TAIWANESE JOURNAL OF MATHEMATICS
Vol. 9, No. 2, pp. 313-330, June 2005
This paper is available online at http://www.math.nthu.edu.tw/tjm/

DIMENSION PROPERTIES OF RANDOM FRACTALS WITH OVERLAPS

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Abstract. We consider random fractals generated by random recursive constructions with overlaps. Our construction allows some overlaps among sets in the same generation. We introduce a certain "limited overlaps condition". Under this condition, we prove that the Hausdorff dimension of the generated fractal satisfies the expectation equation (upon non-extinction), which was studied previously by Falconer, Graf, Mauldin and Williams under open set condition. We also prove that the generated fractal is regular in the sense that its Hausdorff and upper box dimension are equal to a non-random constant (this result holds without assumption of limited overlaps condition).

1. Introduction

In this paper we consider a general type of random fractal and some dimension properties associated with it. The random fractals considered are generated by some random recursive constructions which have been studied by Falconer (1986), Graf (1987), Mauldin-Williams (1986). The significant difference is that in their definitions and investigations open set condition always plays an essential role; while we do not impose open set condition in this paper. We shall prove that the expectation equation for the Hausdorff dimension, established in the above papers, still holds under a certain "limited overlaps condition"; which allows some overlaps among sets in the same generation. Our result also holds in the deterministic case; it asserts that Moran's formula holds under limited overlaps condition. For Moran's formula under open set condition, see Hutchinson (1981). The viewpoint of validating Moran's formula in the overlapping structure has not been considered in the previous literatures, as we know. Lau-Nagi (1999) introduced weak separation condition for an iterated function system (in the deterministic case); but Moran's

Received March 4, 2004; accepted May 6, 2004.
Communicated by Valentin Afraimovich.
2000 *Mathematics Subject Classification*: 60G17.
Key words and phrases: Random fractal, Limited overlaps condition, Hausdorff dimension, Upper box dimension, Regularity.

formula for the Hausdorff dimension does not hold(we thank to K. S. Lau for reminding us this). Indeed, in He-Lau-Rao (2003) a very different formula is established for some particular cases. We also mention Rao-Wen (1998) and Nagi-Wang (2001) for related works.

The second result in this paper concerns with the almost sure equality for the upper box dimension and the Hausdorff dimension of the generated fractal. It seems good to term a fractal to be *regular* if its upper box dimension is equal to its Hausdorff dimension (and thus the various dimension indices are equal). By Falconer (1997, Chapter 3), the invariant set for an iterated function system of similarities with open set condition is a regular fractal. Under the assumptions of geometrical self-similarity and constant branching, that random fractals generated by random recursive constructions are a.s. regular have been proved by Liu-Wu (2002). In Berlinkov-Mauldin (2002, Theorem 1), it is proved that a random fractal is a.s. regular under the assumptions of open set condition and constant branching. We prove that, without any non-overlapping and limited overlaps condition, nor geometric self-similarity condition, the fractal in our construction is regular and the dimension is equal almost surely to a non-random constant, upon non-extinction.

The remaining part of this paper is organized as follows. In Section 2, we review some notations and random recursive constructions. In Section 3, we introduce limited overlaps condition, and prove that the expectation equation for the Huasdorff dimension holds, under this condition together with some other mild conditions in Falconer (1986). In Section 4, we prove the dimensional regularity for the generated fractal. In final Section 5, we give some examples.

2. Random Fractals

We begin with the definition of code space. Let $\mathcal{N} = \{0, 1, 2, \cdots\}$ be the collection of all non-negative integers. Set the code space $\Sigma_* = \bigcup_{n=0}^{\infty} \mathcal{N}^n (\mathcal{N}^0 = \{\emptyset\})$ and $\Sigma = \mathcal{N}^\infty$. For $\sigma \in \Sigma_*$, let $|\sigma|$ be the length of σ, that is, if $\sigma = (\sigma_1, \cdots, \sigma_n)$, then $|\sigma| = n$. For $n \leq |\sigma|$ and $\sigma = (\sigma_1, \cdots, \sigma_n, \cdots, \sigma_{|\sigma|})$, write $\sigma|n = (\sigma_1, \cdots, \sigma_n)$. For any $\sigma \in \Sigma_*$ and $\tau \in \Sigma_* \cup \Sigma$, let $\sigma * \tau$ be the juxtaposition of σ and τ. For any $\sigma, \tau \in \Sigma_* \cup \Sigma$, we denote $\sigma \prec \tau \iff \tau||\sigma| = \sigma$. For any $\sigma, \tau \in \Sigma$, let $\sigma \wedge \tau$ be the longest sequence ξ such that $\xi \in \Sigma_*$ and $\xi = \sigma|k = \tau|k$ for some integer k. The tree metric is defined by $\rho(\sigma, \tau) = e^{|\sigma \wedge \tau|}$. Note that the metric space $(\partial\Gamma, \rho)$. is compact.

Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space which supports the randomization used in our paper, and let $\{N_\sigma\}$ be a family of independent random variables defined on Ω, indexed by $\sigma \in \Sigma_*$. The tree $\Gamma = \Gamma_\omega$ associated with $\{N_\sigma\}$ is a random subset of Σ_* which is characterized as follows. The root $\emptyset \in \Gamma$ and, if $\sigma \in \Gamma$ and $i \in \{1, 2, 3, \cdots\}$, then $\sigma * i \in \Gamma$ if and only if $1 \leq i \leq N_\sigma$. We write

$\Gamma_n = \{\sigma \in \Gamma : |\sigma| = n\} \subset \Sigma_*$ the set of vertices of the tree in the nth generation. The *boundary* $\partial\Gamma$ of Γ is then $\partial\Gamma = \{\sigma \in \Sigma : \sigma|n \in \Gamma_n, \forall_n\}$.

A subset $A \subset \Gamma$ is called a *cutset* of $\partial\Gamma$ if, for any $\sigma, \sigma' \in A$, neither $\sigma \prec \sigma'$ nor $\sigma' \prec \sigma$, and A separates the root and the boundary in the sense that, for each $\sigma \in \partial\Gamma$, there esists a unique $\tau \in A$ such that $\tau \prec \sigma$. Obviously, for any $n \geq 1$, Γ_n is a cutset; yet we shall need some cutsets which constist of vertices with different generations.

Let $|\cdot|$ denote the diameter of subsets of R^d. Let $\{I_\sigma(\omega) : \sigma \in \Gamma_\omega\}$, $\omega \in \Omega$, be a random collection of non-empty compact subsets of R^d and satisfy the following properties:

(a) $I_\sigma = \overline{intI_\sigma}, \forall \sigma \in \Gamma$;

(b) the open sets $\{intI_\sigma : \sigma \in \Gamma\}$ form a net, that is

$$intI_\sigma \supset intI_\tau \quad \text{if} \quad \sigma \prec \tau;$$

(c) define L_σ to be $|I_\sigma| = L_\sigma |I_{\sigma_1, \cdots, \sigma_{k-1}}|$, $\sigma = (\sigma_1, \cdots, \sigma_k) \in \Gamma_k$, the random elements $W(\sigma) = (N(\sigma); L_{\sigma*1}, \cdots, L_{\sigma*N(\sigma)})$ for $\sigma \in \partial\Gamma$ are i.i.d. with the same distribution as $W(\emptyset) = (N(\emptyset); L_1, \cdots, L_{N(\emptyset)})$, and

(2.1) $1 < m := EN(\emptyset) < \infty$.

