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ON A RELATION BETWEEN CARLEMAN'S INEQUALITY AND VAN DER CORPUT'S INEQUALITY

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Abstract. By introducing a parameter $\lambda \in [0,1]$, we give an inequality relating Carleman's inequality with Van der Corput's inequality. In particular, a generalization of Carleman's inequality with a best constant factor $e^{\frac{1}{1-\lambda}}$, $\lambda \in [0,1)$ is considered.

1. Introduction

If $a_n \geq 0\,(n \in N)$ with $0 < \sum_{n=1}^{\infty} a_n < \infty$, then the famous Carleman's inequality is:

$$\sum_{n=1}^{\infty} \big(\prod_{k=1}^{n} a_k\big)^{1/n} < e \sum_{n=1}^{\infty} a_n, \tag{1.1}$$

where the constant factor e is the best possible (see [1]). On the other hand, if $S_n = \sum_{k=1}^{n} \frac{1}{k}$, and $a_n \geq 0\,(n \in N)$ with $0 < \sum_{n=1}^{\infty}(n+1)a_n < \infty$, then we have the following Van der Corput's inequality:

$$\sum_{n=1}^{\infty} \big(\prod_{k=1}^{n} a_k^{1/k}\big)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty}(n+1)a_n, \tag{1.2}$$

where the constant factor $e^{1+\gamma}$ (γ is Euler constant) is the best possible (see [5]).

Recently, Yang et al. [8] gave a strengthened version of (1.1) as follows.

$$\sum_{n=1}^{\infty} \big(\prod_{k=1}^{n} a_k\big)^{1/n} < e \sum_{n=1}^{\infty} \big[1 - \frac{1}{2(n+1)}\big] a_n. \tag{1.3}$$

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Some other strengthened version of (1.1) were given by [6,9]. Hu [3] gave an improvement of (1.2):

$$\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n} a_k^{1/k}\Big)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty}\Big(n-\frac{1}{4n}\ln n\Big)a_n. \tag{1.4}$$

The main objective of this paper is to establish a relation between (1.1) and (1.2) with a parameter $\lambda \in [0,1]$ and a series as

$$\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n} a_k^{1/k^{\lambda}}\Big)^{1/S_n(\lambda)}\ \Big(S_n(\lambda)=\sum_{k=1}^{n}\frac{1}{k^{\lambda}}\Big). \tag{1.5}$$

For this, we need the following Euler-Maclaurin's formula:

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)dx + \frac{1}{2}(f(n)+f(1)) + \int_1^n \rho_1(x)f'(x)dx, \tag{1.6}$$

where $\rho_1(x) = x-[x]+\frac{1}{2}$ is Bernoulli's function, and $f \in C^1[1,\infty)$. If $(-1)^i f^{(i)}(x) > 0\,(x \in [n,\infty))$, and $f^{(i)}(\infty) = 0\,(i = 1,2,3)$, we still have (see [7, (1.7)-(1.9)]):

$$\int_n^{\infty} \rho_1(x)f'(x)dx = -\frac{1}{12}f'(n)\varepsilon\,(0<\varepsilon<1). \tag{1.7}$$

2. Some Lemmas

Lemma 2.1. *If $\lambda \in (0,1)$, setting $S_n(\lambda) = \sum_{k=1}^{n}\frac{1}{k^{\lambda}}$, then we have*

$$\frac{1}{S_n(\lambda)}\sum_{k=1}^{n}\frac{\ln k}{k^{\lambda}} = -\frac{1}{1-\lambda} + \ln n + \alpha_n\,(\alpha_n = o(1)\,(n\to\infty)). \tag{2.1}$$

Proof. Setting $f(x) = \frac{\ln x}{x^{\lambda}}$ $(x \in [1,\infty))$, we have $f(1) = 0,\ f(n) = \frac{\ln n}{n^{\lambda}}$, and

$$\int_1^n f(x)dx = \frac{n^{1-\lambda}\ln n}{1-\lambda} - \frac{n^{1-\lambda}}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2}. \tag{2.2}$$

