

**CONTROLLABILITY OF NEUTRAL FUNCTIONAL  
INTEGRODIFFERENTIAL INFINITE DELAY SYSTEMS  
IN BANACH SPACES**

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**Abstract.** Sufficient conditions for controllability of neutral functional integrodifferential infinite delay systems in Banach spaces are established. The results are obtained by using the analytic semigroup theory and the Nussbaum fixed point theorem. An example is provided to illustrate the theory.

1. INTRODUCTION

The problem of controllability of abstract functional differential equations has received much attention in recent years. Balachandran et al. [1] studied the controllability problem for nonlinear functional differential systems in Banach spaces. Park and Han [9] derived a set of sufficient conditions for the controllability of nonlinear functional integrodifferential systems in Banach spaces whereas in [5] Han et al. investigated the controllability problem of integrodifferential systems by considering the initial condition in some abstract phase space. Balachandran and Sakthivel [2] discussed the controllability of neutral functional integrodifferential systems in Banach spaces by using the semigroup theory and the Schaefer fixed point theorem. Recently in [3] Dauer and Balachandran obtained the existence results for nonlinear neutral integrodifferential equations in a Banach space and as an application the controllability problem for the neutral system is discussed. The purpose of this paper is to establish a set of sufficient conditions for the controllability of neutral functional integrodifferential infinite delay systems in Banach spaces. The results are established using the Nussbaum fixed point theorem.

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## 2. PRELIMINARIES

Consider the neutral integrodifferential control system of the form

$$(1) \quad \frac{d}{dt}[x(t) - h(t, x_t)] = Ax(t) + f(t, x_t) + \int_{-\infty}^t g(t, s, x_s) ds + Bu(t), \quad t \geq 0,$$

$$x_0 = \phi \in \Omega,$$

where  $\Omega$  is an open subset of a phase space  $\mathcal{B}$  which will be defined later, the state  $x(\cdot)$  takes values in a Banach space  $X$  with norm  $\|\cdot\|$ ,  $x_t$  represents the function  $x_t : (-\infty, 0] \rightarrow X$  defined by  $x_t(\theta) = x(t + \theta)$ ,  $-\infty < \theta \leq 0$  which belongs to  $\mathcal{B}$ , the control  $u(\cdot)$  is given in  $L^2(J, U)$ , a Banach space of admissible control functions with  $U$  as a Banach space and  $J = [0, b]$  and  $B$  is a bounded linear operator from  $U$  into  $X$ . The nonlinear operators  $h : J \times \Omega \rightarrow X$ ,  $f : J \times \Omega \rightarrow X$  and  $g : J \times J \times \Omega \rightarrow X$  are continuous. Here  $A : D(A) \rightarrow X$  is the infinitesimal generator of an analytic semigroup  $T(t)$  of bounded linear operators on  $X$ . Then the fractional power  $(-A)^\alpha$  can be defined [10] for  $0 \leq \alpha \leq 1$  and  $(-A)^\alpha$  is a closed linear invertible operator with domain  $D((-A)^\alpha)$  dense in  $X$ . The closedness of  $(-A)^\alpha$  implies that  $D((-A)^\alpha)$  endowed with the graph norm  $\|x\|_X = \|x\| + \|A^\alpha x\|$  is a Banach space. Since  $(-A)^\alpha$  is invertible its graph norm  $\|\cdot\|_X$  is equivalent to the norm  $\|x\| = \|A^\alpha x\|$ . Thus  $D(-A^\alpha)$  equipped with the norm  $\|\cdot\|$  is a Banach space which we denote by  $X_\alpha$ . Also it is clear that  $0 < \alpha < \beta$  implies  $X_\alpha \supset X_\beta$  and that the imbedding is continuous.

**Lemma.** *Suppose that  $(-A)^\alpha$  satisfies the above conditions, then*

(i) *For every  $b > 0$ , there exists a positive constant  $C$  such that*

$$\|(-A)^\alpha T(t)\| \leq \frac{C}{t^\alpha}, \quad 0 < t \leq b.$$

(ii) *For every  $b > 0$ , there exists a positive constant  $C^*$  such that*

$$\|(T(t) - I)(-A)^{-\alpha}\| \leq C^* t^\alpha, \quad 0 < t \leq b.$$

The proof of the above lemma can be found in [10].

In our study we follow the axiomatic definition for the phase space  $\mathcal{B}$  introduced by Hale and Kato [4] and the terminology used in [7]. The phase space  $\mathcal{B}$  is a linear space of functions mapping  $(-\infty, 0]$  into  $X$  endowed with the seminorm  $|\cdot|_{\mathcal{B}}$  and assume that  $\mathcal{B}$  satisfies the following axioms:

(A1) If  $x : (-\infty, b) \rightarrow X$ ,  $b > 0$  is continuous on  $[0, b)$  and  $x_0 \in \mathcal{B}$ , then for every  $t$  in  $[0, b)$  the following conditions hold:

- (i)  $x_t$  is in  $\mathcal{B}$ .
- (ii)  $\|x(t)\| \leq H|x_t|_{\mathcal{B}}$ .
- (iii)  $|x_t|_{\mathcal{B}} \leq K(t) \sup\{\|x(s)\| : 0 \leq s \leq t\} + N(t)|x_0|_{\mathcal{B}}$ .

