in \mathbf{N}$, $\alpha \in (0, \infty), \beta \in [0, 1)$ and $\gamma \in (0, 1)$. Then every positive solution of the equation converges.

Similarly to Theorem 2 we can prove the following theorem.

Theorem 2 (a). *Consider the difference equation*

$$x_{n+1} = \frac{\alpha + \beta x_n}{f(x_n, \ldots, x_{n-k})}, \quad n = 0, 1, 2, \ldots \tag{4}$$

where $\alpha > 0$, $\beta \in [0, 1)$, $k \in \mathbf{N}$, *the initial conditions* $x_{-k}, x_{-k+1}, \ldots, x_{-1}$ *and* x_0 *of Eq.* (4) *are arbitrary positive numbers,* $f : \mathbf{R}_+^k \to \mathbf{R}_+$ *is a continuous function nondecreasing in each variable and where the corresponding function* $g(x)$ *satisfies all conditions as in Theorem 2. Then every positive solution of Eq.* (4) *converges.*

4. The Case $\beta > 1$

In this section we assume that $\beta > 1$ and show that there exist unbounded solutions of Eq. (1). Although it is interesting to know the behavior of all solutions of Eq. (1), we provide results only for the case $k = 2m + 1$ and the odd terms of the solutions do not affect the denumerator in Eq. (1). This means that we focus our attention on the case

$$f(u_1, u_2, u_3, \ldots, u_{2m-1}, u_{2m}, u_{2m+1}) = F(u_1, u_3, \ldots, u_{2m-1}, u_{2m+1}),$$

and so Eq. (1) takes the form

$$x_{n+1} = \frac{\alpha + \beta x_{n-2m-1}}{F(x_n, x_{n-2}, x_{n-4} \ldots, x_{n-2m})}. \tag{5}$$

We assume that $F : \mathbf{R}_+^{m+1} \to \mathbf{R}_+$ is a given continuous function, nondecreasing in each variable, increasing in at least one and $F(0, \ldots, 0) = 1$. Hence $g(x)$ is a real continuous function defined on the interval $[0, \infty)$ which satisfies conditions *(a)* and *(b)* in Theorem 2 and consequently g^{-1} is continuous increasing function on $[0,\infty)$.

In this case we show that there exist positive solutions of Eq. (5), which are unbounded, moreover we show that there exist positive solutions of Eq. (5) such that the subsequence $x_{2n+1}\to 0$ as $n\to\infty$ and $x_{2n}\to\infty$ as $n\to\infty$. Choose

$$x_{-(2m+1)},...,x_{-1}\in(0,g^{-1}(\beta))$$

and

$$\min\{x_{-2m},...,x_0\}>g^{-1}\left(\frac{\alpha}{g^{-1}(\beta)}+\beta\right).$$

It is clear that

$$x_1=\frac{\alpha+\beta x_{-(2m+1)}}{F(x_0,...,x_{-2m})}<\frac{\alpha+\beta g^{-1}(\beta)}{g(\min\{x_0,...,x_{-2m}\})}<g^{-1}(\beta)$$

and

$$x_2=\frac{\alpha+\beta x_{-2m}}{F(x_1,...,x_{-(2m-1)})}>\frac{\alpha+\beta x_{-2m}}{F(g^{-1}(\beta),...,g^{-1}(\beta))}=x_{-2m}+\frac{\alpha}{\beta}.$$

Similarly we have

$$x_{2s-1}=\frac{\alpha+\beta x_{-(2(m-s+1)+1)}}{F(x_{2s-2},...,x_{2s-2m-2})}<\frac{\alpha+\beta g^{-1}(\beta)}{g(\min\{x_{2s-2},...,x_{2s-2m-2}\})}<g^{-1}(\beta)$$

and

$$x_{2s}=\frac{\alpha+\beta x_{-2(m-s+1)}}{F(x_{2s-1},...,x_{2s-(2m+1)})}>\frac{\alpha+\beta x_{-2(m-s+1)}}{F(g^{-1}(\beta),...,g^{-1}(\beta))}=x_{-2(m-s+1)}+\frac{\alpha}{\beta}.$$

for $s=2,...,m+1$.

By induction we obtain

$$x_{2n-1}<g^{-1}(\beta)\quad\text{and}\quad x_{2s+(2m+2)l}>x_{2s+(2m+2)(l-1)}+(l+1)\frac{\alpha}{\beta},$$

for $l\geq 0$ and $s=1,2,...,m+1$. Hence $\lim_{n\to\infty}x_{2n}=\infty$ and consequently $\lim_{n\to\infty}x_{2n+1}=0$.

