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SHAPE AND STRUCTURE OF THE BIFURCATION CURVE OF A BOUNDARY BLOW-UP PROBLEM

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Abstract. We study the shape and the structure of the bifurcation curve $f_a(\rho)$ $(=\sqrt{\lambda})$ with $\rho := \min_{x\in(0,1)} u(x)$ of (sign-changing and nonnegative) solutions of the boundary blow-up problem

$$\begin{cases} -u''(x) = \lambda f(u(x)),\ 0 < x < 1, \\ \lim\limits_{x\to 0^+} u(x) = \infty = \lim\limits_{x\to 1^-} u(x), \end{cases}$$

where λ is a positive bifurcation parameter and the Lipschitz continuous conacve function

$$f = f_a(u) = \begin{cases} -|u|^p & \text{if } u \le -a^{1/p}, \\ -a & \text{if } -a^{1/p} < u < a^{1/p}, \\ -|u|^p & \text{if } u \ge a^{1/p}, \end{cases}$$

with constants $p > 1$ and $a > 0$. We mainly show that the bifurcation curve $G_{f_a}(\rho)$ satisfies $\lim_{\rho\to\pm\infty} G_{f_a}(\rho) = 0$ and $G_{f_a}(\rho)$ has a exactly one critical point, a maximum, on $(-\infty,\infty)$. Thus we are able to determine the exact number of (sign-changing and nonnegative) solutions of the problem for each $\lambda > 0$.

1. INTRODUCTION

In this paper we study the shape and the structure of the bifurcation curve of (sign-changing and nonnegative) solutions of the boundary blow-up problem

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$$\begin{cases} -u''(x) = \lambda f(u(x)),\ 0 < x < 1, \\ \lim\limits_{x\to 0^+} u(x) = \infty = \lim\limits_{x\to 1^-} u(x), \end{cases} \tag{1.1}$$

where λ is a positive bifurcation parameter and the Lipschitz continuous function

$$f = f_a(u) := \begin{cases} -|u|^p & \text{if } u \le -a^{1/p}, \\ -a & \text{if } -a^{1/p} < u < a^{1/p}, \\ -|u|^p & \text{if } u \ge a^{1/p}, \end{cases} \tag{1.2}$$

with constants $p > 1$ and $a > 0$. Note that $f_a(u)$ satisfies

(i) $f_a(0) = -a < 0,\ f_a(u) < 0$ for $u > 0$,

(ii) $f_a(u) = f_a(-u)$ for $u > 0$,

(iii) $f_a(u)$ is a *decreasing* function on $(0, \infty)$ and a *concave* function on $(-\infty, \infty)$.

Let

$$f_0(u) := -|u|^p,\ p > 1,\ -\infty < u < \infty.$$

For fixed $p > 1$, it is important to note that

$$\begin{cases} f_a(u) = f_0(u) \text{ if } |u| \ge a^{1/p}, \\ f_a(u) < f_0(u) \text{ if } -a^{1/p} < u < a^{1/p}, \end{cases} \tag{1.3}$$

and $f_a(u) \to f_0(u)$ uniformly in u as $a \to 0^+$. For $f = f_0(u) = -|u|^p$, the bifurcation curve of solutions of (1.1) has been studied in [14].

Blow-up solutions of the boundary value problem

$$\begin{cases} -\Delta u = f(u) \text{ in } \Omega, \\ u = \infty \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded domain in $\mathbf{R}^N (N \ge 1)$ have been extensively studied; see [1-10, 13-4]. A problem of this type was first considered by Bieberbach [3] in 1916, where $f(u) = -e^u$ and $N = 2$. Bieberbach proved that if Ω is a bounded domain in $\mathbf{R}^2$ such that $\partial\Omega$ is a C^2 submanifold of $\mathbf{R}^2$, then there exists a unique $u \in C^2(\Omega)$ such that $\Delta u(x) = e^u$ in Ω and $\left|u(x) - \ln(d(x))^{-2}\right|$ is bounded on Ω. Here $d(x)$ denotes the distance from a point x to $\partial\Omega$. Rademacher [10] extended the idea of Bieberbach to smooth bounded domain in $\mathbf{R}^3$. Keller [4] studied the existence, but not uniqueness, of positive solutions of (1.4) under the assumptions that f is continuous and decreasing on $[0, \infty)$, $f(0) = 0$ and $\int^\infty (-F)^{-1/2} < \infty$, where

$$F(s) := \int_0^s f(t)dt.$$

For $f(u) = -u^p$ with $p > 1$, problem (1.4) is of interest in the study of the subsonic motion of a gas when $p = 2$ (see [9]). Pohozaev [9] proved the existence, but not the uniqueness, of positive solutions for (1.4), when $f(u) = -u^2$. For the case where $f(u) = -u^{(N+2)/(N-2)}$ $(N > 2)$, Loewener and Nirenberg [5] proved that if $\partial\Omega$ consists of the disjoint union of finitely compact C^∞ manifolds, each having codimension less than $N/2 + 1$, then there exists a unique positive solution of (1.4). Marcus and Véron [6] proved the uniqueness of the positive solution of (1.4) for $f(u) = -u^p$ with $p > 1$, when $\partial\Omega$ is compact and is locally the graph of a continuous function defined on an $(N-1)$-dimensional space.