Here  $H \geq 0$  is a constant,  $K, N : [0, \infty) \rightarrow [0, \infty)$ ,  $K$  is locally bounded and  $H, K$  and  $N$  are independent of  $x(\cdot)$ .

(A2) For the function  $x(\cdot)$  in (A1),  $x_t$  is a  $\mathcal{B}$  valued continuous function on  $[0, b)$ .

(A3) The space  $\mathcal{B}$  is complete.

We need the following fixed point theorem due to Nussbaum [8].

**Nussbaum Fixed Point Theorem.** Let  $S$  be a closed, bounded and convex subset of a Banach space  $X$ . Let  $\Phi_1, \Phi_2$  be continuous mappings from  $S$  into  $X$  such that

- (i)  $(\Phi_1 + \Phi_2)S \subset S$ .
- (ii)  $\|\Phi_1 w_1 - \Phi_1 w_2\| \leq k\|w_1 - w_2\|$  for all  $w_1, w_2 \in S$ , where  $k$  is a constant and  $0 \leq k \leq 1$ .
- (iii)  $\overline{\Phi_2(S)}$  is compact.

Then the operator  $\Phi_1 + \Phi_2$  has a fixed point in  $S$ .

Let  $B_r[x] \subset \Omega$  be the closed ball centered at  $x$  with radius  $r > 0$ . We shall assume the following hypotheses:

- (B1)**  $A$  is the infinitesimal generator of an analytic semigroup  $T(t)$  and there exists a constant  $M > 0$  such that  $\|T(t)\| \leq M$ .
- (B2)** There exist constants  $\beta \in (0, 1)$  and  $L \geq 0$  such that the function  $h$  is  $X_\beta$  valued and satisfies the Lipschitz condition

$$\|(-A)^\beta h(t, \psi_1) - (-A)^\beta h(s, \psi_2)\| \leq L[|t - s| + \|\psi_1 - \psi_2\|]$$

for every  $0 \leq s, t \leq b$  and  $\psi_1, \psi_2 \in \Omega$  and

$$L\|(-A)^{-\beta}\|K(0) < 1.$$

- (B3)**  $(-A)^\beta h$  and  $f$  are continuous and there exist  $M_1, M_2 \geq 0$  such that

$$\|(-A)^\beta h(t, \psi)\| \leq M_1 \text{ and } \|f(t, \psi)\| \leq M_2$$

for every  $0 \leq s \leq t \leq b$  and  $\psi \in B_r[\phi]$ .

**(B4)**  $g : J \times J \times \mathcal{B} \rightarrow X$  is continuous and there exists  $N_1 \geq 0$  such that

$$\|g(t, s, \eta)\| \leq N_1$$

for every  $0 \leq s \leq t \leq b$  and  $\eta \in B_r[\phi]$ .

**(B5)** For each  $\phi \in \mathcal{B}$

$$q(t) = \lim_{a \rightarrow \infty} \int_{-a}^0 g(t, s, \phi(s)) ds$$

exists and it is continuous. Further, there exists  $N_2 \geq 0$  such that  $\|q(t)\| \leq N_2$ .

**(B6)** There is a compact set  $V \subseteq X$  such that  $T(t)f(s, \psi)$ ,  $T(t)Bu(s)$ ,  $T(t)g(s, \tau, \eta)$  and  $T(t)q(s) \in V$  for all  $\psi, \eta \in B_r[\phi]$  and  $0 \leq \tau \leq s \leq t \leq b$ .

**(B7)** The linear operator  $W : L^2([0, b], U) \rightarrow X$  defined by

$$Wu = \int_0^b T(b-s)Bu(s)ds,$$

induces an invertible operator  $\tilde{W}$  defined on  $L^2([0, b], U)/\ker W$  and there exist positive constants  $K_1, K_2 > 0$  such that  $\|\tilde{W}^{-1}\| \leq K_1$  and  $\|B\| \leq K_2$ . See[11].

**(B8)** Let  $K^* = \max\{K(t) : 0 \leq t \leq b\}$  and for fixed  $\epsilon$ , the following conditions hold:

- (i)  $|y_t - \phi|_{\mathcal{B}} \leq \epsilon$ .
- (ii)  $\|(T(t) - I)h(t, y_t)\| \leq \epsilon$ .
- (iii)  $0 \leq L^* = LK^*[\|(-A)^{-\beta}\| + \frac{C}{\beta}\tau^\beta] < 1$ .