Define the random set $K(\omega)$ by

$$K(\omega) = \bigcap_{k=0}^{\infty} \bigcup_{\sigma \in \Gamma_k(\omega)} I_\sigma(\omega).$$

In Mauldin-Williams (1986), $K(\omega)$ is referred as a fractal generated by a *random recursive construction*; while in Falconer (1986) $K(\omega)$ is called a *random net fractal*. The (c) is referred as *stochastic ratio self-similarity* in Mauldin-Williams (1986). However, in their definitions open set condition ((d) below) **is always** built-in, yet we **do not assume** this condition in our construction. Nor we assume that I_σ is geometrically similar to $I_\emptyset$.

The Hausdorff dimension of a Borel $A \subset R^d$ is denoted by *dimA*; we refer to Falconer (1990) for definitions and basic properties of various dimension indices. The following result is proved in Falconer (1986), essentially the same result appeared in Graf (1987) and Mauldin-Williams (1986). If, in addition to conditions (a)(b)(c), the following assumptions are satisfied:

(d) Open Set Condition (OSC for brevity): $intI_\sigma \cap intI_\tau = \emptyset$ whenever neither $\sigma \prec \tau$ nor $\tau \prec \sigma$;

(e) $|I_{\sigma|n}(\omega)| \to 0 (n \to \infty)$ if $\sigma \in \partial\Gamma$;

(f) there is a constant $a > 0$ such that $a|I_\sigma| \leq |I_{\sigma*i}| \leq |I_\sigma|$ if $\sigma * i \in \Gamma$;

(g) there is a non-random constant $\lambda > 0$. independent of σ, such that $inradius(I_\sigma) \geq \lambda |I_\sigma|$; then K is non-empty with probability $1 - q$, $0 < q < 1$, and

$$dimK = \beta,$$

conditional on $\{K \neq \emptyset\}$, where $\beta = \min\{s : E \sum_{i=1}^{N(\emptyset)} L_i^s \leq 1\}$.

In the above, q is the extinction probability of the Galton-Watson branching process generated by $N(\emptyset)$.

In the above result, OSC plays an essential role. Without OSC, it is much more difficult to handle the situation; as we have remarked in Section 1, there may appear very different formula for the Hausdorff dimension of invariant set of iterated function system with overlaps.

Throughout this paper, besides (2.1), we always assume that

(2.2) there are two constants $0 < r_1 \leq r_2 < 1$ such that, for almost all ω, $r_1 \leq L_i(\omega) \leq r_2$ for all $i = 1, \cdots, N(\emptyset)$.

Without loss of generality, we assume that $|I_\emptyset| = 1$ throughout this paper.

3. Hausdorff Dimension of Random Fractals with Limited Overlaps

In this section we consider the Hausdorff dimension of random fractals which satisfy a separation condition in which it allows overlaps in the structure. We define a *Limited Overlaps Condition* (LOC for brevity) as follows.

Definition 3.1. That $\{I_\sigma : \sigma \in \Gamma\}$ satisfies limited overlaps condition, if the following condition holds with probability one. There exists $\delta > 0$, such that, for all $\sigma \in \Gamma, I_\sigma$ contains a ball $B_{\delta|I_\sigma|}$ of radius $\delta|I_\sigma|$. Moreover, $B_{\delta|I_\sigma|} \cap I_\tau = \emptyset$, whenever $\sigma, \tau \in \Gamma_n$ and $\sigma \neq \tau, \forall n \geq 1$. (Note that δ may depend on the realization ω, i.e. it is a random variable).

Remark The above definition applies well to the deterministic case. In fact, the main result of this section, Theorem 3.1, holds in the deterministic case too. Thus, we obtain Moran's formula under LOC.

An important observation is that, we have $B_{\delta|I_\sigma|} \cap B_{\delta|I_\tau|} = \emptyset$ whenever neither $\sigma \prec \tau$ nor $\tau \prec \sigma$; this is due to the net condition (b) in our construction. Indeed, suppose that $\sigma \not\prec \tau$ and $|\sigma| < |\tau|$, then $\sigma \neq \tau||\sigma|$. Thus $B_{\delta|I_\tau|} \subset I_{\tau||\sigma|}$ by (b) and the latter is disjoint from $B_{\delta|I_\sigma|}$ by LOC.

If $\{I_\sigma, \sigma \in \Gamma\}$ satisfies LOC, then under the assumption (2.2), it can be seen that, with probability one, there exists $N_0 = N_0(\omega)$ such that

$$N(\sigma) \leq N_0, \forall \sigma \in \Gamma.$$

In fact,

$$E\sum_{i=1}^{N(\sigma)} vol(B_{\delta|I_{\sigma*i}|}) \leq vol(I_\sigma) \leq vol(B(x,|I_\sigma|)),$$

where $x \in I_\sigma$. Here vol means the Lebesgue measure in R^d. Note that $|I_{\sigma*i}| = L_{\sigma*i}|I_\sigma| \geq r_1|I_\sigma|$ and that $B_{\delta|I_{\sigma*i}|}$ are disjoint, we then have

$$N(\sigma) \leq \frac{1}{r_1^d\delta^d} := N_0.$$

Lemma 3.1. *Under the assumptions* (2.1) *and* (2.2), *there is a unique* $\alpha > 0$ *such that* $E\sum_{i=1}^{N(\emptyset)} L_i^\alpha = 1$. *Moreover,*

(*) $$\frac{\ln m}{\ln r_1^{-1}} \leq \alpha \leq \frac{\ln m}{\ln r_2^{-1}}.$$

Proof. Write

$$\Phi(s) = E\sum_{i=1}^{N(\emptyset)} L_i^s.$$

Then $\Phi(0) = m$, so that by (2.1) $1 < \Phi(0) < \infty$. Thus $s \to \Phi(s)$ is continuous and strictly decreasing. Note that $r_1^s m \leq \Phi(s) \leq r_2^s m$ for $\forall s \geq 0$. by (2.2).

Let $s_1 = \frac{lnm}{lnr_1^{-1}}$, $s_2 = \frac{lnm}{lnr_2^{-1}}$, for sufficiently small $\varepsilon > 0, 1 < \Phi(s_1 - \varepsilon)$ while $\Phi(s_2 + \varepsilon) < 1$. So, there exists a unique α such that $\Phi(\alpha) = 1$; moreover, (*) holds, since $\Phi(\cdot)$ is decreasing. ■

For any n, let $\mathcal{F}_n$ denote the $\sigma-$algebra of subsets of Ω:

$$\mathcal{F}_n = \sigma(\mathcal{F}_{n-1}; N(\sigma) : \sigma \in \Gamma_{n-i}; L_{\sigma*i} : 1 \leq i \leq N(\sigma)),$$

where $\mathcal{F}_1 = \sigma(N(\emptyset); L_i : 1 \leq i \leq N(\emptyset))$ and assume that $\vee_{i=1}^\infty \mathcal{F}_i \subset \mathcal{F}$. Let α be the unique positive number in Lemma 3.1, namely $\Phi(\alpha) = 1$.