The first result of nonuniqueness of (sign-changing and nonnegative) solutions for (1.4) was obtained by McKenna *et al.* [7], in the special case when the domain Ω is a ball and $f(u) = -|u|^p$. They proved that for $1 < p < N^*$ (note that $N^* = (N+2)/(N-2)$ for $N \geq 3$ and $N^* = \infty$ for $N = 1, 2$), there are exactly two blow-up solutions: one is positive and the other one is sign-changing. For $p \geq N^*$, there is a unique blow-up solution and it is positive. Subsequently, Aftalion and Reichel [1] studied

$$(1.5) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

in bounded C^2-domain Ω in $\mathbf{R}^N$ $(N \geq 1)$. They assumed $\max_{\mathbf{R}} f(u) < 0$ and gave growth conditions of f on $\pm\infty$, and they proved that there exists a positive constant $\tilde{\lambda}$ depending on f and Ω such that (1.5) has at least two solutions for $0 < \lambda < \tilde{\lambda}$, and has no solutions for $\lambda > \tilde{\lambda}$.

For general nonlinearities $f(u)$ and in one space dimension, Anuradha *et al.* [2] studied (1.1) based on building a quadrature method for such boundary blow-up solutions as follows:

Define

$$I_{\mathbf{R}} = \{s \in \mathbf{R} : \ f(s) < 0 \text{ and } F(s) > F(u) \text{ for all } u > s\}.$$

Suppose that u is a solution of (1.1). Let

$$\rho := \min_{x \in (0,1)} u(x).$$

Thus solution u is nonnegative if $\rho \geq 0$ (u is positive if $\rho > 0$) and is sign-changing if $\rho < 0$.

The next lemma is mainly due to Anuradha *et al.* [2, Lemma 2.1] after slight generalization.

Lemma 1.1. *Given $\lambda > 0$ and f Lipschitz continuous, there exists a unique* *solution to (1.1) with* $\min_{x\in(0,1)} u(x) = \rho$ *if and only if*

$$G(\rho) := \sqrt{2}\int_{\rho}^{\infty} \frac{1}{\sqrt{F(\rho)-F(u)}}du = \sqrt{\lambda} \quad \text{for} \quad \rho \in I_{\mathbf{R}}. \tag{1.6}$$

Wang [13, Theorems 2.2-2.3] improved some results of Anuradha *et al.* [2] as follows.

Theorem 1.2. *Let f be a Lipschitz continuous function in $\mathbf{R}$. If f satisfies*

$$\liminf_{u\to\infty} \frac{-f(u)}{u(\ln u)^3} = L \ (0 < L \le \infty), \tag{1.7}$$

then there exist solutions to (1.1) for some $\lambda > 0$. Furthermore, $G(\rho)$ is well defined and continuous for all $\rho \in I_{\mathbf{R}}$. In addition, $\lim_{\rho\to\infty} G(\rho) = 0$.

For $f = f_0(u)$, we let $F_0(u) := \int_0^u f_0(t)dt$ and

$$G_{f_0}(\rho) := \sqrt{2}\int_{\rho}^{\infty} \frac{1}{\sqrt{F_0(\rho)-F_0(u)}}du \text{ for } \rho \in I_{\mathbf{R}} = (-\infty,0)\cup(0,\infty). \tag{1.8}$$

Recall the Gamma function as follows (see e.g. [11, p. 9]):

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt, \ (z > 0).$$

Recently, Wang *et al.* [14, Theorem 2.4] computed explicitly $G_{f_0}(\rho)$ for $f = f_0(u) = -|u|^p$ with $p > 1$.