**(B9)** Choose  $\rho = \frac{r - \epsilon}{K^*}$  and  $\omega = (1 + bMK_1K_2)\|(-A)^{-\beta}\|LK^*$ .

**(B10)**  $b[1 + bMK_1K_2]\|(-A)^{-\beta}\|ML + MK_1K_2(\|x_1\| + M\|\phi(0)\|$   
 $+ bM(M_2 + N_2 + bN_1) + \frac{CM_1}{\beta}b^\beta) + M(M_2 + N_2 + bN_1)]$   
 $+ \epsilon(1 + bMK_1K_2)(\|(-A)^{-\beta}\|ML + 1) + \frac{CM_1}{\beta}b^\beta \leq (1 - \omega)\rho$ .

Then the mild solution of the system (1) is given by [6]

$$(2) \quad \begin{aligned} x(t) = & T(t)[\phi(0) - h(0, \phi)] + h(t, x_t) + \int_0^t AT(t-s)h(s, x_s)ds \\ & + \int_0^t T(t-s)[Bu(s) + f(s, x_s) + q(s) + \int_0^s g(s, \tau, x_\tau)d\tau]ds. \end{aligned}$$

**Definition.** The system (1) is said to be controllable on the interval  $[0, b]$ , if for every initial function  $\phi \in \Omega$  and  $x_1 \in X$ , there exists a control  $u \in L^2([0, b], U)$  such that the solution  $x(\cdot)$  of (1) satisfies  $x(b) = x_1$ .

### 3. MAIN RESULT

**Theorem.** *If the hypotheses (B1) - (B10) are satisfied, then the system (1) is controllable on  $J$ .*

*Proof.* Using the hypothesis (B7) for an arbitrary function  $x(\cdot)$ , define the control

$$\begin{aligned} u(t) = & \tilde{W}^{-1}[x_1 - T(b)[\phi(0) - h(0, \phi)] - h(b, x_b) \\ & - \int_0^b AT(b-s)h(s, x_s)ds - \int_0^b T(b-s)f(s, x_s)ds \\ & - \int_0^b T(b-s)q(s)ds - \int_0^b \int_0^s T(b-s)g(s, \tau, x_\tau)d\tau ds](t). \end{aligned}$$

Let  $y(\cdot, \phi) : (-\infty, b) \rightarrow X$  be the function defined by

$$y(t) = \begin{cases} \phi(t) & -\infty < t < 0, \\ T(t)\phi(0) & t \geq 0. \end{cases}$$

From the axioms of  $\mathcal{B}$ , we see that the map  $t \rightarrow y_t$  is continuous. Let  $Y = C(J, X)$ . Define the set

$$Q = \{z \in Y : z(0) = 0, \|z(t)\| \leq \rho, 0 \leq t \leq b\}.$$

Clearly  $Q$  is a nonempty, bounded, convex and closed subset of  $Y$ . For each  $z \in Y$ , define the function  $\hat{z}$  by

$$\hat{z}(t) = \begin{cases} 0 & t \leq 0, \\ z(t) & 0 \leq t \leq b. \end{cases}$$

If  $x(t)$  satisfies (2), then we can write it as  $x(t) = z(t) + y(t)$ ,  $0 \leq t \leq b$ , which implies that  $x_t = \hat{z}_t + y_t$  for every  $0 \leq t \leq b$  and that the function  $z(t)$  satisfies

$$\begin{aligned}
z(t) = & -T(t)h(0, \phi) + h(t, \hat{z}_t + y_t) + \int_0^t AT(t-s)h(s, \hat{z}_s + y_s)ds \\
& + \int_0^t T(t-\eta)B\tilde{W}^{-1}[x_1 - T(b)[\phi(0) - h(0, \phi)] - h(b, \hat{z}_b + y_b) \\
& - \int_0^b AT(b-s)h(s, \hat{z}_s + y_s)ds - \int_0^b T(b-s)f(s, \hat{z}_s + y_s)ds \\
& - \int_0^b T(b-s)q(s)ds - \int_0^b \int_0^s T(b-s)g(s, \tau, \hat{z}_\tau + y_\tau)d\tau ds]d\eta \\
& + \int_0^t T(t-s)[f(s, \hat{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \hat{z}_\tau + y_\tau)d\tau]ds.
\end{aligned}$$

Define the operator  $\Psi = \Psi_1 + \Psi_2$  on  $Q$  by

$$\Psi_1 z(t) = -T(t)h(0, \phi) + h(t, \hat{z}_t + y_t) + \int_0^t AT(t-s)h(s, \hat{z}_s + y_s)ds$$

and

$$\begin{aligned}
\Psi_2 z(t) = & \int_0^t T(t-\eta)B\tilde{W}^{-1}[x_1 - T(b)[\phi(0) - h(0, \phi)] - h(b, \hat{z}_b + y_b) \\
& - \int_0^b AT(b-s)h(s, \hat{z}_s + y_s)ds - \int_0^b T(b-s)f(s, \hat{z}_s + y_s)ds \\
& - \int_0^b T(b-s)q(s)ds - \int_0^b \int_0^s T(b-s)g(s, \tau, \hat{z}_\tau + y_\tau)d\tau ds]d\eta \\
& + \int_0^t T(t-s)[f(s, \hat{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \hat{z}_\tau + y_\tau)d\tau]ds.
\end{aligned}$$

Now we shall show that the operator  $\Psi$  has a fixed point. In order to apply the Nussbaum fixed point theorem for the operator  $\Psi$  we prove the following assertions:

- (i)  $\Psi_1$  and  $\Psi_2$  are well defined.
- (ii)  $\Psi_1$  satisfies Lipschitz condition.
- (iii)  $\Psi_2$  is relatively compact.
- (iv)  $(\Psi_1 + \Psi_2)Q \subseteq Q$ .