Let

$$\bar{L}_\sigma^\alpha(\omega) = \prod_{i=1}^{|\sigma|} L_{\sigma|i}^\alpha(\omega),$$

and define

$$S_n(\omega) = \sum_{\sigma\in\Gamma_n(\omega)} \prod_{i=1}^{|\sigma|} L_{\sigma|i}^\alpha(\omega) = \sum_{\sigma\in\Gamma_n(\omega)} \bar{L}_\sigma^\alpha(\omega)$$

$$S_{\sigma,n}(\omega) = S_n(\omega(\sigma)) = \sum_{\tau\in\Gamma_n(\omega(\sigma))} \prod_{i=1}^{|\tau|} L_{\sigma*(\tau|i)}^\alpha(\omega) = \sum_{\tau\in\Gamma_n(\omega(\sigma))} \bar{L}_\tau^\alpha(\omega(\sigma)),$$

where $\Gamma(\omega(\sigma))$ denotes the shifted tree at vertex σ. It is seen that $\{S_n, \mathcal{F}_n\}$ and $\{S_{\sigma,n}, \mathcal{F}_{|\sigma|+n}\}$ are martingales. The following lemma is of essential importance in Mauldin-Williams (1986), which we see that it is still valid without any non-overlapping assumption.

Lemma 3.2.

(1) $\lim_{n\to\infty} S_n(\omega) := X(\omega)$ *exists a.s. and* $EX \le 1$;

(2) $\lim_{n\to\infty} S_{\sigma,n}(\omega) := X_\sigma(\omega)$ *exists a.s. and* $EX_\sigma \le 1$;

(3) *let* A *be a cutset, then* $\{X_\sigma, \sigma \in A\}$ *are i.i.d with the same distribution of* X;

(4) $X(\omega) = \sum_{\sigma\in\Gamma_k} \overline{L}_\sigma^\alpha(\omega) X_\sigma(\omega), \forall k \ge 1; \overline{L}_\sigma^\alpha(\omega) X_\sigma(\omega) = \sum_{j=1}^{N(\sigma)} \overline{L}_{\sigma*j}^\alpha(\omega) X_{\sigma*j}(\omega)$ *a.s.*;

(5) *when* $EN^p(\emptyset) < \infty$ *for all integer* $p \ge 2$ *and* (2.2) *holds,* $X \in L^p(dP)$, *for all integer* $p \ge 2$, *and* $EX = 1$.

Proof. That (1)(2) follow from martingale convergence theorem. By our definition of recursive constructions, we have (3). Moreover, (4) follows by the definitions of $X(\omega)$ and $X_\sigma(\omega)$; see Mauldin-Williams (1986, (3.4)).

As for (5), this is a direct consequence of Mauldin-Williams (1986, Theorem 2.1). We note that, for any integer $p \ge 2$,

$$E\left[\left(\sum_{i=1}^{N(\emptyset)} L_i^\alpha\right)^p\right] \le r_2^{\alpha p} E(N(\emptyset)^p) < \infty;$$

thus their Theorem 2.1 is applicable to ensure that $X \in L^p(dP), \forall p \ge 2$. That $EX = 1$ follows from the case $p = 2$ and martingale convergence theorem. ∎

Based on Lemma 3.2, we may employ the idea in Mauldin-Williams (1986, Theorems 3.1 and 3.2) to define, for almost all ω, a bounded Borel measure μ_ω, on R^d supported on $K(\omega)$ such that μ_ω has total mass $X(\omega)$. The measure μ_ω can be expressed as

$$\mu_\omega(A) = \lim_{n\to\infty} \sum_{\substack{\alpha\in\Gamma_n \\ I_\sigma\cap A\neq\emptyset}} \overline{L}_\sigma^\alpha(\omega) X_\sigma(\omega) \ a.s. \tag{3.1}$$

for any compact $A \subset R^d$. Moreover, the limit in (3.1) is indeed a decreasing limit.

We remark that, in Mauldin-Williams (1986), their Theorems 3.1 and 3.2 are in fact valid without the assumptions of open set condition and geometric self-similarity.

Let $Z_n(\omega)$ be the number of $\Gamma_n(\omega)$; apparently Z_n is a Galton-Watson branching process. The following lemma is adapted from Mauldin-William (1986, Theorem 3.4 case A). We omit the proof.

Lemma 3.3. *For almost all* ω

"$Z_n(\omega) \to \infty (n \to \infty)$" *if and only if* "$K(\omega) \neq \emptyset$" *if and only if* "$X(\omega) > 0$".

We remark that, under the conditions of Lemma 3.2(5), $X(\omega) > 0$ for ω in a set of positive probability since $EX = 1$. Indeed, $P\{Z_n \to \infty\} = P\{K \neq \emptyset\} = P\{X > 0\} = 1 - q$, where $q : 0 < q < 1$ is the extinction probability of the Galton-Watson branching process determined by the offspring distribution $N(\emptyset)$.

The following lemma is also motivated from some technical arguments in Mauldin-Williams (1986, p 338).

Lemma 3.4. *Assume the conditions in Lemma 3.2(5). For any $\beta < \alpha$ $P\{\omega : \exists N(\omega)$ such that $\forall n \geq N(\omega)$, if $\sigma \in \Gamma_n$, then $\overline{L}_\sigma^\alpha X_\sigma \leq \overline{L}_\sigma^\beta\} = 1$.*

Proof. For any $\beta < \alpha$ and integer $p \geq 1$,

$$\begin{aligned}
& P\{\omega : \exists \sigma \in \Gamma_n s.t. \bar{L}_\sigma^\alpha X_\sigma > \bar{L}_\sigma^\beta\} \\
= \ & EI_{\{\omega:\exists\sigma\in\Gamma_n s.t. \bar{L}_\sigma^\alpha X_\sigma > \bar{L}_\sigma^\beta\}} \\
\leq \ & E \sum_{\sigma\in\Gamma_n(\omega)} \bar{L}_\sigma^{p(\alpha-\beta)} X_\sigma^p \\
= \ & E \sum_{i_1=1}^{N(\emptyset)} L_{i_1}^{(\alpha-\beta)p} \cdot \sum_{i_2=1}^{N(i_1)} L_{i_1 * i_2}^{(\alpha-\beta)p} \cdots \sum_{i_n=1}^{N(i_1 * i_2 * \cdots * i_{n-1})} L_{i_1 * i_2 * \cdots * i_n}^{(\alpha-\beta)p} X_{i_1 * i_2 * \cdots * i_n}^p \\
= \ & M_p E \sum_{i_1=1}^{N(\emptyset)} L_{i_1}^{(\alpha-\beta)p} \cdot \sum_{i_2=1}^{N(i_1)} L_{i_1 * i_2}^{(\alpha-\beta)p} \cdots E \sum_{i_n=1}^{N(i_1 * i_2 * \cdots * i_{n-1})} L_{i_1 * i_2 * \cdots * i_n}^{(\alpha-\beta)p},
\end{aligned}$$

where $M_p = EX^p$.

Choose a large p_0 such that $(\alpha - \beta)p_0 > \alpha$, so that $a_0 := E \sum_{i=1}^{N(\emptyset)} L_i^{(\alpha-\beta)p_0} < 1$.