Theorem 1.3. *Let $f = f_0(u) = -|u|^p$ with $p > 1$. Then*

$$G_{f_0}(\rho) = M_p\rho^{\frac{1-p}{2}} \text{ and } G_{f_0}(-\rho) = N_p\rho^{\frac{1-p}{2}} \text{ for } \rho > 0,$$

where

$$M_p = \sqrt{\frac{2\pi}{p+1}}\frac{\Gamma\left(\frac{p-1}{2p+2}\right)}{\Gamma\left(\frac{p}{p+1}\right)} > 0 \text{ for } p > 1$$

and

$$N_p = \sqrt{\frac{2}{\pi(p+1)}}\Gamma\left(\frac{1}{p+1}\right)\left(\Gamma\left(\frac{p-1}{2p+2}\right) + \frac{\pi}{\Gamma\left(\frac{p+3}{2p+2}\right)}\right) > 0 \text{ for } p > 1.$$

Furthermore,

$$\frac{G_{f_0}(-\rho)}{G_{f_0}(\rho)} = \frac{N_p}{M_p} = \sin\frac{p\pi}{2(p+1)}\csc\frac{\pi}{2(p+1)} > 1 \text{ for } \rho > 0.$$

For general nonlinearity $f = h(u)$ a *negative concave* function on $(-\infty, \infty)$ satisfying (1.7) and $\lim_{u\to-\infty} h(u)/u = \infty$, some numerical simulations in [13, Fig. 3] suggest that $G_h(\rho)$ has exactly one critical point, a maximum, on $(-\infty, \infty)$. In the next section, we verify it for the class of nonlinearities $f = f_a(u)$ defined in (1.2).

2. Main Results

For $f = f_a(u)$ defined in (1.2) with fixed $p > 1$, we let $F_a(u) := \int_0^u f_a(t)dt$ and

$$(2.1) \quad G_{f_a}(\rho) := \sqrt{2}\int_\rho^\infty \frac{1}{\sqrt{F_a(\rho) - F_a(u)}}du \ (= \sqrt{\lambda}) \quad \text{for } \rho \in I_{\mathbf{R}} = (-\infty, \infty),$$

(see (1.6)). First, in Theorem 2.1, we show that $G_{f_a}(\rho)$ satisfies $\lim_{\rho\to\pm\infty} G_{f_a}(\rho) = 0$ and $G_{f_a}(\rho)$ has exactly one critical point at ρ^*, a maximum, on $(-\infty, \infty)$. Subsequently, in Corollary 2.2, we are able to determine the exact number of (sign-changing and nonnegative) solutions of (1.1) for each $\lambda > 0$.

Recall that the hypergeometric function as follows (see e.g. [11, p. 45]):

$$F(\alpha, \beta; \gamma; z) = \sum_{k=0}^\infty \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)\beta(\beta+1)\cdots(\beta+k-1)}{k!\gamma(\gamma+1)\cdots(\gamma+k-1)}z^k.$$

Theorem 2.1. *(See Figs. 1 and 2) Let $f = f_a(u)$ be defined in (1.2) with $p > 1$ and $a > 0$. Then $G_{f_a}(\rho)$ satisfies*

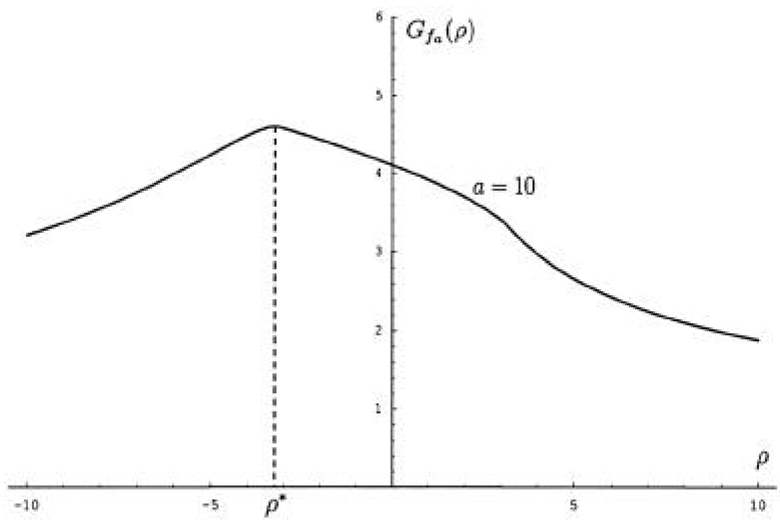

Fig. 1. A numerical simulation of $G_{f_a}(\rho)$ for $f_a(u)$ with $p = 2$, $a = 10$. $\rho^* \cong -3.2297$.

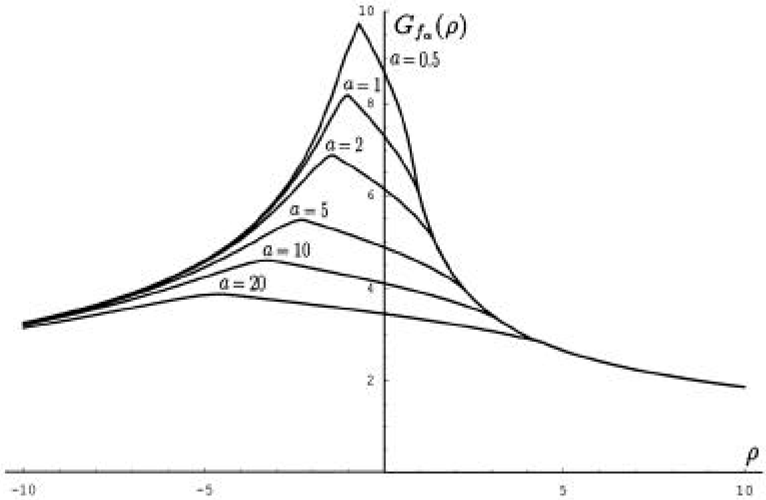

Fig. 2. Numerical simulations of $G_{f_a}(\rho)$ for $f_a(u)$ with $p=2$. $a=0.5,\ 1, 2, 5, 10, 20$.

