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SOME APPLICATIONS OF GENERALIZED FRACTIONAL CALCULUS OPERATORS TO A NOVEL CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. In the present paper, by making use of certain operators of generalized fractional calculus, we introduce a novel class $\mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$ of functions which are analytic and univalent in the open unit disk $\mathcal{U}$. A necessary and sufficient condition for a function to be in the class $\mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$ is obtained. In addition, this paper includes distortion theorems involving generalized fractional integrals (and generalized fractional derivatives), radii of close-to-convexity, starlikeness, and convexity. Relevance with some new (or known) special cases are also pointed out.

1. Introduction and Definitions

Let $\mathcal{T}(n)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0; n \in \mathcal{N} = \{1,2,3,\cdots\}), \tag{1.1}$$

which are analytic and univalent in the unit open disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by $\mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$ the subclass of functions $f(z) \in \mathcal{T}(n)$ which also satisfy the inequality:

$$\left| \frac{z J_{0,z}^{1+\mu,1+\varphi,1+\eta}\{f(z)\} + \lambda z^2 J_{0,z}^{2+\mu,2+\varphi,2+\eta}\{f(z)\}}{(1-\lambda) J_{0,z}^{\mu,\varphi,\eta}\{f(z)\} + \lambda z J_{0,z}^{1+\mu,1+\varphi,1+\eta}\{f(z)\}} - (1-\varphi) \right| < \alpha, \tag{1.2}$$

$$(z \in \mathcal{U}; n \in \mathcal{N}; 0 < \alpha \leq 1; 0 \leq \mu < 1; \varphi, \eta \in \mathbb{R}; \varphi < 1; \eta > max\{\mu, \varphi\} - 2).$$

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Here and throughout this paper, $J_{0,z}^{\mu,\varphi,\eta}$ denotes an operator of fractional calculus, which is defined as follows (cf., [2] and [4]):

Definition 1. The fractional integral of order μ of a function $f(z)$ is defined by

$$D_z^{-\mu}\{f(z)\} = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\mu}} d\xi, \quad (\mu > 0), \tag{1.3}$$

where $f(z)$ is analytic in a simply- connected region of the $z-$ plane containing the origin, and the multiplicity of $(z-\xi)^{\mu-1}$ is removed by requiring $log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 2. The fractional derivative of order μ of a function $f(z)$ is defined by

$$D_z^{\mu}\{f(z)\} = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\mu}} d\xi, \quad (0 \leq \mu < 1), \tag{1.4}$$

where $f(z)$ it is choosen as in (1.3), it and the multiplicity of $(z-\xi)^{-\mu}$ is removed, as in Definition 1.

Definition 3. Let $\mu > 0$ and $\eta,\ \beta \in \mathbb{R}$. Then, in terms of familiar (Gauss's) hypergeometric function ${}_2F_1$, the generalized fractional integral operator $I_{0,z}^{\mu,\beta,\eta}$ of a function $f(z)$ is defined by

$$I_{0,z}^{\mu,\beta,\eta}\{f(z)\} = \frac{z^{-\mu-\beta}}{\Gamma(\mu)} \int_0^z (z-\xi)^{\mu-1} f(\xi) \cdot {}_2F_1\left(\mu+\beta, -\eta; 1-\frac{\xi}{z}\right) d\xi, \tag{1.5}$$

where the function $f(z)$ is analytic in a simply-connected region of the $z-$ plane containing the origin, with the order

$$f(z) = O(|z|^{\epsilon}), \quad (z \to 0), \tag{1.6}$$

for

$$\epsilon > \max\{0, \beta-\eta\} - 1, \tag{1.7}$$

and the multiplicty of $(z-\xi)^{\mu-1}$ is removed by requiring $log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 4. Let $0 \leq \mu < 1$ and $\eta,\ \beta \in \mathbb{R}$. Then, the generalized fractional derivative operator $J_{0,z}^{\mu,\beta,\eta}$ of a function $f(z)$ is defined by

$$\begin{aligned} J_{0,z}^{\mu,\beta,\eta}\{f(z)\} &= \frac{1}{\Gamma(1-\mu)} \\ &\cdot \frac{d}{dz}\left\{ z^{\mu-\beta} \int_0^z (z-\xi)^{-\mu} f(\xi) \cdot {}_2F_1\left(\beta-\mu, 1-\eta; 1-\mu; 1-\frac{\xi}{z}\right) d\xi \right\}, \end{aligned} \tag{1.8}$$

where the function $f(z)$ is analytic in a simply-connected region of the $z-$ plane containing the origin, with the order as given by (1.6), and the multiplicity of $(z-\xi)^{-\mu}$ is removed by requiring $log(z-\xi)$ to be real when $z-\xi>0$.