We can easily see that if  $z(\cdot) \in Q$ , then  $\hat{z}_t + y_t \in B_r[\phi]$  for all  $0 \leq t \leq b$ . From axioms (A1) and (B8) we see that

$$\begin{aligned}
|\hat{z}_t + y_t - \phi|_{\mathcal{B}} & \leq |\hat{z}_t|_{\mathcal{B}} + |y_t - \phi|_{\mathcal{B}} \\
& \leq K^* \rho + \epsilon \\
& \leq r - \epsilon + \epsilon = r.
\end{aligned}$$

Now, for  $0 \leq t \leq b$ ,

$$\begin{aligned} \|\Psi_1 z(t)\| &= \left\| -T(t)h(0, \phi) + h(t, \hat{z}_t + y_t) + \int_0^t AT(t-s)h(s, \hat{z}_s + y_s) ds \right\| \\ &\leq \|(-A)^{-\beta}T(t)\| \|(-A)^\beta h(t, y_t) - (-A)^\beta h(0, \phi)\| \\ &\quad + \|(I - T(t))h(t, y_t)\| \\ &\quad + \|(-A)^{-\beta}\| \|(-A)^\beta h(t, \hat{z}_t + y_t) - (-A)^\beta h(t, y_t)\| \\ &\quad + \int_0^t \|(-A)^{1-\beta}T(t-s)\| \|(-A)^\beta h(s, \hat{z}_s + y_s) ds\| \\ &\leq \|(-A)^{-\beta}\| ML(t + |y_t - \phi|_{\mathcal{B}}) + \epsilon + \|(-A)^{-\beta}\| L|\hat{z}_t|_{\mathcal{B}} + \int_0^t \frac{CM_1}{(t-s)^{1-\beta}} ds \\ &\leq \|(-A)^{-\beta}\| ML(b + \epsilon) + \epsilon + \|(-A)^{-\beta}\| LK^* \rho + \frac{CM_1}{\beta} b^\beta \end{aligned}$$

and

$$\begin{aligned} \|\Psi_2 z(t)\| &= \left\| \int_0^t T(t-\eta)B\tilde{W}^{-1}[x_1 - T(b)[\phi(0) - h(0, \phi)] - h(b, \hat{z}_b + y_b) \right. \\ &\quad - \int_0^b AT(b-s)h(s, \hat{z}_s + y_s) ds - \int_0^b T(b-s)f(s, \hat{z}_s + y_s) ds \\ &\quad - \int_0^b T(b-s)q(s) ds - \int_0^b \int_0^s T(b-s)g(s, \tau, \hat{z}_\tau + y_\tau) d\tau ds \left. \right] d\eta \\ &\quad + \int_0^t T(t-s)[f(s, \hat{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \hat{z}_\tau + y_\tau) d\tau] ds \left\| \right. \\ &\leq \left\| \int_0^t T(t-\eta)B\tilde{W}^{-1}[x_1 - T(b)\phi(0) \right. \\ &\quad + (-A)^{-\beta}T(b)[(-A)^\beta h(0, \phi) - (-A)^\beta h(b, y_b)] + (T(b) - I)h(b, y_b) \\ &\quad + (-A)^{-\beta}[(-A)^\beta h(b, y_b) - (-A)^\beta h(b, \hat{z}_b + y_b)] \\ &\quad + \int_0^b (-A)^{1-\beta}T(b-s)(-A)^\beta h(s, \hat{z}_s + y_s) ds \\ &\quad - \int_0^b T(b-s)[f(s, \hat{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \hat{z}_\tau + y_\tau) d\tau] ds \left. \right] d\eta \left\| \right. \\ &\quad + \left\| \int_0^t T(t-s)[f(s, \hat{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \hat{z}_\tau + y_\tau) d\tau] ds \right\| \\ &\leq \int_0^t \|T(t-\eta)\| \|B\| \|\tilde{W}^{-1}\| [\|x_1\| + \|T(b)\phi(0)\| \\ &\quad + \|(-A)^{-\beta}\| \|T(b)\| \|(-A)^\beta h(0, \phi) - (-A)^\beta h(b, y_b)\| \end{aligned}$$