For $x > e^{1/\lambda}$, $f'(x) = -\frac{\lambda\ln x-1}{x^{\lambda+1}} < 0$, and by induction, we obtain

$$(-1)^i f^{(i)}(x) = \frac{\lambda(\lambda+1)\cdots(\lambda+i-1)\ln x - \phi_i(\lambda)}{x^{\lambda+i}}\ (i=1,2,\cdots),$$

where $\phi_i(\lambda)\,(i = 1, 2, \cdots)$ are positive constants. It follows that there exists $n_0 > e^{1/\lambda}$ such that for $x \in [n_0, \infty)$ f(x) possesses the condition of (1.7). Hence for $n > n_0$, we find

$$0 < \int_n^\infty \rho_1(x) f'(x)dx < -\frac{1}{12} f'(n) = \frac{\lambda \ln n - 1}{12n^{\lambda+1}}, \text{ and}$$

$$\beta_n = \frac{\ln n}{2n^\lambda} - \int_n^\infty \rho_1(x) f'(x)dx = o(1)\,(n \to \infty). \tag{2.3}$$

By (1.6), we have

$$\sum_{k=1}^n f(k) = \int_1^n f(x)dx + \frac{\ln n}{2n^\lambda} + \int_1^n \rho_1(x) f'(x)dx, \text{ and} \tag{2.4}$$

$$C_\lambda = \lim_{n\to\infty} \Big[\sum_{k=1}^n f(k) - \int_1^n f(x)dx\Big] = \int_1^\infty \rho_1(x) f'(x)dx$$

$$= \int_1^n \rho_1(x) f'(x)dx + \int_n^\infty \rho_1(x) f'(x)dx. \tag{2.5}$$

Setting $C = \frac{1}{(1-\lambda)^2} + C_\lambda$, by (2.2), (2.3), and (2.5), we reduce (2.4) as

$$\sum_{k=1}^n \frac{\ln k}{k^\lambda} = \frac{n^{1-\lambda}\ln n}{1-\lambda} - \frac{n^{1-\lambda}}{(1-\lambda)^2} + C + \beta_n \;(\beta_n = o(1)\,(n \to \infty)). \tag{2.6}$$

For $\lambda \in (0, 1)$, by (1.6) and (1.7), we have

$$\frac{n^{1-\lambda}}{1-\lambda} - \frac{1}{1-\lambda} = \int_1^n \frac{1}{x^\lambda}dx$$

$$< \sum_{k=1}^n \frac{1}{k^\lambda} < \int_0^n \frac{1}{x^\lambda}dx = \frac{n^{1-\lambda}}{1-\lambda}, \text{ and}$$

$$\sum_{k=1}^n \frac{1}{k^\lambda} = \frac{n^{1-\lambda}}{1-\lambda} + O(1)(n \to \infty). \tag{2.7}$$

Hence by (2.6) and (2.7), we have

$$-\ln n + \frac{1}{S_n(\lambda)} \sum_{k=1}^n \frac{\ln k}{k^\lambda}$$

$$= -\ln n + \frac{\dfrac{n^{1-\lambda}\ln n}{1-\lambda} - \dfrac{n^{1-\lambda}}{(1-\lambda)^2} + C + \beta_n}{\frac{n^{1-\lambda}}{1-\lambda} + O(1)}$$

$$= \frac{-\ln n O(1) - \dfrac{n^{1-\lambda}}{(1-\lambda)^2} + C + \beta_n}{\dfrac{n^{1-\lambda}}{1-\lambda} + O(1)}$$

$$= \frac{-\frac{\ln n}{n^{1-\lambda}} O(1) - \dfrac{1}{(1-\lambda)^2} + \dfrac{1}{n^{1-\lambda}}(C + \beta_n)}{\dfrac{1}{1-\lambda} + \dfrac{1}{n^{1-\lambda}} O(1)} \to \frac{-1}{1-\lambda} \ (n \to \infty).$$

It follows that (2.1) is valid. The lemma is proved.