By comparing (1.3) with (1.5), we obtain the following relationship:

$$I_{0,z}^{\mu,-\mu,\eta}\{f(z)\}=D_z^{-\mu}\{f(z)\} \quad (\mu>0). \tag{1.9}$$

Similarly, by comparing (1.4) with (1.8), we find

$$J_{0,z}^{\mu,\mu,\eta}\{f(z)\}=D_z^{\mu}\{f(z)\} \quad (0\leq\mu<1). \tag{1.10}$$

From the general class $\mathcal{T}_\lambda^{\mu,\varphi,\eta}(n;\alpha)$ defined by (1.2), we obtain the following important subclasses:

$$V^{\mu,\varphi,\eta}(n;\alpha)=\mathcal{T}_0^{\mu,\varphi,\eta}(n;\alpha) \quad (n\in\mathcal{N};0\leq\mu<1;0<\alpha\leq1), \tag{1.11}$$

$$W^{\mu,\varphi,\eta}(n;\alpha)=\mathcal{T}_1^{\mu,\varphi,\eta}(n;\alpha) \quad (n\in\mathcal{N};0\leq\mu<1;0<\alpha\leq1), \tag{1.12}$$

$$\Omega_\mu(n;\alpha)=V^{\mu,\mu,\eta}(n;\alpha) \quad (n\in\mathcal{N};0\leq\mu<1;0<\alpha\leq1), \tag{1.13}$$

$$\triangle_\mu(n;\alpha)=W^{\mu,\mu,\eta}(n;\alpha) \quad (n\in\mathcal{N};0\leq\mu<1;0<\alpha\leq1), \tag{1.14}$$

$$\mathcal{S}_n(\alpha)=\Omega_0(n;\alpha) \quad (n\in\mathcal{N};0<\alpha\leq1), \tag{1.15}$$

$$\mathcal{C}_n(\alpha)=\triangle_0(n;\alpha) \quad (n\in\mathcal{N};0<\alpha\leq1), \tag{1.16}$$

$$\mathcal{S}^*(\alpha)=S_1(\alpha) \quad (0<\alpha\leq1), \tag{1.17}$$

$$\mathcal{C}^*(\alpha)=C_1(\alpha) \quad (0<\alpha\leq1). \tag{1.18}$$

The classes $\mathcal{S}_n(\alpha)$ and $\mathcal{S}^*(\alpha)$ consist of starlike functions, of order $1-\alpha$ $(0\leq\alpha<1)$ and the classes $\mathcal{C}_n(\alpha)$ and $\mathcal{C}^*(\alpha)$ consist of convex functions, of order $1-\alpha$ $(0\leq\alpha<1)$. These classes are of much importance in the Geometric Function Theory (cf., [9]). Some other interesting papers involving fractional calculus operators are the ones by Altintaş *et al.* ([1], [2]), Chen *et al.* ([3], [4]), Irmak [6], and Raina and Srivastava [7].

2. Some Properties of the Class $\mathcal{T}_\lambda^{\mu,\varphi,\eta}(n;\alpha)$

We begin by establishing a necessary and sufficient condition for a function $f(z)\in\mathcal{T}(n)$ to be in the class $\mathcal{T}_\lambda^{\mu,\varphi,\eta}(n;\alpha)$. This result is contained in the following:

Theorem 1. *Let a function $f(z) \in \mathcal{T}(n)$. Then, the function $f(z)$ belongs to the class $\mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{(k+\alpha-1)\Gamma(k+1)\Gamma(k-\varphi+\eta+1)[1+\lambda(k-\varphi-1)]}{\Gamma(k-\varphi+1)\Gamma(k-\mu+\eta+1)} a_k \leq \frac{\alpha(1-\lambda\varphi)\Gamma(2-\varphi+\eta)}{\Gamma(2-\varphi)\Gamma(2-\mu+\eta)}, \tag{2.1}$$

$$(n \in \mathcal{N}; 0<\alpha\leq 1; 0\leq \mu<1; \varphi,\eta\in\mathbb{R}; \varphi<1; \eta>max\{\mu,\varphi\}-2).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{\alpha(1-\lambda\varphi)\Gamma(2-\varphi+\eta)\Gamma(n-\varphi+2)}{(n+\alpha)[1+\lambda(n-\varphi)]\Gamma(2-\varphi)\Gamma(2-\mu+\eta)} \cdot \frac{\Gamma(n-\mu+\eta+2)}{\Gamma(n+2)\Gamma(n-\varphi+\eta+2)} z^{n+1}, \quad (n\in\mathcal{N}). \tag{2.2}$$

Proof. Suppose that the function $f(z)$ is defined by (1.1), and the inequality (2.1) holds true. Then, invoking [9, p. 15, Eq. (2.2)], we have

$$\begin{aligned}
&|zJ_{0,z}^{1+\mu,1+\varphi,1+\eta}\{f(z)\} + \lambda z^2 J_{0,z}^{2+\mu,2+\varphi,2+\eta}\{f(z)\} \\
&\quad -(1-\varphi)\{(1-\lambda)J_{0,z}^{\mu,\varphi,\eta}\{f(z)\} + \lambda z J_{0,z}^{1+\mu,1+\varphi,1+\eta}\}| \\
&\quad -\alpha\left|(1-\lambda)J_{0,z}^{\mu,\varphi,\eta}\{f(z)\} + \lambda z J_{0,z}^{1+\mu,1+\varphi,1+\eta}\{f(z)\}\right| \\
&= \left|\sum_{k=n+1}^{\infty} \tfrac{(k-1)[1+\lambda(k-\varphi-1)]\Gamma(k+1)\Gamma(k-\varphi+\eta+1)}{\Gamma(k-\varphi+1)\Gamma(k-\mu+\eta+1)} a_k z^{k-1}\right| \\
&\quad -\alpha\left|\tfrac{(1-\lambda\varphi)\Gamma(2-\varphi+\eta)}{\Gamma(2-\varphi)\Gamma(k-\mu+\eta)} - \sum_{k=n+1}^{\infty} \tfrac{[1+\lambda(k-\varphi-1)]\Gamma(k+1)\Gamma(k-\varphi+\eta+1)}{\Gamma(k-\varphi+1)\Gamma(k-\mu+\eta+1)} a_k z^{k-1}\right| \\
&\leq \sum_{k=n+1}^{\infty} \tfrac{(k+\alpha-1)[1+\lambda(k-\varphi-1)]\Gamma(k+1)\Gamma(k-\varphi+\eta+1)}{\Gamma(k-\varphi+1)\Gamma(k-\mu+\eta+1)} a_k - \tfrac{\alpha(1-\lambda\varphi)\Gamma(2-\varphi+\eta)}{\Gamma(2-\varphi)\Gamma(k-\mu+\eta)} \\
&\leq 0 \;\; (n\in\mathcal{N}; 0\leq\alpha\leq 1; 0\leq\mu<1; \varphi,\eta\in\mathbb{R}; \varphi<1; \eta>max\{\mu,\varphi\}-2).
\end{aligned}$$

Hence, by maximum modulus theorem, the function $f(z)$ defined by (1.1) belongs to the class $\mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$.