Then

$$\sum_{n=0}^{\infty} P\{\omega : \exists \sigma \in \Gamma_n s.t. \bar{L}_\sigma^\alpha X_\sigma > \bar{L}_\sigma^\beta\} \leq M_{p0} \sum_{n=0}^{\infty} a_0^n < \infty.$$

By Borel-Cantelli lemma, the assertion holds. ■

To prove the Hausdorff dimension of $K(\omega)$, we need the following standard result giving a lower bound for the Hausdorff dimension of a set, see Falconer (1990, p 55).

Proposition A (mass distribution principle). *Let ν be a Borel measure on a compact metric space and let F be a Borel subset with $O < \nu(F) < \infty$. If there exists $c > 0$ and $r_0 > 0$ such that*

$$\nu(B(x, r)) < cr^s$$

for all $x \in F$ and $r \leq r_0$, then $dimF \geq s$.

For $\beta : 0 < \beta < \alpha$ and $N(\omega)$: the generation number in Lemma 3.4, let

$$\Omega_0(\beta) = \{\omega : \text{ if } \forall n \geq N(\omega) \text{ and } \sigma \in \Gamma_n(\omega), \text{ then } \overline{L}_\sigma^\alpha(\omega) X_\sigma(\omega) \leq \overline{L}_\sigma^\beta(\omega)\}.$$

Then, by Lemma 3.4, $P(\Omega_0(\beta)) = 1$. Moreover,

$$\Omega_0(\beta) \subset \Omega_0(\beta'), if \beta > \beta'.$$

We are now ready to state and prove the main result in this section.

Theorem 3.1. *Assume that* $\{I_\sigma, \sigma \in \Gamma\}$ *satisfies LOC, (2.1) and (2.2), and that* $EN(\emptyset)^p < \infty$ *for all integer* $p \geq 2$. *Then, conditional on* $K(\omega) \neq \emptyset$, *with probability one* $dim K(\omega) = \alpha$, *where* α *is the unique positive number for which* $E \sum_{i=1}^{N(\emptyset)} L_i^\alpha = 1$.

Remark. As we have remarked in the above, LOC is also defined in the deterministic case. Theorem 3.1 holds too, and then we have Moran's formula: $dim_H K = \alpha$, where α satisfies $\sum_{i=1}^{N} L_i^\alpha = 1$, and L_i are the geometric similarity ratios. That is, $F_i = L_i F_0$, $i = 1, \cdots, N$ and $K = \bigcap_{i=1}^{\infty} \bigcup_{\substack{i_1, \cdots, i_k \\ \in \{1, \cdots, N\}}} L_{i_1} L_{i_2} \cdots L_{i_k} F_0$.

Proof. $\{I_\sigma, \sigma \in \Gamma_n\}$ is a cover of $K(\omega)$ and $|I_\sigma| \to 0 (n \to \infty)$. So

$$\mathcal{H}^\alpha(K(\omega)) \leq \lim_{n\to\infty} \sum_{\sigma\in\Gamma_n} |I_\sigma|^\alpha = X(\omega).$$

Hence $E\mathcal{H}^\alpha(K(\omega)) \leq EX(\omega) \leq 1$. Thus, $\mathcal{H}^\alpha(K(\omega)) < \infty$ a.s. which means $dim K(\omega) \leq \alpha$ a.s. The above arguments do not depend on any assumption on the overlaps of I_σ.

Let μ be the measure defined by (3.1). By Lemma 3.3, since $\mu(K(\omega)) = X(\omega)$, we have

$$P(\mu(K(\omega)) > 0 | K(\omega) \neq \emptyset) = 1.$$

For $\forall \omega \in \Omega_0(\beta)$ and any $O < r < \min\{|I_\sigma| : \sigma \in \Gamma_{N(\omega)}\}$, define

$$A_r(\omega) = \{\sigma \in \Gamma : \sigma \text{ is of the least value } |\sigma| \text{ for which } r_1 r \leq |I_\sigma| < r\},$$

where r_1 is the lower bound in assumption (2.2). Then A_r is a cutset of Γ; moreover, for any $\sigma \in A_r$, $|\sigma| \geq N(\omega)$.

Let $n_r = \max\{|\sigma| : \sigma \in A_r\}$. Since $\bar{L}_\sigma^\alpha(\omega) X_\sigma(\omega) = \sum_{j=1}^{N(\sigma)} \bar{L}_{\sigma * j}^\alpha(\omega) X_{\sigma * j}(\omega)$ a.s., for any $x \in K(\omega)$ and $0 < r < \min\{|I_\sigma| : \sigma \in \Gamma_{N(\omega)}\}$, we have

$$\sum_{\substack{\sigma\in\Gamma_{n_r}(\omega) \\ I_\sigma \cap B(x,r) \neq \emptyset}} \bar{L}_\sigma^\alpha X_\sigma \leq \sum_{\substack{\sigma\in A_r \\ I_\sigma \cap B(x,r) \neq \emptyset}} \bar{L}_\sigma^\alpha X_\sigma.$$

Thus we have, by Lemma 3.4, for $\omega \in \Omega_0(\beta)$

$$\begin{aligned}
\mu_\omega(B(x,r)) &\leq \sum_{\substack{\sigma\in\Gamma_{n_r}(\omega) \\ I_\sigma \cap B(x,r) \neq \emptyset}} \bar{L}_\sigma^\alpha X_\sigma \\
&\leq \sum_{\substack{\sigma\in A_r \\ I_\sigma \cap B(x,r) \neq \emptyset}} \bar{L}_\sigma^\alpha X_\sigma \\
&\leq \sum_{\substack{\sigma\in A_r \\ I_\sigma \cap B(x,r) \neq \emptyset}} \bar{L}_\sigma^\beta \leq \sum_{\substack{\sigma\in A_r \\ I_\sigma \cap B(x,r) \neq \emptyset}} r^\beta \\
&\leq r^\beta \sharp\{\sigma \in A_r : I_\sigma \cap B(x,r) \neq \emptyset\}.
\end{aligned}$$

We show that, under LOC and assumption (2.2), there is a positive $c(\omega)$ such that, for almost all ω,

$$\sup_{\substack{0<r<1\\ x\in I_\emptyset}} \sharp\{\sigma \in A_r : I_\sigma \cap B(x,r) \neq \emptyset\} = c(\omega) < \infty.$$

Denote the sets of $\{I_\sigma, \sigma \in A_r\}$ which meet $B(x,r)$ by

$$I_{\sigma^1}, \cdots, I_{\sigma^{N_r}}.$$

Then $|I_{\sigma^i}| < r, \forall i = 1, \cdots N_r$, and $I_{\sigma^i} \cap B(x,r) \neq \emptyset$. Thus, $I_{\sigma^i} \subset B(x, 2r)$.