Lemma 2.2. *If* $o_n = o(1)$ $(n \to \infty)$*, then we have*

$$\frac{\sum_{n=1}^{N} \dfrac{o_n}{n}}{\sum_{n=1}^{N} \dfrac{1}{n}} = o(1) \ (N \to \infty). \tag{2.8}$$

Proof. For any $\varepsilon > 0$, there exists $N_0 > 1$, such that for any $n > N_0$ $|o_n| < \varepsilon/2$. Setting $M = \max\{|o_1|, |o_2|, \cdots, |o_{N_0}|\}$, since we find

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N_0} \dfrac{M}{n}}{\sum_{n=1}^{N} \dfrac{1}{n}} = 0,$$

there exists $N_1 > N_0$, such that for any $N > N_1$,

$$\frac{\sum_{n=1}^{N_0} \dfrac{M}{n}}{\sum_{n=1}^{N} \dfrac{1}{n}} < \frac{\varepsilon}{2}.$$

Then for any $N > N_1$,

$$\left|\frac{\sum_{n=1}^{N}\frac{o_n}{n}}{\sum_{n=1}^{N}\frac{1}{n}}\right| \le \frac{\sum_{n=1}^{N}\frac{|o_n|}{n}}{\sum_{n=1}^{N}\frac{1}{n}}$$

$$< \frac{\sum_{n=1}^{N_0}\frac{M}{n}+\frac{\varepsilon}{2}\sum_{n=N_0+1}^{N}\frac{1}{n}}{\sum_{n=1}^{N}\frac{1}{n}} < \frac{\sum_{n=1}^{N_0}\frac{M}{n}}{\sum_{n=1}^{N}\frac{1}{n}}+\frac{\varepsilon}{2}<\varepsilon.$$

Hence we have (2.8). The lemma is proved.

3. Main Results

Theorem 3.1. *If* $\lambda \in [0,1]$, $S_n(\lambda)=\sum_{k=1}^{n}\frac{1}{k^\lambda}$, *and* $a_n \ge 0$ $(n \in N)$, *then we have*

$$\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n}a_k^{1/k^\lambda}\Big)^{1/S_n(\lambda)} \le e\sum_{n=1}^{\infty}e^{\lambda n^{\lambda-1}S_n(\lambda)}a_n. \tag{3.1}$$

Proof. Setting $c_n>0$, such that

$$\Big(\prod_{k=1}^{n}c_k^{1/k^\lambda}\Big)^{-1/S_n(\lambda)} = \frac{1}{(n+1)^\lambda S_{n+1}(\lambda)}, \tag{3.2}$$

then we find $\prod_{k=1}^{n}c_k^{1/k^\lambda}=\big[(n+1)^\lambda S_{n+1}(\lambda)\big]^{S_n(\lambda)}$, $\prod_{k=1}^{n-1}c_k^{1/k^\lambda}=\big[n^\lambda S_n(\lambda)\big]^{S_{n-1}(\lambda)}$, and

$$c_n=\frac{\big[(n+1)^\lambda S_{n+1}(\lambda)\big]^{n^\lambda S_n(\lambda)}}{\big[n^\lambda S_n(\lambda)\big]^{n^\lambda S_{n-1}(\lambda)}}\ (n\in N,\ S_0(\lambda)=0). \tag{3.3}$$

By using the arithmetic-geometric average inequality (see [2, Th. 9], we have

$$\Big[\prod_{k=1}^{n}(c_k a_k)^{1/k^\lambda}\Big]^{1/S_n(\lambda)} \le \sum_{k=1}^{n}\frac{1}{k^\lambda S_n(\lambda)}c_k a_k. \tag{3.4}$$

Since we have (see [6, (5)])

$$\Big(1+\frac{1}{x}\Big)^x < e\Big[1-\frac{1}{2(x+1)}\Big] < e\ (for\, x>0), \tag{3.5}$$

then by (3.4), (3.2), (3.3) and (3.5), we find

(3.6)