In order to prove the converse, we assume that the function $f(z) \in \mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$. Then

$$\left|\frac{zJ_{0,z}^{1+\mu,1+\varphi,1+\eta}\{f(z)\} + \lambda z^2 J_{0,z}^{2+\mu,2+\varphi,2+\eta}\{f(z)\}}{(1-\lambda)J_{0,z}^{\mu,\varphi,\eta}\{f(z)\} + \lambda z J_{0,z}^{1+\mu,1+\varphi,1+\eta}\{f(z)\}} - (1-\varphi)\right|$$

$$(2.3) = \left|\frac{\sum_{k=n+1}^{\infty} \frac{(k-1)[1+\lambda(k-\varphi-1)]\Gamma(k+1)\Gamma(k-\varphi+\eta+1)}{\Gamma(k-\varphi+1)\Gamma(k-\mu+\eta+1)} a_k z^{k-1}}{\frac{[1+\lambda(k-\varphi-1)]\Gamma(k-\varphi+\eta+1)}{\Gamma(k-\varphi+1)\Gamma(k-\mu+\eta)} - \sum_{k=n+1}^{\infty} \frac{(k-1)[1+\lambda(k-\varphi-1)]\Gamma(k+1)\Gamma(k-\varphi+\eta+1)}{\Gamma(k-\varphi+1)\Gamma(k-\mu+\eta+1)} a_k z^{k-1}}\right| < \alpha.$$

Since $|\Re(z)| \leq |z|$ for any z , choosing z to be real and letting $z \to$ 1- through real values, (2.3) yields

$$\sum_{k=n+1}^{\infty} \frac{(k-1)[1+\lambda(k-\varphi-1)]\Gamma(k+1)\Gamma(k-\varphi+\eta+1)}{\Gamma(k-\varphi+1)\Gamma(k-\mu+\eta+1)} a_k$$
$$\leq \alpha \left\{ \frac{(1-\lambda\varphi)\Gamma(2-\varphi+\eta)}{\Gamma(2-\varphi)\Gamma(k-\mu+\eta)} - \sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-\varphi-1)]\Gamma(k+1)\Gamma(k-\varphi+\eta+1)}{\Gamma(k-\varphi+1)\Gamma(k-\mu+\eta+1)} a_k \right\}$$

which is the desired assertion (2.1) of Theorem 1.

Finally, by observing that the function $f(z)$ given by (2.2) is indeed an extremal function for the assertion (2.1), we complete the proof of Theorem 1.

Corollary 1.1. *Let a function $f(z) \in \mathcal{T}(n)$. Then, the function $f(z)$ belongs to the class $V^{\mu,\varphi,\eta}(n;\alpha)$ if and only if*

$$(2.4) \qquad \sum_{k=n+1}^{\infty} \frac{k!(k+\alpha-1)\Gamma(k-\varphi+\eta+1)}{\Gamma(k-\varphi+1)\Gamma(k-\mu+\eta+1)} a_k \leq \frac{\alpha\Gamma(2-\varphi+\eta)}{\Gamma(2-\varphi)\Gamma(2-\mu+\eta)},$$

$$(n \in \mathcal{N}; 0 \leq \alpha \leq 1; 0 \leq \mu < 1; \varphi, \eta \in \mathbb{R}; \varphi < 2; \eta > max\{\mu, \varphi\} - 2).$$

Corollary 1.2. *Let a function $f(z) \in \mathcal{T}(n)$. Then, the function $f(z)$ belongs to the class $W^{\mu,\varphi,\eta}(n;\alpha)$ if and only if*

$$(2.5) \qquad \sum_{k=n+1}^{\infty} \frac{k!(k+\alpha-1)\Gamma(k-\varphi+\eta+1)}{\Gamma(k-\varphi)\Gamma(k-\mu+\eta+1)} a_k \leq \frac{\alpha\Gamma(2-\varphi+\eta)}{\Gamma(1-\varphi)\Gamma(2-\mu+\eta)},$$

$$(n \in \mathcal{N}; 0 \leq \alpha \leq 1; 0 \leq \mu < 1; \varphi, \eta \in \mathbb{R}; \varphi < 1; \eta > max\{\mu, \varphi\} - 2).$$

Corollary 1.3. *Let a function $f(z) \in \mathcal{T}(n)$. Then, the function $f(z)$ belongs to the class $\Omega_{\mu}(n;\alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{k!(k+\alpha-1)}{\Gamma(k-\mu+1)} a_k \leq \frac{\alpha}{\Gamma(2-\mu)}, \quad (n \in \mathcal{N}; 0 < \alpha \leq 1; 0 \leq \mu < 1). \tag{2.6}$$