When $\{I_\sigma, \sigma \in \Gamma\}$ satisfies LOC, then I_{σ^i} contains a ball $B_{\delta|I_{\sigma^i}|}$, and, since any two vertices in A_r are incomparable under $\prec$, $B_{\delta|I_{\sigma^i}|} \cap B_{\delta|I_{\sigma^j}|} = \emptyset$ if $i \neq j$. Thus, we have

$$\sum_{i=1}^{N_r} volB_{\sigma|_{\sigma^i}|} \le volB(x, 2r).$$

By assumption (2.2), $\delta|I_{\sigma^i}| \ge \delta r_1 r$, which implies that

$$N_r \le \frac{2^d}{\delta^d r_1^d} := c(\omega),$$

which is what we want. Hence, for any $\omega \in \Omega_0(\omega)$, $x \in K(\omega)$ and $0 < r < r_0(\omega) := \min\{|I_\sigma| : \sigma \in \Gamma_{N(\omega)}\}$, we have

$$\mu_\omega(B(x,r)) \le c(\omega) r^\beta.$$

Write $\Omega_{0,n} = \Omega_0(\alpha - \frac{1}{n})$ and let $\Omega_0 = \bigcap_{n=1}^{\infty} \Omega_{0,n}$. Then $P(\Omega_0) = \lim_{n\to\infty} P(\Omega_{0,n}) = 1$.

By Proposition A, conditional on $K(\omega) \neq \emptyset$ and $\omega \in \Omega_0$, $dimK(\omega) \ge \alpha$ a.s. ω. This proves the theorem. ■

4. Regularity of Random Fractals

In this section, in addition to (2.2), we assume that
(4.1) there is $\lambda(\omega) > 0$, independent of σ, such that

$$inradius\ I_\sigma \ge \lambda \cdot |I_\sigma|, \forall \sigma \in \Gamma.$$

However, we **do not assume** any non-overlapping or limited overlaps condition in this section. As for (2.1), we impose a stronger assumption that $N(\emptyset)$ is of finite mean and $P\{N(\emptyset) = 0\} = P\{N(\emptyset) = 1\} = 0$. i.e. each vertex has at least two children.

For $\forall \sigma \in \Gamma$. let $\Gamma_{\omega(\sigma)}$ denote the shifted tree at vertex σ. Then for $\sigma \in \Gamma_\omega$ and $\tau \in \Gamma_{\omega(\sigma)}$, we have

$$L_\tau(\omega(\sigma)) = L_{\sigma*\tau}(\omega).$$

Given $0 < r < 1$. For $\forall \sigma \in \Gamma$, write

$$\Gamma_{r,k}(\omega(\sigma)) = \{\tau \in \Gamma_{\omega(\sigma)} : \bar{L}_\tau(\omega(\sigma)) r^k, \bar{L}_{\tau||\tau|-1}(\omega(\sigma)) > r^k\}.$$

For convenience, write

$$\Gamma_{r,k}(\omega) = \Gamma_{r,k}(\omega(\emptyset)) : \{\tau \in \Gamma_\omega : \bar{L}_\tau(\omega) \le r^k, \bar{L}_{\tau||\tau|-1}(\omega) > r^k\}.$$

Obviously,

$$\Gamma_{r,k}(\omega(\sigma)) = \{\tau \in \Gamma_{\omega(\sigma)} : \bar{L}_{\sigma*\tau}(\omega) \le \bar{L}_\sigma(\omega) r^k, \bar{L}_{\sigma*(\tau||\tau|-1} > \bar{L}_\sigma(\omega) r^k\}.$$

Observe that neither $\xi \prec \eta$ nor $\eta \prec \xi$ whenever $\xi \ne \eta, \xi, \eta \in \Gamma_{r,k}(\omega(\sigma))$. In what follows $\Gamma^*_{r,k}(\omega(\sigma))$ denotes the subset of $\Gamma_{r,k}(\omega(\sigma))$ which satisfies that

$$I_\eta \cap I_\xi = 0, \quad \text{for any } \eta \ne \xi, \quad \xi, \eta \in \Gamma^*_{r,k}(\omega(\sigma))$$

and for any $\xi \in \Gamma_{r,k}(\omega(\sigma)) \backslash \Gamma^*_{r,k}(\omega(\sigma))$ there exists a $\eta \in \Gamma^*_{r,k}(\omega(\sigma))$ such that $I_\xi \cap I_\eta \ne \emptyset$.

Write $Z^*_{r,k}(\omega) = \sharp\Gamma^*_{r,k}(\omega)$. From our assumption on $N(\emptyset)$ and the first part in the proof of Lemma 4.1 below, $Z^*_{r,k} \ge 2$ when r is small enough and k is large enough. Let $M_\varepsilon(K)$ be the smallest number of closed balls of radius ε that cover K, and let $N_\varepsilon(K)$ be the maximum number of disjoint balls of radius ε with center in K. Recall that the upper box dimension of a compact $K \subset R^d$ is defined by

$$\overline{dim_B} K = \limsup_{\varepsilon \to 0} \frac{\ln N_\varepsilon(K)}{-\ln \varepsilon} = \limsup_{\varepsilon \to 0} \frac{\ln M_\varepsilon(K)}{-\ln \varepsilon}$$

and note that ε can be replaced by a decreasing sequence $\varepsilon_k = r^k, 0 < r < 1$.

Lemma 4.1. *Let $K(\omega)$ be the fractal generated by our construction in Section 2. Under the assumptions (2.2), (4.1) and the above assumption on $N(\emptyset)$, for each $r : 0 < r < 1$*

$$\overline{dim_B} K(\omega) = \limsup_{k \to \infty} \frac{\ln Z^*_{r,k}(\omega)}{-k \ln r}$$

Proof. We prove that $N_{r^k} \le Z^*_{r,k} \le const \cdot M_{r^k}$, which will imply the assertion. Let $B(x_1, r^k), B(x_2, r^k), \cdots, B(x_{N_{r^k}}, r^k)$ be N_{r^k} disjoint balls of radius r^k and centers x_i in $K(\omega)$. For any $1 \le i \le N_{r^k}$, by the assumption that $L_i \le r_2 < 1$, there exists $\sigma_i = \sigma(x_i) \in \partial\Gamma_\omega$ such that

$$\bigcap_{n=1}^{\infty} I_{\sigma_i|n}(\omega) = x_i.$$

Choose n_0 such that

$$I_{\sigma_i|n_0}(\omega) \subset B(x_i, r^k), \quad I_{\sigma_i|n_0-1}(\omega) \not\subset B(x_i, r^k).$$

Then we have $|I_{\sigma_i|n_0-1}| > r^k$.

If $|I_{\sigma_i|n_0}(\omega)| = \overline{L}_{\sigma_i|n_0} \le r^k$, then we choose $\tau^i = \sigma_i|n_0$; while if $|I_{\sigma_i|n_0}(\omega)| = \overline{L}_{\sigma_i|n_0} > r^k$, then we can find $l \ge 1$ such that

$$\bar{L}_{\sigma_i|n_0+l-1} > r^k, \bar{L}_{\sigma_i|n_0+l} \le r^k.$$

Then we choose $\tau^i = \sigma_i|n_0 + l$. As $I_{\tau_i} \subset B(x_i, r^k)$ and $B(x_i, r^k)$ are disjoint, so I_{τ_i} are disjoint, therefore we have

$$Z^*_{r,k} \geq N_{r^k}.$$

On the other hand, let $I_{\tau^i}, i = 1, \cdots, Z^*_{r,k}$ be the $Z^*_{r,k}$ disjoint sets satisfying for any $1 \leq i \leq Z^*_{r,k}, \overline{L}_{\tau^i} \leq r^k, \overline{L}_{\tau^i||\tau^i|-1} > r^k$. By assumption (2.2), $|I_{\tau^i}| = \overline{L}_{\tau^i} > r_1 r^k$.