$$\begin{aligned}
& + \|(-A)^{-\beta}\| \|(-A)^\beta h(b, y_b) - (-A)^\beta h(b, \hat{z}_b + y_b)\| \\
& + \int_0^b \|(-A)^{1-\beta} T(b-s)\| \|(-A)^\beta h(s, \hat{z}_s + y_s)\| ds \\
& + \|(T(b) - I)h(b, y_b)\| + \int_0^b \|T(b-s)\| \|f(s, \hat{z}_s + y_s)\| ds \\
& + \int_0^b \|T(b-s)\| [\|q(s)\| + \int_0^s \|g(s, \tau, \hat{z}_\tau + y_\tau)\| d\tau] ds d\eta \\
& + \int_0^t \|T(t-s)\| [\|f(s, \hat{z}_s + y_s)\| + \|q(s)\| + \int_0^s \|g(s, \tau, \hat{z}_\tau + y_\tau)\| d\tau] ds \\
\leq & bMK_1K_2[\|x_1\| + M\|\phi(0)\|] + \|(-A)^{-\beta}\| ML(b + |\phi - y_b|_{\mathcal{B}}) \\
& + \|(-A)^{-\beta}\| L\rho \max_{0 \leq t \leq b} K(t) + \int_0^b \frac{CM_1}{(b-s)^{1-\beta}} ds \\
& + \epsilon + bM(M_2 + N_2 + bN_1)] + bM(M_2 + N_2 + bN_1) \\
\leq & bMK_1K_2[\|x_1\| + M\|\phi(0)\|] + \|(-A)^{-\beta}\| ML(b + \epsilon) \\
& + \|(-A)^{-\beta}\| L\rho K^* + \frac{CM_1}{\beta} b^\beta + \epsilon + bM(M_2 + N_2 + bN_1)] \\
& + bM(M_2 + N_2 + bN_1).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|\Psi z(t)\| & \leq \|\Psi_1 z(t)\| + \|\Psi_2 z(t)\| \\
& \leq (1 + bMK_1K_2) \|(-A)^{-\beta}\| LK^*\rho + b[1 + bMK_1K_2] \|(-A)^{-\beta}\| ML \\
& \quad + MK_1K_2[\|x_1\| + M\|\phi(0)\| + bM(M_2 + N_2 + bN_1) + \frac{CM_1}{\beta} b^\beta] \\
& \quad + M(M_2 + N_2 + bN_1)] + (1 + bMK_1K_2) (\|(-A)^{-\beta}\| ML + 1)\epsilon + \frac{CM_1}{\beta} b^\beta \\
& \leq \omega\rho + (1 - \omega)\rho = \rho.
\end{aligned}$$

Hence  $\Psi(Q) \subset Q$ . Next we shall prove that the operator  $\Psi_1$  satisfies the Lipschitz condition. Let  $v, w \in Q$  and for each  $0 \leq t \leq b$  we have

$$\begin{aligned}
\|\Psi_1 v(t) - \Psi_1 w(t)\| & \leq \int_0^t \|(-A)^{1-\beta} T(t-s) (-A)^\beta [h(s, \hat{v}_s + y_s) - h(s, \hat{w}_s + y_s)]\| ds \\
& \quad + \|h(t, \hat{v}_t + y_t) - h(t, \hat{w}_t + y_t)\| \\
& \leq \int \frac{CL|\hat{v}_s - \hat{w}_s|_{\mathcal{B}}}{(t-s)^{1-\beta}} ds + \|(-A)^{-\beta}\| L|\hat{v}_t - \hat{w}_t|_{\mathcal{B}} \\
& \leq \max_{0 \leq t \leq b} K(t) L \left[ \frac{C}{\beta} b^\beta + \|(-A)^{-\beta}\| \right] \max_{0 \leq s \leq b} \|v(s) - w(s)\|
\end{aligned}$$



and so

$$\|\Psi_1 v - \Psi_1 w\| \leq L^* \|v - w\|.$$

Thus  $\Psi_1$  is Lipschitz continuous.

Finally, we prove that  $\Psi_2$  is relatively compact in  $Q$ . To prove this, first we shall show that  $\Psi_2$  maps  $Q$  into a precompact subset of  $Q$ . We now show that for every fixed  $t \in J$  the set  $Q(t) = \{\Psi_2 z(t) : z \in Q\}$  is precompact in  $X$ .

Obviously for  $t = 0$ ,  $Q(0) = \{\Psi_2(0)\}$ . Let  $t > 0$  be fixed and for  $0 < \epsilon < t$  define

$$\begin{aligned} \Psi_{2\epsilon} z(t) = & \int_0^{t-\epsilon} T(t-\eta) B \tilde{W}^{-1} [x_1 - T(b)[\phi(0) - h(0, \phi)] - h(b, \hat{z}_\tau + y_\tau) \\ & - \int_0^b AT(b-s)h(s, \hat{z}_s + y_s) ds - \int_0^b T(b-s)f(s, \hat{z}_s + y_s) ds \\ & - \int_0^b T(b-s)q(s) ds - \int_0^b \int_0^s T(b-s)g(s, \tau, \hat{z}_\tau + y_\tau) d\tau ds] d\eta \\ & + \int_0^{t-\epsilon} T(t-s)f(s, \hat{z}_s + y_s) ds + \int_0^{t-\epsilon} T(t-s)q(s) ds \\ & + \int_0^{t-\epsilon} \int_0^s T(t-s)g(s, \tau, \hat{z}_\tau + y_\tau) d\tau ds. \end{aligned}$$