Corollary 1.4. *Let a function $f(z) \in \mathcal{T}(n)$. Then, the function $f(z)$ belongs to the class $\triangle_{\mu}(n;\alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{k!(k+\alpha-1)}{\Gamma(k-\mu)} a_k \leq \frac{\alpha}{\Gamma(1-\mu)}, \quad (n \in \mathcal{N}; 0 < \alpha \leq 1; 0 \leq \mu < 1). \tag{2.7}$$

Corollary 1.5 [cf., [1] and [10]]. *Let a function $f(z) \in \mathcal{T}(n)$. Then, the function $f(z)$ belongs to the class $\mathcal{S}_n(\alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty} (k+\alpha-1) a_k \leq \alpha, \quad (n \in \mathcal{N}; 0 < \alpha \leq 1). \tag{2.8}$$

Corollary 1.6 [cf., [1] and [10]]. *Let a function $f(z) \in \mathcal{T}(n)$. Then, the function $f(z)$ belongs to the class $\mathcal{C}_n(\alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty} k(k+\alpha-1) a_k \leq \alpha, \quad (n \in \mathcal{N}; 0 < \alpha \leq 1). \tag{2.9}$$

We next prove two distortion inequalities (Theorems 2 and 3 below) involving the fractional operators $I_{0,z}^{\mu,\beta,\eta}$ and $J_{0,z}^{\mu,\beta,\eta}$, respectively.

Theorem 2. *Let $\beta \in \mathbb{R}_+$ and $\gamma, \eta \in \mathbb{R}$ such that $\eta > max\{-\beta, \gamma\} - 2$. If n is a positive integer such that*

$$n \geq \frac{\gamma(\beta+\eta)}{\beta} - 2 \tag{2.10}$$

and, if $f(z) \in \mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$, then

$$\begin{aligned} &\left| \left| I_{0,z}^{\beta,\gamma,\eta}\{f(z)\} \right| - \frac{\Gamma(2-\gamma+\eta)}{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)} |z|^{1-\gamma} \right| \\ &\leq \frac{\alpha(1-\lambda\varphi)\Gamma(2-\varphi+\eta)\Gamma(n-\varphi+2)\Gamma(n-\mu+\eta+2)}{(n+\alpha)\Gamma(2-\varphi)\Gamma(2-\mu+\eta)\Gamma(n+\beta+\eta+2)\Gamma(n-\gamma+2)} \\ &\cdot \frac{\Gamma(n-\gamma+\eta+2)}{\Gamma(n-\varphi+\eta+2)[1+\lambda(n-\varphi)]} |z|^{n-\gamma+1} \end{aligned} \tag{2.11}$$

for $z \in \mathcal{U}$ if $\gamma \leq 1$ and *$z \in \mathcal{D}$ if $\gamma > 1$. The result* (2.11) *is sharp for the function $f(z)$ given by* (2.2).

Proof. Let $f(z) \in \mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$. It follows from the inequality (2.1) that

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\alpha(1-\lambda\varphi)\Gamma(2-\varphi+\eta)\Gamma(n-\varphi+2)}{(n+\alpha)[1+\lambda(n-\varphi)]\Gamma(2-\varphi)\Gamma(2-\mu+\eta)} \cdot \frac{\Gamma(n-\mu+\eta+2)}{\Gamma(n+2)\Gamma(n-\varphi+\eta+2)}. \tag{2.12}$$

From (1.1), (1.8), and a known result due to Srivastava et al. [10, p. 415, Eq. (2.3)], we have

$$I_{0,z}^{\beta,\gamma,\eta}\{f(z)\} = \frac{\Gamma(2-\gamma+\eta)}{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)} z^{1-\gamma} - \sum_{k=n+1}^{\infty} \Psi(k) a_k z^{k-\gamma}, \tag{2.13}$$

where

$$\Psi(k) = \frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)}, \quad (k \geq n+1; n \in \mathcal{N}).$$

The function $\Psi(z)$ is non-increasing for integers k $(k \geq n+1; n \in \mathcal{N})$, under the hypotheses of Theorem 2 including the constraint (2.10). Therefore, we have

$$0 < \Psi(k) \leq \Psi(n+1) = \frac{\Gamma(n+2)\Gamma(n-\gamma+\eta+2)}{\Gamma(n-\gamma+2)\Gamma(n+\beta+\eta+2)}, \quad (n \in \mathcal{N}). \tag{2.14}$$

The the desired assertion (2.11) of the theorem follows on combining the two inequalities which straigthforwardly emerge from (2.13) upon suitably using (2.12) and (2.14) in the process.