(*i*) $\lim_{\rho\to\pm\infty} G_{f_a}(\rho)=0$ *and* $G_{f_a}(-\rho)>G_{f_a}(\rho)$ *for* $\rho>0$.

(*ii*) $G_{f_a}(\rho)$ *has exactly one critical point at* $\rho=\rho^*$*, a maximum, on* $(-\infty,\infty)$.

(*iii*)

$$-\left(\frac{3p+1}{p-1}\right)^{1/(p+1)} a^{1/p}<\rho^*\leq -a^{1/p}. \tag{2.2}$$

(*iv*)

$$G_{f_a}(-a^{1/p})\leq G_{f_a}(\rho^*)=\max_{-\infty<\rho<\infty} G_{f_a}(\rho)<G_{f_0}(-a^{1/p}), \tag{2.3}$$

where

$$G_{f_0}(-a^{1/p})=\sqrt{\frac{2}{\pi(p+1)}}\Gamma\left(\frac{1}{p+1}\right)\left[\Gamma\left(\frac{p-1}{2p+2}\right)+\frac{\pi}{\Gamma\left(\dfrac{p+3}{2p+2}\right)}\right]a^{\frac{1-p}{2p}} \tag{2.4}$$

and

$$\begin{aligned}G_{f_a}(-a^{1/p})=&\left\{4+\sqrt{\frac{2}{\pi}}(p+1)^{1/2}(2p+1)^{\frac{1-p}{2p+2}}\Gamma\left(\frac{p+2}{p+1}\right)\Gamma\left(\frac{p-1}{2p+2}\right)\right.\\&\left.-\sqrt{2}(p+1)^{1/2}(2p+1)^{-1/2}F\left(\frac{1}{p+1},\frac{1}{2};\frac{p+2}{p+1};\frac{-1}{2p+1}\right)\right\}a^{\frac{1-p}{2p}}.\end{aligned} \tag{2.5}$$

(*v*) *For fixed* $p>1$, $\rho^*=\rho^*(a)$ *satisfies* $\lim_{a\to 0^+}\rho^*(a)=0$, $\lim_{a\to\infty}\rho^*(a)=-\infty$.

(*vi*) *For fixed* $p>1$, $G_{f_a}(\rho^*(a))$ *is a strictly decreasing functions of* $a>0$, *and* $\lim_{a\to 0^+} G_{f_a}(\rho^*(a))=\infty$ *and* $\lim_{a\to\infty} G_{f_a}(\rho^*(a))=0$.

Conjecture. *Some numerical simulations as in Fig. 2 suggest that $\rho^*(a)$ is a strictly decreasing functions of $a > 0$. But we are not able to give a proof.*

By (2.1), it can be computed that

$$\begin{aligned} G_{f_a}(0) &= \sqrt{2}\int_0^\infty \frac{1}{\sqrt{-F_a(u)}}du \\ &= a^{\frac{1-p}{2p}}\left\{2\sqrt{2}+\sqrt{2p+2}p^{\frac{1-p}{2p+2}}\left[\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{p+2}{p+1}\right)\Gamma\left(\frac{p-1}{2p+2}\right)\right.\right. \\ &\left.\left.\quad -p^{\frac{-1}{p+1}}F\left(\frac{1}{p+1},\frac{1}{2};\frac{p+2}{p+1};\frac{-1}{p}\right)\right]\right\}. \end{aligned}$$

We omit the detailed computation; see (3.10) for $F_a(u)$. Theorem 2.1 implies immediately the next corollary.