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n}a_k^{1/k^{\lambda}}\Big)^{1/S_n(\lambda)}=\sum_{n=1}^{\infty}\Big[\prod_{k=1}^{n}(c_ka_k)^{1/k^{\lambda}}\Big]^{1/S_n(\lambda)}\Big(\prod_{k=1}^{n}c_k^{1/k^{\lambda}}\Big)^{-1/S_n(\lambda)}\\
&\le\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k^{\lambda}S_n(\lambda)}c_ka_k\frac{1}{(n+1)^{\lambda}S_{n+1}(\lambda)}=\sum_{k=1}^{\infty}\frac{1}{k^{\lambda}}c_ka_k\sum_{n=k}^{\infty}\frac{1}{(n+1)^{\lambda}S_{n+1}(\lambda)S_n(\lambda)}\\
&=\sum_{k=1}^{\infty}\frac{1}{k^{\lambda}}c_ka_k\sum_{n=k}^{\infty}\Big[\frac{1}{S_n(\lambda)}-\frac{1}{S_{n+1}(\lambda)}\Big]=\sum_{k=1}^{\infty}\frac{1}{k^{\lambda}}c_ka_k\frac{1}{S_k(\lambda)}\\
&=\sum_{k=1}^{\infty}\Big[\frac{(k+1)^{\lambda}S_{k+1}(\lambda)}{k^{\lambda}S_k(\lambda)}\Big]^{k^{\lambda}S_k(\lambda)}a_k\\
&\le\sum_{k=1}^{\infty}\Big[(1+\frac{1}{k})^{k}\Big]^{\lambda k^{\lambda-1}S_k(\lambda)}\Big[1+\frac{1}{(k+1)^{\lambda}S_k(\lambda)}\Big]^{(k+1)^{\lambda}S_k(\lambda)}a_k\\
&\le e\sum_{k=1}^{\infty}\Big[(1+\frac{1}{k})^{k}\Big]^{\lambda k^{\lambda-1}S_k(\lambda)}a_k\le e\sum_{k=1}^{\infty}\Big\{e\Big[1-\frac{1}{2(k+1)}\Big]\Big\}^{\lambda k^{\lambda-1}S_k(\lambda)}a_k.
\end{aligned}$$

Hence, we obtain (3.1). The theorem is proved.

Remark 1. For $\lambda = 1$, by (1.6) and (1.7), we find the following Franel's inequality (see [4]):

$$\sum_{k=1}^{n}\frac{1}{k}<\ln n+\frac{1}{2n}+\gamma,\ \text{and} \tag{3.7}$$

$$S_n=S_n(1)=\sum_{k=1}^{n+1}\frac{1}{k}-\frac{1}{n+1}$$

$$<\ln(n+1)-\frac{1}{2(n+1)}+\gamma<\ln(n+1)+\gamma. \tag{3.8}$$

Hence, for $\lambda=1$, by (3.8), inequality (3.1) reduces to (1.2). It is obvious that for λ = 0, (3.1) reduces to (1.1). It follows that (3.1) is a relation between (1.1) and (1.2).

Theorem 3.2. *If $a_n \ge 0$ $(n \in N)$, such that $0 < \sum_{n=1}^{\infty} a_n < \infty$, $\lambda \in [0,1)$, and $S_n(\lambda) = \sum_{k=1}^{n}\frac{1}{k^{\lambda}}$, then we have*

$$\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n}a_k^{1/k^{\lambda}}\Big)^{1/S_n(\lambda)}<e^{\frac{1}{1-\lambda}}\sum_{n=1}^{\infty}a_n, \tag{3.9}$$

where the constant factor $e^{\frac{1}{1-\lambda}}$ is the best possible. We also have its strengthened version as:

$$\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n} a_k^{1/k^{\lambda}}\Big)^{1/S_n(\lambda)} < e^{\frac{1}{1-\lambda}}\sum_{n=1}^{\infty}\Big[1-\frac{1}{2(n+1)}\Big]^{\frac{\lambda}{1-\lambda}} a_n. \tag{3.10}$$