Theorem 3. *Let* $0 \leq \beta < 1$ *and* $\gamma, \eta \in \mathbb{R}$ *such that* $\gamma < 2,\ \eta > max\{\beta, \gamma\} - 2$. *If* n *is a positive integer such that*

$$n \geq \frac{\gamma(\beta-\eta)}{\beta} - 2, \tag{2.15}$$

and, if $f(z) \in \mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$, *then*

$$\left| \left| J_{0,z}^{\beta,\gamma,\eta}\{f(z)\} \right| - \frac{\Gamma(2-\gamma+\eta)}{\Gamma(2-\gamma)\Gamma(2-\beta+\eta)} |z|^{1-\gamma} \right| \leq \frac{\alpha(1-\lambda\varphi)\Gamma(2-\varphi+\eta)\Gamma(n-\varphi+2)\Gamma(n-\mu+\eta+2)}{(n+\alpha)\Gamma(2-\varphi)\Gamma(2-\mu+\eta)\Gamma(n-\beta+\eta+2)\Gamma(n-\gamma+2)} \cdot \frac{\Gamma(n-\gamma+\eta+2)}{\Gamma(n-\varphi+\eta+2)[1+\lambda(n-\varphi)]} |z|^{n-\gamma+1} \tag{2.16}$$

for $z \in \mathcal{U}$ if $\gamma \leq 1$ and $z \in \mathcal{D}$ if $\gamma > 1$. *The result is sharp for the function* $f(z)$ *given by* (2.2).

Proof. Under the hypotheses of Theorem 3, we have from (1.1) and a known result due to Raina and Srivastava [7, p. 15, Eq. (2.2)]:

$$J_{0,z}^{\beta,\gamma,\eta}\{f(z)\} = \frac{\Gamma(2-\gamma+\eta)}{\Gamma(2-\gamma)\Gamma(2-\beta+\eta)} z^{1-\gamma} - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k-\beta+\eta+1)} z^{k-\gamma}. \tag{2.17}$$

Making use of (2.17), and the following steps similar to those given in the proof of Theorem 2, the assertion (2.16) of Theorem 3 is easily arrived at.

A number of simpler distortion properties can easily be deduced from Theorems 2 and 3 by suitably specializing the parameters γ, φ, λ, α, and n in the equations (2.11) and (2.16). For $\gamma = -\beta$ in Theorem 2 and $\gamma = \beta$ in Theorem 3, we get following corollaries.

Corollary 1.7. *If* $f(z) \in \mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$, *then*

$$\left| \left| D_z^{-\beta}\{f(z)\} \right| - \frac{1}{\Gamma(2+\beta)} |z|^{1+\beta} \right| \leq \frac{\alpha(1-\lambda\varphi)\Gamma(2-\varphi+\eta)\Gamma(n-\varphi+2)}{(n+\alpha)\Gamma(2-\varphi)\Gamma(2-\mu+\eta)} \cdot \frac{\Gamma(n-\mu+\eta+2)}{\Gamma(n+\beta+2)\Gamma(n-\varphi+\eta+2)[1+\lambda(n-\varphi)]} |z|^{n+\beta+1} \tag{2.18}$$

for all β $(\beta > 0)$, $z \in U$ *and* $n \in \mathcal{N}$.