Since  $AT(t)h(s, \hat{z}_s + y_s)$ ,  $T(t)f(s, \hat{z}_s + y_s)$ ,  $T(t)g(s, \tau, \hat{z}_\tau + y_\tau)$  and  $T(t)q(s)$  belong to the compact set  $V$ , the set

$$Q_\epsilon(t) = \{\Psi_{2\epsilon} z(t); x \in Q\}$$

is precompact in  $X$  for every  $\epsilon$ ,  $0 < \epsilon < t$ . Further for  $z \in Q$ , we have

$$\begin{aligned} & \|\Psi_2 z(t) - \Psi_{2\epsilon} z(t)\| \\ & \leq \left\| \int_{t-\epsilon}^t T(t-\eta) B \tilde{W}^{-1} [x_1 - T(b)[\phi(0) - h(0, \phi)] - h(b, \hat{z}_\tau + y_\tau) \right. \\ & \quad - \int_0^b AT(b-s)h(s, \hat{z}_s + y_s) ds - \int_0^b T(b-s)f(s, \hat{z}_s + y_s) ds \\ & \quad - \int_0^b T(b-s)q(s) ds - \int_0^b \int_0^s T(b-s)g(s, \tau, \hat{z}_\tau + y_\tau) d\tau ds] d\eta \left. \right\| \\ & \quad + \left\| \int_{t-\epsilon}^t T(t-s)[f(s, \hat{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \hat{z}_\tau + y_\tau) d\tau] ds \right\| \\ & \leq \int_{t-\epsilon}^t \|T(t-\eta)\| \|B\| \|\tilde{W}^{-1}\| [\|x_1\| + \|T(b)\phi(0)\| \\ & \quad + \|(-A)^{-\beta}\| \|T(b)\| \|(-A)^\beta h(0, \phi) - (-A)^\beta h(b, y_b)\| \\ & \quad + \|(T(b) - I)h(b, y_b)\| + \|(-A)^{-\beta}\| \|(-A)^\beta h(b, y_b) - (-A)^\beta h(b, \hat{z}_b + y_b)\| \end{aligned}$$

$$\begin{aligned}
& + \int_0^b \|(-A)^{1-\beta}T(b-s)\| \|(-A)^\beta h(s, \hat{z}_s + y_s)\| ds \\
& + \int_0^b \|T(b-s)\| \|f(s, \hat{z}_s + y_s)\| ds + \int_0^b \|T(b-s)\| \|q(s)\| ds \\
& + \int_0^b \int_0^s \|T(t-s)\| \|g(s, \tau, \hat{z}_\tau + y_\tau)\| d\tau ds] d\eta \\
& + \int_{t-\epsilon}^t \|T(t-s)\| \|f(s, \hat{z}_s + y_s)\| ds + \int_{t-\epsilon}^t \|T(t-s)\| \|q(s)\| ds \\
& + \int_{t-\epsilon}^t \int_0^s \|T(t-s)\| \|g(s, \tau, \hat{z}_\tau + y_\tau)\| d\tau ds \\
& \leq \epsilon MK_1 K_2 [\|x_1\| + M\|\phi(0)\| + \|(-A)^{-\beta}\| ML(b + |\phi - y_b|_B) \\
& + \|(T(b) - I)h(b, y_b)\| + \|(-A)^{-\beta}\| L\rho K^* + \int_0^b \frac{CM_1}{(b-s)^{(1-\beta)}} ds \\
& + bM(M_2 + N_2 + bN_1)] + \epsilon M(M_2 + N_2 + bN_1).
\end{aligned}$$

Since there are precompact sets arbitrarily close to the set  $Q(t)$ , it is totally bounded, that is, precompact in  $X$ . We now show that the image of  $Q$ ,

$$\Psi_2(Q) = \{\Psi_2 z : z \in Q\}$$

is an equicontinuous family of functions. Let  $0 < t_1 < t_2$ ,

$$\begin{aligned}
& \|\Psi_2 z(t_1) - \Psi_2 z(t_2)\| \\
& \leq \left\| \int_0^{t_1} [T(t_1 - \eta) - T(t_2 - \eta)] B\tilde{W}^{-1} [x_1 - T(b)[\phi(0) - h(0, \phi)] \right. \\
& \quad - h(b, \hat{z}_b + y_b) - \int_0^b AT(b-s)h(s, \hat{z}_s + y_s) ds - \int_0^b T(b-s)f(s, \hat{z}_s + y_s) ds \\
& \quad - \int_0^b T(b-s)q(s) ds - \int_0^b \int_0^s T(b-s)g(s, \tau, \hat{z}_\tau + y_\tau) d\tau ds] d\eta \\
& \quad - \int_{t_1}^{t_2} T(t_2 - \eta) B\tilde{W}^{-1} [x_1 - T(b)[\phi(0) - h(0, \phi)] - h(b, \hat{z}_b + y_b) \\
& \quad - \int_0^b AT(b-s)h(s, \hat{z}_s + y_s) ds - \int_0^b T(b-s)f(s, \hat{z}_s + y_s) ds \\
& \quad \left. - \int_0^b T(b-s)q(s) ds - \int_0^b \int_0^s T(b-s)g(s, \tau, \hat{z}_\tau + y_\tau) d\tau ds] d\eta \right\| \\
& + \left\| \int_0^{t_1} [T(t_1 - s) - T(t_2 - s)] [f(s, \hat{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \hat{z}_\tau + y_\tau) d\tau] ds \right. \\
& \quad \left. - \int_{t_1}^{t_2} T(t_2 - s) [f(s, \hat{z}_s + y_s) + q(s) + \int_0^s g(s, \tau, \hat{z}_\tau + y_\tau) d\tau] ds \right\|
\end{aligned}$$