In particular, for $\lambda = 1/2$, *we have* $S_n(\frac{1}{2}) = \sum_{k=1}^{n}\frac{1}{\sqrt{k}}$, *and*

$$\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n} a_k^{1/\sqrt{k}}\Big)^{1/S_n(1/2)} < e^{2}\sum_{n=1}^{\infty}\Big[1-\frac{1}{2(n+1)}\Big] a_n. \tag{3.11}$$

Proof. For $\lambda = 0$, since $S_n(0) = n$, (3.9) reduces to (1.1). We only consider $\lambda \in (0,1)$ in the following. Since we have

$$S_n(\lambda) < \int_0^n \frac{1}{x^{\lambda}}dx = \frac{n^{1-\lambda}}{1-\lambda}, \text{ for } \lambda \in (0,1),$$

then by (3.1) and (3.6), we obtain (3.9) and (3.10).

Setting $\tilde{a}_n$ $(n \in N)$ as:

$$\tilde{a}_n = \frac{1}{n}, \text{ for } n \le N; \tilde{a_n} = 0, \text{ for } n > N,$$

then by (2.1), for $n \le N$, since $\alpha_n = o(1)$ $(n \to \infty)$, we find

$$\Big(\prod_{k=1}^{n} \tilde{a}_k^{1/k^{\lambda}}\Big)^{1/S_n(\lambda)} = \exp\Big\{\ln\Big[\prod_{k=1}^{n}\Big(\frac{1}{k}\Big)^{1/k^{\lambda}}\Big]^{1/S_n(\lambda)}\Big\}$$
$$= \exp\Big\{-\tfrac{1}{S_n(\lambda)}\textstyle\sum_{k=1}^{n}\frac{\ln k}{k^{\lambda}}\Big\} = \exp\Big\{\frac{1}{1-\lambda} - \ln n - \alpha_n\Big\}$$

$$= \frac{1}{n}\exp\Big\{\frac{1}{1-\lambda}\Big\}\exp\{\ln(1+o_n)\} = \frac{1+o_n}{n}\exp\Big\{\frac{1}{1-\lambda}\Big\}, \tag{3.12}$$

where $o_n = o(1)$ $(n \to \infty)$.

If there exists $\lambda \in (0,1)$, such that the constant factor $e^{\frac{1}{1-\lambda}}$ in (3.9) is not the best possible, then there exists positive number $K < e^{\frac{1}{1-\lambda}}$, such that (3.9) is still valid if we replace $e^{\frac{1}{1-\lambda}}$ by K. In particular, we have

$$\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n} \tilde{a}_k^{1/k^{\lambda}}\Big)^{1/S_n(\lambda)} < K\sum_{n=1}^{\infty}\tilde{a}_n. \tag{3.13}$$

Hence we find

$$K > \frac{1}{\sum_{n=1}^{N}\frac{1}{n}}\sum_{n=1}^{N} exp\Big\{\ln\Big[\prod_{k=1}^{n}\Big(\frac{1}{k}\Big)^{1/k^{\lambda}}\Big]^{1/S_n(\lambda)}\Big\}$$

$$= \frac{1}{\sum_{n=1}^{N} \frac{1}{n}} \sum_{n=1}^{N} \frac{1+o_n}{n} \exp\{\frac{1}{1-\lambda}\} = e^{\frac{1}{1-\lambda}} [1 + \frac{\sum_{n=1}^{N} \frac{o_n}{n}}{\sum_{n=1}^{N} \frac{1}{n}}],$$

and $K \geq e^{\frac{1}{1-\lambda}},\ for\ N \to \infty$, by (2.8). This contradicts the face that $K < e^{\frac{1}{1-\lambda}}$. Hence the constant factor $e^{\frac{1}{1-\lambda}}$ in (3.9) is the best possible. The theorem is proved.

Remark 2. For $\lambda = 0$, by (3.9) or (3.10), we have (1.1). Inequality (3.9) is a generalization of Carleman's inequality with a best constant factor $e^{\frac{1}{1-\lambda}}$ $(\lambda \in [0,1])$; So is (3.10).

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