If $\alpha=0$, then as above $x_{2n-1}\in(0,g^{-1}(\beta))$ for all $n\in\mathbf{N}$ and $x_{2s+(2m+2)l}>x_{2s+(2m+2)(l-1)}>g^{-1}(\beta)$ for all $l\in\mathbf{N}\cup\{0\}$ and $s=1,2,...,m+1$. Hence the limits $\lim_{l\to\infty}x_{2s+(2m+2)l}$ are finite or $+\infty$. Assume that all these limits are finite, say p_s. Since the sequence (x_{2n-1}) is bounded there is finite $\limsup_{n\to\infty}x_{2n-1}=L\geq 0$. Assume that $L>0$. Then letting $n=2r\to\infty$ in (5) we obtain

$$L\leq\frac{\beta L}{F(p_s,...,p_{s-m-1})}<\frac{\beta L}{F(g^{-1}(\beta),...,g^{-1}(\beta))}=L,$$

which is a contradiction. Hence $L=0$.

5. The Case $\beta = 1$, $\alpha > 0$

In this section we consider the equation:

$$x_{n+1} = \frac{\alpha + x_{n-k}}{f(x_n, ..., x_{n-k+1})} \quad n \in \mathbf{N}. \tag{6}$$

The equilibrium points $\bar{x}$ of Eq. (6) satisfy the equation $\bar{x}g(\bar{x}) = \bar{x} + \alpha$. Since the function $x(g(x) - 1)$ is increasing there is the unique positive equilibrium $\bar{x} = x^*$, of Eq. (6).

Before we formulate and prove the main result we need an auxiliary result which is incorporated in the following lemma.

Lemma 1. *Let h be a function which satisfies the conditions*

(a) $h(0) \geq 0$.

(b) $h(x)$ *is increasing on* $[0, \infty)$;

(c) $x/h(x)$ *is nondecreasing on* $[0, \infty)$. *Then for given* l, L, $\alpha > 0$ *such that* $l < L$, *there exist* l_0 *and* L_0 *such that*

1. $0 < l_0 \leq l$ *and* $L \leq L_0$;
2. $l_0 h(L_0) \leq \alpha \leq h(l_0) L_0$.

Proof. Since $l < L$ and by (c) we see that $lh(L) \leq h(l)L$. Now we look at the number α. There are three cases.

(1) (1) $lh(L) \leq \alpha \leq h(l)L$. There is nothing to prove.

(2) $\alpha < lh(L)$. Now we may fix L and decrease the number l. By continuity of the functions $xh(L)$ and $h(x)L$, and since $xh(L) \leq h(x)L$ for $x < L$, we obtain that there exists $l_0 > 0$ such that $l_0 h(L) \leq \alpha \leq h(l_0)L$, as desired.

(3) $h(l)L < \alpha$. Now we may fix l and increase the number L. By continuity of the functions $lh(x)$ and $h(l)x$, and since $lh(x) \leq h(l)x$ for $l < x$, we obtain that there exists L_0 such that $lh(L_0) \leq \alpha \leq h(l)L_0$.

Theorem 3. *Consider Eq.* (6), *where* $\alpha > 0$, $k \in \mathbf{N}$, *the initial conditions* $x_{-k}, x_{-k+1}, ..., x_{-1}$ *and* x_0 *of Eq.* (6) *are arbitrary positive numbers,* $f : \mathbf{R}_+^k \to \mathbf{R}_+$ *is a continuous function nondecreasing in each variable and increasing in at least one, the function* $x/(g(x) - 1)$ *is nondecreasing and* $g(0) \geq 1$. *Then every solution of Eq.* (6) *is bounded and persists.*

Proof. Choose l and L such that $L > \max\{x_{-k}, ..., x_0\}$ and $\min\{x_{-k}, ..., x_0\} > l > 0$. By Lemma 1 applied on $h(x) = g(x) - 1$, we may also assume that $l(g(L) - 1) \le \alpha \le (g(l) - 1)L$. Now we may use mathematical induction to prove the result. Assume the statement is true for $x_{-k}, ..., x_0, ..., x_n$, that is,

$$l \le x_i \le L \qquad \text{for all} \quad i = -k, ..., 0, 1, ..., n.$$

Then

$$x_{n+1} = \frac{\alpha + x_{n-k}}{f(x_n, ..., x_{n-k+1})} \le \frac{\alpha + L}{g(l)}.$$

We claim that $\frac{\alpha+L}{g(l)} \le L$. But this is obvious since

$$\alpha + L \le g(l)L \; \Leftrightarrow \; \alpha \le (g(l) - 1)L.$$

Similarly,

$$x_{n+1} = \frac{\alpha + x_{n-k}}{f(x_n, ..., x_{n-k+1})} \ge \frac{\alpha + l}{g(L)}.$$

It is easy to see that $x_{n+1} \ge l$. The proof is therefore complete.