Corollary 2.2. *(See Fig. 1) Let $f = f_a(u)$ be defined in (1.2) with $p > 1$ and $a > 0$. Then*

- (i) *for $0 < \lambda \le (G_{f_a}(0))^2$, problem (1.1) has exactly one nonnegative solution and exactly one sign-changing solution,*
- (ii) *for $(G_{f_a}(0))^2 < \lambda < (G_{f_a}(\rho^*))^2$, problem (1.1) has exactly two sign-changing solutions,*
- (iii) *for $\lambda = (G_{f_a}(\rho^*))^2$, problem (1.1) has exactly one sign-changing solutions, and*
- (iv) *for $\lambda > (G_{f_a}(\rho^*))^2$, problem (1.1) has no solution.*

Example 1. (See Fig. 1) Let $f = f_a(u)$ with $p = 2$ and $a = 10$. Then $G_{f_a}(0) \cong 4.1087$, and (2.2) reduces to

$$-6.0492 \cong -7^{1/3}\sqrt{10} < \rho^* \cong -3.2297 \le -\sqrt{10} \cong -3.1623,$$

and (2.3) reduces to

$$4.6016 \cong G_{f_a}(-\sqrt{10}) \le G_{f_a}(\rho^*) \cong 4.6046 < G_{f_0}(-\sqrt{10}) \cong 5.7943.$$

3. Proof of Theorem 2.1

First, in Theorem 2.1(i), the assertion that $G_{f_a}(-\rho) > G_{f_a}(\rho)$ for $\rho > 0$ follows from the next lemma which is of independent interest.

Lemma 3.1. *Consider (1.1). In addition to (1.7), suppose that f satisfies*

- (i) *$f(u) < 0$ on $(0,\infty)$ and $f(u)$ is decreasing on $(0,\infty)$ and is strictly decreasing on (K,∞) for some $K \ge 0$,*
- (ii) *$f(-u) = f(u)$ for $u > 0$.*

Then for $G(\rho)$ defined in (1.6), $G(-\rho) > G(\rho)$ for $\rho > 0$, $\rho \in I_{\mathbf{R}}$.

Proof of Lemma 3.1. From (1.6), for $\rho \in I_{\mathbf{R}}$,

$$\begin{aligned} G(\rho) &= \sqrt{2}\int_\rho^\infty \frac{du}{\sqrt{F(\rho)-F(u)}} \\ &= \sqrt{2}(\int_\rho^{2\rho+K} + \int_{2\rho+K}^\infty)\frac{du}{\sqrt{F(\rho)-F(u)}} \end{aligned}$$

and

$$\begin{aligned} G(-\rho) &= \sqrt{2}\int_{-\rho}^\infty \frac{du}{\sqrt{F(-\rho)-F(u)}} \\ &= \sqrt{2}\int_{-\rho}^\infty \frac{du}{\sqrt{-F(\rho)-F(u)}} \quad (F(-\rho) = -F(\rho) \text{ by assumption (ii)}) \\ &= \sqrt{2}\int_{\rho}^\infty \frac{dt}{\sqrt{-F(\rho)-F(t-2\rho)}} \quad (\text{set } u = t-2\rho) \\ &= \sqrt{2}(\int_\rho^{2\rho+K} + \int_{2\rho+K}^\infty)\frac{du}{\sqrt{-F(\rho)-F(u-2\rho)}}. \end{aligned}$$

To prove $G(-\rho) > G(\rho)$ for $\rho > 0$, $\rho \in I_{\mathbf{R}}$, it suffices to show that

$$\begin{cases} F(\rho) - F(u) \geq -F(\rho) - F(u-2\rho) \text{ for } 0 < \rho < u \leq 2\rho + K, \\ F(\rho) - F(u) > -F(\rho) - F(u-2\rho) \text{ for } u > 2\rho + K; \end{cases}$$

i.e.,

$$\begin{cases} 2F(\rho) + F(u-2\rho) \geq F(u) \text{ for } 0 < \rho < u \leq 2\rho + K, \\ 2F(\rho) + F(u-2\rho) > F(u) \text{ for } u > 2\rho + K. \end{cases} \tag{3.1}$$

Case (I) $(0 <)\ \rho < u < 2\rho$. By the Mean Value Theorem, we have

$$\begin{cases} F(\rho) + F(u-2\rho) = F(\rho) - F(2\rho - u) = (u-\rho)f(c_1), \\ F(u) - F(\rho) = (u-\rho)f(c_2), \end{cases}$$

where $c_1 \in (0, \rho)$, $c_2 \in (\rho, 2\rho)$. Thus $f(c_1) \geq f(c_2)$ which implies

$$2F(\rho) + F(u-2\rho) \geq F(u) \text{ for } 0 < \rho < u < 2\rho.$$

Case (II) $u = 2\rho$. By assumption (i),

$$2F(\rho) \geq F(2\rho) \text{ for } u = 2\rho.$$

Case (III) $2\rho < u \le 2\rho + K$. By assumption (i),

$$\begin{aligned} 2F(\rho) + F(u-2\rho) &\ge F(2\rho) + F(u-2\rho) \\ &= \int_0^{2\rho} f(t)dt + \int_0^{u-2\rho} f(t)dt \\ &\ge \int_0^{2\rho} f(t)dt + \int_{2\rho}^{u} f(t)dt \\ &= F(u) \text{ for } 2\rho < u \le 2\rho + K. \end{aligned}$$