Let $B(x_1, r^k), \cdots, B(x_{M_{r^k}}, r^k)$ be M_{r^k} closed balls of radius r^k such that $\bigcup_{i=1}^{M_{r^k}} B(x_i, r^k) \supset K_\omega$. For any $1 \leq i \leq M_{r^k}$, define

$$H_i = \{j : 1 \leq j \leq Z^*_{r,k},\ I_{r^j} \cap B(x_i, r^k) \neq \emptyset\}.$$

Then for any $j \in H_i, I_{\tau^j} \subset B(x_i, 2r^k)$. By assumption (4.1), I_{τ^j} contains a ball of radius $\lambda|I_{\tau^j}| = \lambda\overline{L}_{\tau^j}$. Those balls are disjoint since I_{τ^j} are disjoint. Then

$$\begin{aligned} vol(B(x_i, 2r^k)) &\geq \sum_{j \in H_i} vol(I_{r^j}) \\ &\geq c_d(\lambda \bar{L}_{\tau^j})^d \sharp H_i \\ &\geq c_d \lambda^d r_1^d r^{kd} \sharp H_i, \end{aligned}$$

where c_d is the volume of the $d-$dimensional unit sphere.

Thus, letting i be running over $i = 1, \cdots, Z^*_{r,k}$, we have

$$Z^*_{r,k} \leq \frac{2^d}{\lambda^d r_1^d} M_{r^k}. \qquad \blacksquare$$

The main result in this section is

Theorem 4.1. *Assume that* (2.2) *and* (4.1) *hold, and that the distribution of* $N(\emptyset)$ *is of finite mean* m *and* $P\{N(\emptyset) = 0\} = P\{N(\emptyset) = 1\} = 0$. *There is a positive constant* a *such that, for almost every* ω,

$$dim K(\omega) = \overline{dim}_B K(\omega) = a.$$

Proof. For any $k \geq 1$, we construct a series of random subsets $K_k \subset K$ as follows. We fix one $r : 0 < r < 1$, and we have corresponding $\Gamma_{r,k}, \Gamma^*_{r,k}$ and $Z^*_{r,k}$ defined as above. Let

$$\begin{aligned} K_{k,0} &= I_0 \\ K_{k,1} &= \bigcup_{\sigma \in \Gamma^*_{r,k}(\omega)} I_\sigma \\ K_{k,2} &= \bigcup_{\sigma \in \Gamma^*_{r,k}(\omega)} \bigcup_{\tau \in \Gamma^*_{r,k}(\omega(\sigma))} I_{\sigma * \tau} \\ &\vdots \\ K_{k,n} &= \bigcup_{\sigma_1 \in \Gamma^*_{r,k}(\omega)} \bigcup_{\sigma_2 \in \Gamma^*_{r,k}(\omega(\sigma_1))} \cdots \bigcup_{\sigma_n \in \Gamma^*_{r,k}(\omega(\sigma_1 * \sigma_2 * \cdots * \sigma_{n-1}))} I_{\sigma_1 * \sigma_2 * \cdots * \sigma_n} \\ &\vdots \end{aligned}$$

and write $K_k = \bigcap_{n=0}^{\infty} K_{k,n}$.

It is obvious that $K_k \subset K$ for all $k \geq 1$, a.s. By the construction of K_k, it consists of non-overlapping members. By Falconer (1986) or Mauldin-Williams (1986), for almost every ω,

$$dim K_k(\omega) = \alpha_k,$$

where α_k is the solution of the expectation equation

$$E \sum_{\sigma \in \Gamma^*_{r,k}(\omega)} L_\sigma^{\alpha_k} = 1.$$

Note that, since we have assumed that $P\{N(\emptyset) = 0\} = 0$. the extinction probability $q = 0$, and thus the dimension formula holds with probability one.

Let $a = \sup\limits_{k \geq 1} \alpha_k$, as we have seen in Lemma 3.1

$$\frac{\sup\limits_{k \leq 1} \ln E\sharp\Gamma^*_{r,k}}{\ln \frac{1}{r_1}} \leq a \leq \frac{\sup\limits_{k \leq 1} \ln E\sharp\Gamma^*_{r,k}}{\ln \frac{1}{r_2}}$$

Moreover, we have

$$E \sum_{\sigma \in \Gamma^*_{r,k}(\omega)} \bar{L}_\sigma^\alpha \leq 1, \forall k \geq 1.$$

By assumption (2.2), we have

$$(EZ^*_{r,k})(r^k r_1)^a \leq 1, \forall k \geq 1.$$

Hence, for any $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} P\{\omega : Z^*_{r,k} \geq r^{-k(a+\varepsilon)}\} \leq \sum_{k=1}^{\infty} E(Z^*_{r,k}) r^{k(a+\varepsilon)} \leq \frac{1}{r_1^a} \sum_{k=1}^{\infty} (r^k)^\varepsilon < \infty.$$

By the Borel-Cantelli Lemma, we have

$$P\{\omega : Z^*_{r,k} \geq r^{-k(a+\varepsilon)}, i.o.\} = 0.$$

Hence, we have for P-a.s. ω,

$$\overline{dim}_B K(\omega) = \limsup_{k \to \infty} \frac{\ln Z^*_{r,k}}{-k \ln r} =\leq \limsup_{k \to \infty} \frac{\ln r^{-k(a+\epsilon)}}{-K \ln r} = a + \epsilon.$$

Since ε is arbitrary, we have for P-a.s. ω

$$\overline{dim}_B K(\omega) \leq a.$$

However, since $K_k(\omega) \subset K(\omega)$ for all $k \geq 1$, for almost every ω

$$a = \sup_{k \geq 1} dim K_k(\omega) \leq dim K(\omega).$$

Therefore, with probability one,

$$dim K(\omega) = \overline{dim}_B K(\omega) = a.$$

This proves the theorem. ■

Remark. The results and the proofs in this section are motivated from Liu-Wu (2002). However, we have avoided the use of geometrical self-similarity, which is essential in some key arguments in their paper.

5. EXAMPLES

Example 1. (Segments on the line)

Let $I = [0,1], \Gamma_n = \{1,2\}^n, \delta = \frac{1}{6}$. Write $S_1(x) = \frac{1}{3}x, S_2(x) = \frac{1}{3}x + \frac{2}{9}$ and let K be the invariant set with respect to iterated function system $\{S_1, S_2\}$. Obviously, $\{S_1, S_2\}$ does not satisfy open set condition, but we may check that $\{I_\sigma, \sigma \in \Gamma\}$ satisfies LOC with $\delta = \frac{1}{6}$. To see the case $n = 2$, let

$$I_{11} = S_1 \circ S_1[0,1] = \left[0, \frac{1}{9}\right], \quad I_{12} = S_1 \circ S_2[0,1] = \left[\frac{2}{27}, \frac{5}{27}\right]$$

$$I_{21} = S_2 \circ S_1\,[0,1] = \left[\frac{2}{9}, \frac{1}{3}\right], \quad I_{22} = S_2 \circ S_2\,[0,1] = \left[\frac{8}{27}, \frac{11}{27}\right]$$

Note that

$$\|I_{11} \cap I_{12}\| = \frac{1}{27} = \frac{1}{3}|I_{11}| = \frac{1}{3}|I_{12}|,$$

$$|I_{21} \cap I_{22}| = \frac{1}{27} = \frac{1}{3}|I_{21}| = \frac{1}{3}|I_{22}|.$$

Then, we have

$$I_{ij} \supset B_{\delta|I_{ij}|}, \quad B_{\delta|I_{ij}|} \cap B_{\delta|I_{i'j'}|} = \emptyset, \quad B_{\delta|I_{ij}|} \cap I_{i'j'} = \emptyset, (i,j) \neq (i'j').$$

Thus, $dimK = \frac{\ln 2}{\ln 3}$.