$$\begin{aligned}
 &\leq \int_0^{t_1} \|[T(t_1 - \epsilon - \eta) - T(t_2 - \epsilon - \eta)]T(\epsilon)B\tilde{W}^{-1}[x_1 - T(b)[\phi(0) - h(0, \phi)] \\
 &\quad - h(b, \hat{z}_b + y_b) - \int_0^b AT(b-s)h(s, \hat{z}_s + y_s)ds - \int_0^b T(b-s)f(s, \hat{z}_s + y_s)ds \\
 &\quad - \int_0^b T(b-s)q(s)ds - \int_0^b \int_0^s T(b-s)g(s, \tau, \hat{z}_\tau + y_\tau)d\tau ds]d\eta\| \\
 &\quad + (t_1 - t_2)MK_1K_2[\|x_1\| + M\|\phi(0)\| + \|(-A)^{-\beta}\|ML(b + \epsilon) \\
 &\quad + \|(-A)^{-\beta}\|L\rho K^* + \frac{CM_1}{\beta}b^\beta + \epsilon + bM(M_2 + N_2 + bN_1)] \\
 &\quad + (t_1 - t_2)M(M_2 + N_2 + bN_1) \\
 &\quad + \int_0^{t_1} \|[T(t_1 - \epsilon - s) - T(t_2 - \epsilon - s)]T(\epsilon)[f(s, \hat{z}_s + y_s) + q(s) \\
 &\quad + \int_0^s g(s, \tau, \hat{z}_\tau + y_\tau)d\tau]\|ds + (t_1 - t_2)M(M_2 + N_2 + bN_1).
 \end{aligned}$$

Since  $h(s, \hat{z}_s + y_s)$  is continuous and  $T(\epsilon)f(s, \hat{z}_s + y_s)$ ,  $T(\epsilon)g(s, \tau, \hat{z}_\tau + y_\tau)$  and  $T(\epsilon)q(s)$  are included in the compact set  $V$  for all  $0 \leq s \leq b$  and all  $z \in Q$ , the functions  $T(\cdot)z$  for  $z \in V$  are equicontinuous. Hence  $\Psi_2(Q)$  is an equicontinuous family of functions. Also  $\Psi_2(Q)$  is bounded in  $Y$  and so by the Arzela-Ascoli theorem,  $\Psi_2(Q)$  is precompact. Hence it follows from the Nussbaum fixed point theorem, there exists a fixed point  $z \in Q$  such that  $\Psi z(t) = z(t)$ . Since we have  $x(t) = z(t) + y(t)$ , it follows that  $x(t)$  is a mild solution of (1) on  $[0, b]$  satisfying  $x(b) = x_1$ . Thus the system (1) is controllable on  $[0, b]$ .

#### 4. EXAMPLE

Consider the partial integrodifferential equation of the form

$$\begin{aligned}
 (3) \quad &\frac{\partial}{\partial t}[y(t, \xi) + \int_{-\infty}^t \int_0^\pi a(s-t, \eta, \xi)y(s, \eta)d\eta ds] \\
 &= \frac{\partial^2}{\partial \xi^2}y(t, \xi) + k_0(\xi)y(t, \xi) + \int_{-\infty}^t k(s-t)y(s, \xi)ds + \nu(t, \xi), \\
 &y(t, 0) = y(t, \pi) = 0, \quad t \geq 0, \\
 &y(\theta, \xi) = \phi(\theta, \xi), \quad \theta \leq 0, \quad 0 \leq \xi \leq \pi.
 \end{aligned}$$

Let us take  $X = U = L^2[0, \pi]$ ,  $x(t) = y(t, \cdot)$  and  $u(t) = \nu(y, \cdot)$  where  $\nu : J \times [0, \pi] \rightarrow [0, \pi]$  is continuous. Define the operator  $A$  by

$$Az(\xi) = z''(\xi)$$

with domain

$$D(A) = \{z(\cdot) \in L^2([0, \pi]); z'' \in L^2([0, \pi]), z(0) = z(\pi) = 0\}.$$

It is clear that  $A$  generates a strongly continuous semigroup  $T(t)$  which is compact, analytic and self adjoint. Further,  $A$  has discrete spectrum, the eigenvalues are  $-n^2, n \in N$ , with corresponding normalized eigenvectors  $y_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi)$ .