Corollary 2. *Consider the difference equation*

$$x_{n+1} = \frac{\alpha + x_{n-k}}{1 + \sum_{i=0}^{k-1} \beta_i x_{n-i}}, \tag{7}$$

where $k \in \mathbf{N}$, $\alpha > 0, \beta_i \ge 0, i = 0, ..., k-1$, $\sum_{i=0}^{k-1} \beta_i > 0$, *the initial conditions* $x_{-k}, x_{-k+1}, ..., x_{-1}$ *and* x_0 *of Eq.* (7) *are nonnegative numbers. Then every solution of Eq.* (7) *is bounded and persists.*

Now we are in a position to formulate and prove a global convergence result.

Theorem 4. *Consider Eq.* (6), *where* $\alpha > 0$, $k \in \mathbf{N}$, *the initial conditions* $x_{-k}, x_{-k+1}, ..., x_{-1}$ *and* x_0 *of Eq.* (6) *are arbitrary positive numbers,* $f : \mathbf{R}_+^k \to \mathbf{R}_+$ *is a continuous function nondecreasing in each variable and increasing in at least one, the function* $x/(g(x)-1)$ *is increasing and* $g(0) \ge 1$. *Then every solution of Eq.* (6) *converges.*

Proof. By Theorem 3 the sequence (x_n) is bounded. Thus there are the finite $\liminf_{n\to\infty} x_n = l$ and $\limsup_{n\to\infty} x_n = L$, moreover $l > 0$. Letting $\liminf_{n\to\infty}$ and $\limsup_{n\to\infty}$ in (6) we obtain

$$l \ge \frac{\alpha + l}{g(L)} \quad \text{and} \quad L \le \frac{\alpha + L}{g(l)}.$$

From this and since $x/(g(x)-1)$ is increasing we obtain

$$\alpha \geq L(g(l)-1) > l(g(L)-1) \geq \alpha,$$

which is a contradiction. Hence $l = L = x^*$, as desired.

Remark 1. Theorem 4 solves the open problem in [13].

Remark 2. The condition $x/(g(x)-1)$ is increasing, in Theorem 4, cannot be replaced by $x/(g(x)-1)$ is nondecreasing. Indeed, it is easy to see that the difference equation

$$x_{n+1} = \frac{\alpha + x_{n-1}}{1 + x_n}, \qquad n = 0, 1, \dots ,$$

has period two solutions.

6. The Case $\alpha = 0$

In this section we consider Eq. (1) where $\alpha = 0$. The case $\alpha = 0$ and $\beta > 1$ was considered in Section 4. If $\alpha = 0,\ \beta < 1$, then we have

$$x_{n+1} = \frac{\beta x_{n-k}}{f(x_n, \dots, x_{n-k+1})} \quad n \in \mathbf{N}. \tag{8}$$

From which it follows that

$$x_{n+1} < \beta x_{n-k} \quad n \in \mathbf{N}.$$

Thus the zero equilibrium is a geometrically global attractor for all positive solutions of Eq. (8) (see, Definition 1 in [6]).

If $\alpha = 0,\ \beta = 1$, then it is easy to prove the following result:

Theorem 5. *Assume that $\alpha = 0$ and $f(u_1, u_2, \dots, u_k)$ is continuous and increasing with respect to each variable in a right neighborhood of 0. If it holds $g(0) = 1$, then every positive solution of Eq. (8) converges to a period - $(k+1)$ solution of the form*

$$p,\ 0,\ 0,\ \dots,\ 0,\ p,\ 0,\ 0,\ \dots,\ 0,\ p,\ \dots .$$

Remark 3. In [3] the following problem is posed:
Is there a solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n}, \quad x_{-1}, x_0 > 0, \quad n = 0, 1, 2, \dots$$

such that $x_n \to 0$ as $n \to \infty$?

The positive answer to a more general problem is given in [11]. For readers who are interested in this area we leave the following problem:

Open Problem 1. Let the function f be as in Theorem 5 and $g(0) = 1$ and $k \geq 2$. Is there a solution of Eq. (8) such that $x_n \to 0$ as $n \to \infty$?

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