We give a random version of this example as follows:

Suppose that $u_1(\omega), u_2(\omega), \cdots, u_n(\omega), \cdots$ are i.i.d random variables, with the uniform distribution on $\left[\frac{1}{6}, \frac{1}{3}\right]$.

For any $n \geq 1$, write

$$S_n^1(x) = \frac{1}{3}x, \qquad S_n^2(x) = u_n(\omega)x + \frac{2}{9}.$$

For any $\sigma \in \bigcap_{n=1}^{\infty} \Gamma_n$, write

$$I_{\sigma_1,\cdots,\sigma_n} = S_1^{\sigma_1} \circ S_2^{\sigma_2} \circ \cdots \circ S_n^{\sigma_n}\,[0,1].$$

Let $K(\omega) = \bigcup_{\sigma \in \{1,2\}^\infty} \bigcap_{n=1}^{\infty} I_{\sigma|n}$. We may check that $\{I_\sigma, \sigma \in \Gamma\}$ satisfies LOC with $\delta = \frac{1}{12}$, and thus, with probability one $dimK(\omega) = \alpha$, where α is the solution of the equation

$$\left(\frac{1}{3}\right)^\alpha + 6\int_{\frac{1}{6}}^{\frac{1}{3}} x^\alpha dx = 1.$$

Again, we check the case $n = 2$; we have

$I_{11} = S_1^1 \circ S_2^1\,[0,1] = \left[0, \frac{1}{9}\right]$,
$I_{12} = S_1^1 \circ S_2^2\,[0,1] = \left[\frac{2}{27}, \frac{2}{27} + \frac{1}{3}u_2(\omega)\right] \subset \left[\frac{2}{27}, \frac{5}{27}\right]$,
$I_{21} = S_1^2 \circ S_2^1\,[0,1] = \left[\frac{2}{9}, \frac{2}{9} + \frac{1}{3}u_1(\omega)\right]$,
$I_{22} = S_1^2 \circ S_2^2\,[0,1] = \left[\frac{2}{9} + \frac{2}{9}u_1(\omega), \frac{2}{9} + \frac{2}{9}u_1(\omega) + u_1(\omega)u_2(\omega)\right]$.

$$|I_{11} \cap I_{12}| = \frac{1}{27} = \begin{cases} \frac{1}{3}|I_{11}| < \frac{2}{3}|I_{11}| \\ \frac{1}{27}\frac{1}{|I_{12}|}|I_{12}| = \frac{1}{27}\frac{3}{u_2(\omega)}|I_{12}| \le \frac{2}{3}|I_{12}|, \end{cases}$$

$$|I_{21} \cap I_{22}| = \frac{1}{9}u_2(\omega) = \begin{cases} \frac{1}{3}|I_{21}| < \frac{2}{3}|I_{21}| \\ \frac{1}{9}u_1(\omega)\frac{1}{|I_{22}|}|I_{22}| = \frac{1}{9}\frac{1}{u_2(\omega)}|I_{22}| \le \frac{2}{3}|I_{22}| \end{cases}$$

Thus, with $\delta = \frac{1}{12}$, with probability one, we have

$$I_{ij} \supset B_{\delta|I_{ij}|}, \quad B_{\delta|I_{ij}|} \cap B_{\delta|I_{i'j'}|} = \emptyset, \quad B_{\delta|I_{ij}|} \cap I_{i'j'} = \emptyset, (i,j) \neq (i'j').$$

Example 2. (Triangles on the plane) let $I \subset R^2$ be a triangle of edge length 1 and vertex (0, 0), (1, 0), $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Let

$S_1(x,y) = (\frac{1}{2}x, \frac{1}{2}y)$,
$S_2(x,y) = (\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y)$,
$S_3(x,y) = (\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4} - a)$, where $0 < a < \frac{\sqrt{3}}{8}$.

Suppose that K is the invariant set with respect to iterated function system $\{S_1, S_2, S_3\}$. Note that, if $a = 0$. then $\{S_1, S_2, S_3\}$ generates Sierpinski gasket; yet our situation is that a third triangle is topped with overlaps on the first two triangles, as Fig. 5.1 shows.

Let $\Gamma_n = \{1,2,3\}^n, \delta = \frac{\sqrt{3}}{8} - \frac{a}{2}$.

For any $\sigma \in \bigcap_{n=1}^{\infty} \Gamma_n$, write

$$I_{\sigma_1,\cdots,\sigma_n} = S_{\sigma_1} \circ S_{\sigma_2} \circ \cdots \circ S_{\sigma_n} I.$$

Denote the height of the n-th generation sub-triangle $I_\sigma, \sigma \in \Gamma_n$ by h_n. Then $h_n = \frac{\sqrt{3}}{2}\frac{1}{2^n}$.

Note that

$$a + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8} < \frac{\sqrt{3}}{2},$$

which means that $I_3 \cap I_{11} = \emptyset$, $I_3 \cap I_{12} = \emptyset$, $I_3 \cap I_{21} = \emptyset$, $I_3 \cap I_{22} = \emptyset$; note that I_{13} has overlap with I_{31}, but this overlap does not cross over the line $x = \frac{1}{4}$. Moreover

$$\frac{1}{2}a + \frac{\sqrt{3}}{16} < \frac{\sqrt{3}}{8}$$

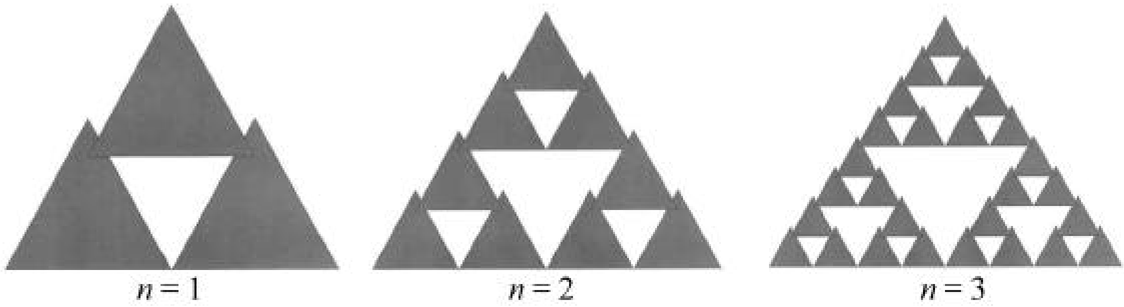

Fig. 5.1.