We shall consider the following properties:

- (i)  $\{y_n : n \in N\}$  is an orthonormal basis of  $X$ .
- (ii) If  $z \in D(A)$  then  $Az = -\sum_{n=1}^{\infty} n^2 \langle z, y_n \rangle y_n$ .
- (iii) For every  $z \in X$ ,  $T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} \langle z, y_n \rangle y_n$ . In particular,  $T(t)$  is a uniformly stable semigroup and  $\|T(t)\| \leq e^{-t}$ .
- (iv) For every  $z \in X$ ,  $(-A)^{-\frac{1}{2}}z = \sum_{n=1}^{\infty} \frac{1}{n} \langle z, y_n \rangle y_n$  and  $\|(-A)^{-\frac{1}{2}}\| = 1$ .
- (v) The operator  $(-A)^{\frac{1}{2}}$  is given by

$$(-A)^{\frac{1}{2}}z = \sum_{n=1}^{\infty} n \langle z, y_n \rangle y_n$$

on the space  $D((-A)^{\frac{1}{2}}) = \{z(\cdot) \in X : \sum_{n=1}^{\infty} n \langle z, y_n \rangle y_n \in X\}$ .

Let  $\mathcal{B}$  denote the space  $C_r \times L^2(g, X)$  with  $r = 0$  as defined in [6]. It is clear that  $\mathcal{B}$  is isomorphic and isometric to the space  $X \times L^2_{\mu}((-\infty, 0] \times [0, \pi])$  where  $\mu$  is the measure  $\mu(\theta, \tau) = g(\theta)d\theta d\tau$ . Next we assume that the following conditions hold:

- (a) The function  $a(\cdot)$  is measurable and

$$\int_0^{\pi} \int_{-\infty}^0 \int_0^{\pi} (a^2(\theta, \eta, \xi)/g(\theta)) d\eta d\theta d\xi < \infty.$$

- (b) The function  $\frac{\partial}{\partial \psi} a(\theta, \eta, \psi)$  is measurable;  $a(\theta, \eta, \pi) = 0$ ;  $a(\theta, \eta, 0) = 0$ , and

$$N_1^* = \int_0^{\pi} \int_{-\infty}^0 \int_0^{\pi} \frac{1}{g(\theta)} \left( \frac{\partial}{\partial \psi} a(\theta, \eta, \psi) \right)^2 d\eta d\theta d\psi < 1.$$

- (c) The function  $k_0 \in L^{\infty}([0, \pi])$ .

- (d) The function  $k(\cdot)$  is measurable and

$$\int_{-\infty}^0 \frac{k^2(\theta)}{g(\theta)} d\theta \leq \infty.$$

(e) The function  $\phi$  defined by  $\phi(\theta)(\xi) = \phi(\theta, \xi)$  belongs to  $\mathcal{B}$ .

Under these conditions we define  $h : [0, \infty) \times \mathcal{B} \rightarrow X$ ,  $f : [0, \infty) \times \mathcal{B} \rightarrow X$  and  $g : [0, \infty) \times [0, \infty) \times \mathcal{B} \rightarrow X$  by  $h(t, \phi) = P_1(\phi)$ ,  $f(t, \phi) = P_2(\phi)$  and  $g(t, s, \phi) = P_3(\phi)$  where

$$(4) \quad P_1(\phi) = \int_{-\infty}^t \int_0^\pi a(\theta, \eta, \xi) \phi(\theta, \eta) d\eta d\theta,$$

$$(5) \quad P_2(\phi) = k_0(\xi) \phi(0, \xi)$$

$$(6) \quad P_3(\phi) = \int_{-\infty}^0 k(\theta) \phi(\theta, \xi) d\theta.$$

From (a) and (c) it is clear that  $P_1, P_2$  and  $P_3$  are bounded linear operators on  $\mathcal{B}$ . Further,  $P_1(\phi) \in D((-A)^{\frac{1}{2}})$  and  $\|(-A)^{\frac{1}{2}} P_1\| \leq N_1^* < 1$ . Also from (4) and (b) it follows that

$$\langle P_1(\phi), y_n \rangle = \frac{1}{n} \sqrt{\frac{2}{\pi}} \langle P(\phi), \cos(n\psi) \rangle,$$

where  $P$  is defined by

$$P(\phi) = \int_{-\infty}^0 \int_0^\pi \frac{\partial}{\partial \psi} a(\theta, \eta, \psi) \phi(\theta, \eta) d\eta d\theta.$$

From (b) it is clear that  $P : \mathcal{B} \rightarrow X$  is a bounded linear operator with  $\|P\| \leq N_1$ . Assume that there exists an invertible operator  $\tilde{W}$  defined on  $L^2(J, U)/\ker W$  by

$$Wu = \int_0^b T(b-s)u(s)ds$$

and satisfies the condition (B7).

With this choice of  $A, h, f, g$  and  $B = I$ , the identity operator we see that the equation (3) is an abstract formulation of (1). Also all the conditions stated in the theorem are satisfied. Hence the system (3) is controllable on  $J$ .

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