Corollary 1.8. *If* $f(z) \in \mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$, *then*

$$\left| \left| D_z^{\beta}\{f(z)\} \right| - \frac{1}{\Gamma(2-\beta)} |z|^{1-\beta} \right| \leq \frac{\alpha(1-\lambda\varphi)\Gamma(2-\varphi+\eta)\Gamma(n-\varphi+2)}{(n+\alpha)\Gamma(2-\varphi)\Gamma(2-\mu+\eta)} \cdot \frac{\Gamma(n-\mu+\eta+2)}{\Gamma(n-\beta+2)\Gamma(n-\varphi+\eta+2)[1+\lambda(n-\varphi)]} |z|^{n-\beta+1} \tag{2.19}$$

for all β $(0 \leq \beta < 1)$, $z \in \mathcal{U}$ *and* $n \in \mathcal{N}$.
Each of these results in (2.18) *and* (2.19) *is sharp for the function* $f(z)$ given by

$$f(z) = z - \frac{\alpha(1-\lambda\varphi)\Gamma(2-\varphi+\eta)\Gamma(n-\varphi+2)}{(n+\alpha)[1+\lambda(n-\varphi)]\Gamma(2-\varphi)\Gamma(2-\mu+\eta)} \cdot \frac{\Gamma(n-\mu+\eta+2)}{\Gamma(n+2)\Gamma(n-\varphi+\eta+2)} z^{n+1}, \ (n \in \mathcal{N}). \tag{2.20}$$

Finally, Theorem 4 below gives the radii of starlikeness, convexity, and close-to-convexity of functions $f(z)$ belonging to the class $\mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$. These results are proven by appropriately using the inequalities (1.2)-(1.4) in the forms:

$$(2.21) \qquad \left|\frac{zf'(z)}{f(z)}-1\right|<1-\delta, \quad (0\le\delta<1; 0<|z|<\Gamma_1),$$

$$(2.22) \qquad \left|\frac{zf''(z)}{f'(z)}\right|<1-\delta, \quad (0\le\delta<1; 0<|z|<\Gamma_2),$$

$$(2.23) \qquad |f'(z)-1|<1-\delta, \quad (0\le\delta<1; 0<|z|<\Gamma_3),$$

respectively, and we omit the details involved, $r_1, r_2,$ and r_3 being defined by (2.25)-(2.27).

Theorem 4. *If* $f(z)\in\mathcal{T}_{\lambda}^{\mu,\varphi,\eta}(n;\alpha)$. *Then, the function* $f(z)$ *is*

(i) *starlike of order* $\delta(0\le\delta<1)$ in $0<|z|<r_1,$

(ii) *convex of order* $\delta(0\le\delta<1)$ in $0<|z|<r_2,$

(iii) *close-to-convex of order* $\delta(0\le\delta<1)$ *in* $0<|z|<r_3,$

where

$$(2.25) \quad r_1=r_1(n;\alpha;\delta;\lambda;\mu;\varphi;\eta)=\inf_{k\ge n+1}\left[(1-\delta)\cdot\Omega(k,n,\lambda,\alpha,\mu,\eta,\varphi)\right]^{\frac{1}{k-1}},$$

$$(2.26) \quad r_2=r_2(n;\alpha;\delta;\lambda;\mu;\varphi;\eta)=\inf_{k\ge n+1}\left[\frac{k(1-\delta)}{k-\delta}\cdot\Omega(k,n,\lambda,\alpha,\mu,\eta,\varphi)\right]^{\frac{1}{k-1}},$$

$$(2.27) \quad r_3=r_3(n;\alpha;\delta;\lambda;\mu;\varphi;\eta)=\inf_{k\ge n+1}\left[\frac{1-\delta}{k-\delta}\cdot\Omega(k,n,\lambda,\alpha,\mu,\eta,\varphi)\right]^{\frac{1}{k-1}},$$

$for\ all\ n\in\mathcal{N},\ when$

$$\Omega(k,n,\lambda,\alpha,\mu,\eta,\varphi)=\frac{[1+\lambda(n-\varphi)]\Gamma(k)\Gamma(2-\varphi)\Gamma(2-\mu+\eta)\Gamma(2-\varphi+\eta)}{\alpha(1-\lambda\varphi)\Gamma(2-\varphi+\eta)\Gamma(k-\mu+\eta+1)}.$$

We conclude by remarking that several results of interest giving the coefficient bounds, distortion inequalities, radii of close-to-convexity, starlikeness and convexity of functions which belong to various subclasses of $\mathcal{T}(n)$ can be obtained by suitable choices of the involved parameters from the results presented in this paper. We skip further details in this regard.

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