Case (IV) $u > 2\rho + K$. By assumption (i),

$$\begin{aligned} 2F(\rho) + F(u-2\rho) &\ge F(2\rho) + F(u-2\rho) \\ &= \int_0^{2\rho} f(t)dt + \int_0^{u-2\rho} f(t)dt \\ &> \int_0^{2\rho} f(t)dt + \int_{2\rho}^{2\rho+K} f(t)dt + \int_{2\rho+K}^{u} f(t)dt \\ &= F(u) \text{ for } u > 2\rho + K. \end{aligned}$$

By above, (3.1) holds and hence $G(-\rho) > G(\rho)$ for $\rho > 0$, $\rho \in I_{\mathbf{R}}$. ■

We are now in position to prove Theorem 2.1.

Proof of Theorem 2.1.

Part (i). By Theorem 1.2, we obtain $\lim_{\rho\to\infty} G_{f_a}(\rho) = 0$. By (1.3), (1.8) and (2.1), we easily obtain the comparison result

$$(0<)\ G_{f_a}(\rho) < G_{f_0}(\rho)\ (= N_p\,(-\rho)^{\frac{1-p}{2}}) \text{ for } \rho < 0; \tag{3.2}$$

we omit the details of the proof. Hence $\lim_{\rho\to-\infty} G_{f_a}(\rho) = 0$ since $\lim_{\rho\to-\infty} G_{f_0}(\rho) = 0$ and $p > 1$. The fact that $G_{f_a}(-\rho) > G_{f_a}(\rho)$ for $\rho > 0$ was proved above by applying Lemma 3.1.

Part (ii). First, since $f_a(u)$ is nonincreasing on $[-a^{1/p}, \infty)$, it follows by [2, Theorem 3.4] that $G_{f_a}(\rho)$ is strictly decreasing on $[-a^{1/p}, \infty)$.

For $G_{f_a}(\rho)$ in (1.6), $G'_{f_a}(\rho)$ can be easily computed, see e.g. [12, p. 273]. We have, for $\rho < -a^{1/p}$,

$$G'_{f_a}(\rho) = 2^{-1/2} \int_\rho^\infty \frac{\theta_a(\rho) - \theta_a(u)}{\rho(\Delta F_a)^{3/2}} du, \tag{3.3}$$

where $\Delta F_a = F_a(\rho) - F_a(u)$ and

$$\theta_a(x) = 2F_a(x) - x f_a(x). \tag{3.4}$$

We compute that

$$\theta'_a(x) = f_a(x) - x f'_a(x), \tag{3.5}$$

$$\theta_a''(x) = -xf_a''(x). \tag{3.6}$$

Moreover, $G''_{f_a}(\rho)$ can be computed from (3.3), cf. [12, p. 273]. We have, for $\rho < -a^{1/p}$,

$$G''_{f_a}(\rho) = 2^{-1/2}\int_\rho^\infty \frac{-\frac{3}{2}\left[\theta_a(\rho)-\theta_a(u)\right](\Delta\tilde{f}_a) + (\Delta F_a)\left[\rho\theta_a'(\rho) - u\theta_a'(u)\right]}{\rho^2(\Delta F_a)^{5/2}}du, \tag{3.7}$$

where $\Delta\tilde{f}_a = \rho f_a(\rho) - uf_a(u)$.

Recalling a result of Smoller and Wasserman [12, p. 282], we obtain

$$\begin{aligned} &G''_{f_a}(\rho) + \frac{M}{2\rho}G'_{f_a}(\rho) \\ &= \int_\rho^\infty \frac{\frac{M}{2}\left[2(\Delta F_a)^2 - (\Delta\tilde{f}_a)(\Delta F_a)\right] + \frac{3}{2}(\Delta\tilde{f}_a)^2 - 2(\Delta\tilde{f}_a)(\Delta F_a) - (\Delta\hat{f}_a')(\Delta F_a)}{\rho^2(\Delta F_a)^{5/2}}du \end{aligned} \tag{3.8}$$

where $\Delta\hat{f}_a' = \rho^2 f_a'(\rho) - u^2 f_a'(u)$ and M is a constant to be chosen. Let the numerator of the integrand of the above integral be Q; i.e.,

$$Q = \frac{M}{2}\left[2(\Delta F_a)^2 - (\Delta\tilde{f}_a)(\Delta F_a)\right] + \frac{3}{2}(\Delta\tilde{f}_a)^2 - 2(\Delta\tilde{f}_a)(\Delta F_a) - (\Delta\hat{f}_a')(\Delta F_a). \tag{3.9}$$