which implies that

$$I_3 \cap I_{132} = \emptyset, I_3 \cap I_{131} = \emptyset, I_3 \cap I_{232} = \emptyset, I_3 \cap I_{231} = \emptyset.$$

Thus, it can be checked that LOC holds with $\delta = \frac{\sqrt{3}}{8} - \frac{a}{2}$, and hence,

$$dimK = \frac{\ln 3}{\ln 2}.$$

Example 3. (Squares on the plane)
Let $I = [0,1]^2$ be the unit square in R^2 and let

$S_1(x,y) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y)$,
$S_2(x,y) = (\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3})$,
$S_3(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{1}{3})$,
$S_4(x,y) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{2}{3})$,
$S_5(x,y) = (\frac{2}{9}x + \frac{2}{9}, \frac{2}{9}y + \frac{2}{9})$,
$S_6(x,y) = (\frac{2}{9}x + \frac{5}{9}, \frac{2}{9}y + \frac{2}{9})$,
$S_7(x,y) = (\frac{2}{9}x + \frac{2}{9}, \frac{2}{9}y + \frac{5}{9})$,
$S_8(x,y) = (\frac{2}{9}x + \frac{5}{9}, \frac{2}{9}y + \frac{5}{9})$.

Write $\Gamma_n = \{1,2,\cdots,8\}^n$. For any $\sigma \in \bigcap_{n=1}^{\infty} \Gamma_n$, write

$$I_{\sigma_1,\cdots,\sigma_n} = S_{\sigma_1} \circ S_{\sigma_2} \circ \cdots \circ S_{\sigma_n}[0,1]^2.$$

Let K be the invariant set with respect to iterated function system $\{S_1, S_2, \cdots, S_8\}$. The situation now is four squares are topped with overlaps on the first four nonoverlapping squares, as Fig. 5.2 shows. Note that, for all $\sigma, \tau \in \{1,2,\cdots,8\}, \sigma \neq \tau$,

$$intI_{\sigma * i} \cap intI_{\tau * j} = \emptyset, \qquad \forall i,j = 1,\cdots,8.$$

Based on this, we can check that $\{I_\sigma, \sigma \in \Gamma\}$ satisfies LOC with $\delta = \frac{1}{8}$.

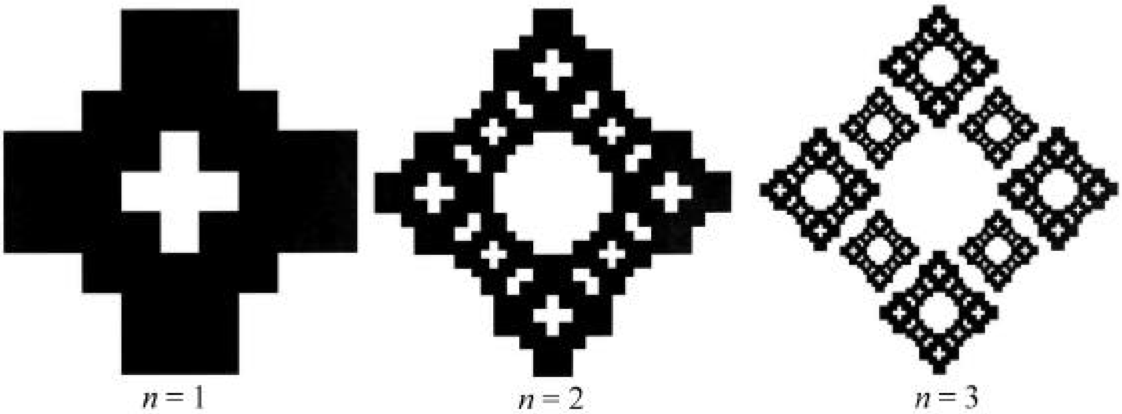

Fig. 5.2.

Thus,

$$dimK = \alpha$$

where α is the solution of equation

$$4\left(\frac{1}{3}\right)^{\alpha} + 4\left(\frac{2}{9}\right)^{\alpha} = 1.$$

Moreover, we can give a random version of this example too. In fact, for any $n \geq 1$, let $u_n^i (i = 1, \cdots, 4)$ be random variables with uniform distribution on $[\frac{1}{9}, \frac{2}{9}]$. Suppose that random vectors $\{(u_n^1, u_n^2, u_n^3, u_n^4), n \geq 1\}$ are i.i.d.

For any $n \geq 1$, write

$S_n^1(x, y) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y)$,
$S_n^2(x, y) = (\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3})$,
$S_n^3(x, y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{1}{3})$,
$S_n^4(x, y) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{2}{3})$,
$S_n^5(x, y) = (u_n^1 x + \frac{1}{3} - \frac{u_n^1}{2}, u_n^1 y + \frac{1}{3} - \frac{u_n^1}{2})$,
$S_n^6(x, y) = (u_n^2 x + \frac{2}{3} - \frac{u_n^2}{2}, u_n^2 y + \frac{1}{3} - \frac{u_n^2}{2})$,
$S_n^7(x, y) = (u_n^3 x + \frac{1}{3} - \frac{u_n^3}{2}, u_n^3 y + \frac{2}{3} - \frac{u_n^3}{2})$,
$S_n^8(x, y) = (u_n^4 x + \frac{2}{3} - \frac{u_n^4}{2}, u_n^4 y + \frac{2}{3} - \frac{u_n^4}{2})$.

For any $\sigma \in \bigcap_{n=1}^{\infty} \Gamma_n$, write

$$I_{\sigma_1, \cdots, \sigma_n} = S_1^{\sigma_1} \circ S_2^{\sigma_2} \circ \cdots \circ S_n^{\sigma_n} [0, 1]^2.$$

Let $K(\omega) = \bigcup_{\sigma \in \{1, \cdots, 8\}^\infty} \bigcap_{n=1}^{\infty} I_{\sigma|n}$. We can check, with $\delta = \frac{1}{16}$, that $\{I_\sigma, \sigma \in \Gamma\}$ satisfies the LOC and thus, with probability one $dimK(\omega) = \alpha$, where α is the solution of the equation

$$4(\frac{1}{3})^{\alpha} + 4 \times 9 \int_{\frac{1}{9}}^{\frac{2}{9}} x^{\alpha} dx = 1.$$

Concluding Remark. We have estabilished Moran's formula under LOC, both in deterministic and in random cases. In a future work, we shall study miltifractal structure under LOC, extending those works of Cawley-Mauldin (1992) and Falconer (1994) from OSC to LOC. We also mention a recent work of Shieh-Yu (2003) on the relation between Galton-Watson tree and iterated function system, which may be extended to hold under LOC.

Added in Proof. It is found that a paper by Y. Pesin and H. Weiss (Comm. Math. Phys. vol. 182 (1996), 105-153) contains a result (Corollary 2 at p. 116) which also validates Moran's formula under overlapping structure. The paper is based on viewpoint of symbolic dynamical systems.

ACKNOWLEDGEMENT

This work is finished while the second author did her postdoctoral research at National Taiwan University. Both authors are supported by a grant NSC 2002/3-2115-M-002-017. We thank to L. C. Huang for drawing Figures 5.1 and 5.2.

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