Now, for $f = f_a(u)$ defined in (1.2), we obtain

$$F_a(u) = \int_0^u f_a(t)dt = \begin{cases} \frac{1}{p+1}(-u)^{p+1} + \frac{p}{p+1}a^{\frac{p+1}{p}} & \text{if } u < -a^{1/p}, \\ -au & \text{if } -a^{1/p} \le u \le a^{1/p}, \\ -\frac{1}{p+1}u^{p+1} - \frac{p}{p+1}a^{\frac{p+1}{p}} & \text{if } u > a^{1/p} \end{cases} \tag{3.10}$$

and

$$f_a'(u) = \begin{cases} p(-u)^{p-1} & \text{if } u < -a^{1/p}, \\ 0 & \text{if } -a^{1/p} < u < a^{1/p}, \\ -pu^{p-1} & \text{if } u > a^{1/p}, \end{cases}$$

where $f_a'(u)$ does not exist only at $u = \pm a^{1/p}$.

We choose $M = p+1$ in (3.8) and (3.9), and we compute that

(a) For $\rho < -a^{1/p}$, $\rho < u < -a^{1/p}$,

$$\begin{aligned} Q &= \frac{M}{2}\left[2(\Delta F_a)^2 - (\Delta\tilde{f}_a)(\Delta F_a)\right] + \frac{3}{2}(\Delta\tilde{f}_a)^2 - 2(\Delta\tilde{f}_a)(\Delta F_a) - (\Delta\hat{f}_a')(\Delta F_a) \\ &= \frac{(1-p)\left[M-(p+1)\right]\left[(-u)^{p+1} - (-\rho)^{p+1}\right]^2}{2(p+1)^2} \\ &= 0 \text{ since } M = p+1. \end{aligned}$$

(b) For $\rho < -a^{1/p}$, $-a^{1/p} < u < a^{1/p}$,

$$\begin{aligned} Q &= \frac{pa}{2(p+1)} \Big\{ (p+1)au^2 + (3p-1) \left[a^{\frac{p+1}{p}} - (-\rho)^{p+1} \right] u \\ &\quad + 2pa^{\frac{p+2}{p}} - a^{1/p}(-\rho)^{p+1} - 3pa^{1/p}(-\rho)^{p+1} \Big\}. \end{aligned}$$

In the above quadratic polynomial in u:

$$(p+1)au^2 + (3p-1) \left[a^{\frac{p+1}{p}} - (-\rho)^{p+1} \right] u + 2pa^{\frac{p+2}{p}} - (3p+1)a^{1/p}(-\rho)^{p+1},$$

the two coefficients $(p+1)a > 0$ and $(3p-1)[a^{\frac{p+1}{p}} - (-\rho)^{p+1}] < 0$ since $p > 1$, $a > 0$, and $\rho < -a^{1/p}$. So

$$\begin{aligned} Q &= \frac{pa}{2(p+1)} \Big\{ (p+1)au^2 + (3p-1) \left[a^{\frac{p+1}{p}} - (-\rho)^{p+1} \right] u \\ &\quad + 2pa^{\frac{p+2}{p}} - (3p+1)a^{1/p}(-\rho)^{p+1} \Big\} \\ &< \frac{pa}{2(p+1)} \Big\{ (p+1)a(-a^{1/p})^2 + (3p-1) \left[a^{\frac{p+1}{p}} - (-\rho)^{p+1} \right] (-a^{1/p}) \\ &\quad + 2pa^{\frac{p+2}{p}} - (3p+1)a^{1/p}(-\rho)^{p+1} \Big\} \\ &= \frac{pa^{\frac{p+1}{p}}}{(p+1)} \left\{ a^{\frac{p+1}{p}} - (-\rho)^{p+1} \right\} \\ &< 0 \text{ (since } \rho < -a^{1/p}). \end{aligned}$$

(c) For $\rho < -a^{1/p}$, $u > a^{1/p}$, it is easy to see that

$$\begin{aligned} Q &= \frac{pa^{\frac{p+1}{p}} \left[-(3p+1)(-\rho)^{p+1} + 4pa^{\frac{p+1}{p}} - (3p+1)u^{p+1} \right]}{(p+1)} \\ &< \frac{pa^{\frac{p+1}{p}} \left[-(3p+1)a^{\frac{p+1}{p}} + 4pa^{\frac{p+1}{p}} - (3p+1)a^{\frac{p+1}{p}} \right]}{(p+1)} \\ &= -2pa^{\frac{2p+2}{p}} \\ &< 0. \end{aligned}$$

By (a)-(c) above, for $\rho < -a^{1/p}(<0)$, we choose $M = p+1$ in (3.8) and we conclude that

$$\begin{aligned} G''_{f_a}(\rho) + \frac{p+1}{2\rho} G'_{f_a}(\rho) &= \int_\rho^\infty \frac{Q}{\rho^2(\triangle F_a)^{5/2}} du \\ &= \int_\rho^{-a^{1/p}} \frac{Q}{\rho^2(\triangle F_a)^{5/2}} du + \int_{-a^{1/p}}^{a^{1/p}} \frac{Q}{\rho^2(\triangle F_a)^{5/2}} du \\ &\quad + \int_{a^{1/p}}^\infty \frac{Q}{\rho^2(\triangle F_a)^{5/2}} du \\ &< 0. \end{aligned}$$

Since $G_{f_a}(\rho)$ is strictly decreasing on $[-a^{1/p}, \infty)$ and $\lim_{\rho\to\pm\infty} G_{f_a}(\rho) = 0$, we conclude that $G_{f_a}(\rho)$ has a exactly one critical point, a maximum, at $\rho = \rho^*$ for some ρ^*, on $(-\infty, -a^{1/p}]$ and hence on $(-\infty, \infty)$. The proof of part (ii) is complete.

Part (iii). In above, we obtain $\rho^* \le -a^{1/p}$. To show

$$-\left(\frac{3p+1}{p-1}\right)^{1/(p+1)} a^{1/p} < \rho^*,$$

it suffices to show

$$G'_{f_a}(\rho) > 0 \text{ for } \rho \le -\left(\frac{3p+1}{p-1}\right)^{1/(p+1)} a^{1/p}, \tag{3.11}$$

which is shown as follows. By (3.4)-(3.6), it is easy to see that, for $f = f_a(u)$ in (1.2),

$$\theta_a(0) = 0, \quad \lim_{u\to-\infty} \theta_a(u) = -\infty, \quad \lim_{u\to\infty} \theta_a(u) = \infty,$$

$$\theta'_a(u) = f_a(u) = -a < 0 \text{ for } -a^{1/p} < u < a^{1/p},$$

$$\theta''_a(u) = -uf''_a(u) = \begin{cases} -p(p-1)(-u)^{p-1} < 0 & \text{if } u < -a^{1/p}, \\ 0 & \text{if } -a^{1/p} < u < a^{1/p}, \\ p(p-1)u^{p-1} > 0 & \text{if } u > a^{1/p}. \end{cases}$$

Thus, it follows that $\theta_a(u)$ has exactly two critical points at $-a^{1/p}$ (a local maximum) and $a^{1/p}$ (a local minimum) such that

$$\theta_a\left(-\left(\frac{3p+1}{p-1}\right)^{1/(p+1)} a^{1/p}\right) = \theta_a(a^{1/p}) = -a^{\frac{p+1}{p}} < \theta_a(0) = 0 < \theta_a(-a^{1/p}) = a^{\frac{p+1}{p}}$$

and $\theta_a(u)$ is strictly increasing on $(-\infty, -a^{1/p})$, strictly decreasing on $(-a^{1/p}, a^{1/p})$, and strictly increasing on $(a^{1/p}, \infty)$. Thus by (3.3), Inequality (3.11) follows. So (2.2) is proved.

Part (iv). Equality (2.4) follows immediately from Theorem 1.3, and Equality (2.5) from (2.1), after some simple computations which we omit.

By parts (ii)-(iii),

$$G_{f_a}(-a^{1/p}) \le G_{f_a}(\rho^*) = \max_\rho G_{f_a}(\rho).$$

By (3.2), we obtain

$$G_{f_a}(\rho) < G_{f_0}(\rho) = \frac{N_p}{(-\rho)^{\frac{p-1}{2}}} \text{ for } \rho < 0.$$

Moreover, since $G_{f_0}(\rho)$ is strictly increasing for $\rho < 0$ and since $\rho^* \le -a^{1/p}$, we obtain

$$G_{f_a}(\rho^*) = \max_\rho G_{f_a}(\rho) < G_{f_0}(\rho^*) \le G_{f_0}(-a^{1/p}).$$

So part (vi) follows.

Part (v). For fixed $p > 1$, Inequality (2.2) implies immediately that $\lim_{a\to 0^+} \rho^*(a) = 0$ and $\lim_{a\to\infty} \rho^*(a) = -\infty$.

Part (vi). For fixed $p > 1$, Inequality (2.3) implies immediately that $\lim_{a\to 0^+} G_{f_a}(\rho^*(a)) = \infty$ and $\lim_{a\to\infty} G_{f_a}(\rho^*(a)) = 0$. By (1.2) and (2.1), for $0 < a_1 < a_2$, it is easy to show the comparison result $G_{f_{a_1}}(\rho) > G_{f_{a_2}}(\rho)$ for $\rho < 0$; we omit the details of the proof. Thus by Parts (ii) and (iii), $G_{f_{a_1}}(\rho^*(a_1)) > G_{f_{a_2}}(\rho^*(a_2))$. Hence $G_{f_a}(\rho^*(a))$ is a strictly decreasing functions of $a > 0$.

The proof of Theorem 2.1 is now complete. ■

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