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CONFORMALLY FLAT HYPERSURFACES IN REAL SPACE FORMS WITH LEAST TENSION

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Abstract. Roughly speaking, an ideal immersion of a Riemannian manifold into a real space form is an isometric immersion which receives the least possible amount of tension imposed on the submanifold from the ambient space. The purpose of this paper is to classify conformally flat ideal hypersurfaces in real space forms; thus we completely determine conformally flat manifolds which admit codimension one isometric immersions into real space forms with least possible tension.

1. INTRODUCTION

The theorema which asserts the invariance of the Gauss curvature under isometric deformations of surfaces M^2 in the Euclidean world E^3 is egregium, as C. F. Gauss labeled it in his general theory of curved surfaces ([18], 1827). Its impact on the development of mathematics has been equally egregium indeed. Immediately, this theorem lead to the distinction between the intrinsic and the extrinsic qualities of such surfaces. Later, the awareness of the existence of an intrinsic geometry of surfaces M^2 in E^3, resulted in the creation of global analysis; extending classical analysis from space R^n to differentiable manifolds M^n; and of general differential geometry as the study of such manifolds endowed in addition with a geometrical structure; most notable a Riemannian metric tensor, as the simplest such structure discussed by B. Riemann in his habilitation lecture (1814). The anticipation of this work was likely Gauss's motivation to be so happy with his theorema egregium. As expressed by S. S. Chern [13], the Riemannian geometry forms the modern differential geometry.

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Once Riemannian spaces were around, the differential geometry of surfaces M^2 in E^3 was generalized to submanifolds M^n of Riemannian manifolds. In this general theory of Riemannian submanifolds, the relations between intrinsic and extrinsic qualities constitute a central theme of study (see [9]).

The crucial characteristics of Riemannian spaces are their curvatures (see [1, 22]). Besides the sectional curvatures and the Ricci curvatures, the scalar curvature has been the most studied invariants on Riemannian manifolds. Among the beautiful results are those linking these local metric invariants with the global-topological nature of the differentiable manifolds.

The abstract mathematical theories of differentiable manifolds and Riemannian manifolds proved their relevance for understanding better many diverse types of experiences related to real world situations in exact, medical and human sciences. More direct through the celebrated theorems of H. Whitney (1935) and J. F. Nash (1954) showing the realizability as submanifolds of the differentiable manifolds in standard manifold E^m. Nash's theorem was aimed for, in particular, in the hope that it would made possible to derive new Riemannian results, taking profit from the fact that if so that Riemannian manifolds could always be considered as submanifolds of Euclidean spaces, this would then yield the opportunity to use extrinsic help. Till when observed as such by M. Gromov [19], this hope had not been materialized however.

Another observation concerning Nash's theorem was expressed by S. T. Yau [23] who pointed to the lack of controls of the extrinsic properties by the known intrinsic data and, related to this, to the difficulties in having criteria to determine nice ways, amongst all possible ones, to shape an actual realization of a given Riemannian manifold in some Euclidean space. In this line of thought, S. S. Chern formulated already in [12] the specific question for new intrinsic obstructions for the existence of minimal immersibility of a Riemannian manifold into a Euclidean space besides positivity of the Ricci curvatures.

The first author gave a first answer to Chern's question in 1993 [5] and many more afterwards [7, 8], in terms of the new scalar valued curvature invariants $\delta(n_1, \dots, n_k)$ on a Riemannian n-manifold M. In the context of Nash's theorem, the first author established for all these invariants optimal pointwise inequalities involving the squared mean curvature. These inequalities give rise to new results on intrinsic spectral properties of homogeneous spaces obtained via extrinsic data [8] which extend a well-known result of T. Nagano [21]. These and subsequently obtained inequalities give prima controls on the most important extrinsic curvature(s), namely, the mean curvature (and the scalar normal curvatures), by the initial intrinsic curvatures of M [7, 8, 14, 15]. These give rise to criteria to naturally consider the best one(s) among all possible realizations of a given Riemannian manifold as submanifolds. In particular, one gets new views on rigidity of submanifolds as expressed in the notion of *ideal immersions.* In physical language: ideal immersions of M are isometric immersions giving M such a shape that at each point the surface tension on this submanifold assumes the least possible value as is determined by the intrinsic curvature characteristics $\delta(n_1,\dots,n_k)$ of M. In variational terms: every ideal immersion is a stable critical point of the total squared mean curvature functional within the class of all isometric immersions.

The main new notions briefly mentioned above are recalled in Section 2. In Section 3 we review some special functions and special families of Riemannian manifolds for later use. In Section 4 we define some special families of Riemannian manifolds. Rotation hypersurfaces in real space forms in the sense of [3] are briefly explained in Section 5. In Section 6 we prove some basic properties of conformally flat ideal hypersurfaces. The complete classification of conformally flat ideal hypersurfaces in real space forms are obtained in the last three sections.

This paper once more illustrates the close ties between special functions of classical analysis and explicit formulas describing basic geometrical objects.

2. Riemannian Invariants, Inequalities and Ideal Immersions

Let M be a Riemannian n-manifold. Denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $e_1,\dots,e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{2.1}$$

When L is a 1-dimensional subspace of T_pM, we put $\tau(L) = 0$. If L is a subspace of T_pM of dimension $r \geq 2$, we define the scalar curvature $\tau(L)$ of L by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha,\beta \leq r, \tag{2.2}$$

where $\{e_1,\dots,e_r\}$ is an orthonormal basis of L.

For an integer $k \geq 0$, denote by $\mathcal{S}(n,k)$ the finite set consisting of unordered k-tuples $(n_1,\dots,n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Let $\mathcal{S}(n)$ is the union $\cup_{k\geq 0}\mathcal{S}(n,k)$. If $n = 2$, we have $k = 0$ and $\mathcal{S}(2) = \{\emptyset\}$.

For each $(n_1,\dots,n_k) \in \mathcal{S}(n)$, the invariant $\delta(n_1,\dots,n_k)$ is defined by [7, 8]:

$$\delta(n_1,\dots,n_k)(p) = \tau(p) - S(n_1,\dots,n_k)(p), \tag{2.3}$$

where

$$S(n_1,\dots,n_k)(p) = \inf\left\{\tau(L_1) + \cdots + \tau(L_k)\right\}$$

and $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \dots, k$. Clearly, the invariant $\delta(\emptyset)$ is nothing but the scalar curvature τ of M.

For a given $(n_1, \dots, n_k) \in \mathcal{S}(n)$, we put

$$b(n_1, \dots, n_k) = \frac{1}{2}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1) \right), \tag{2.4}$$

$$c(n_1, \dots, n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)}. \tag{2.5}$$

For each real number ϵ and each $(n_1, \dots, n_k) \in \mathcal{S}(n)$, the associated normalized invariant $\Delta_\epsilon(n_1, \dots, n_k)$ is defined by

$$\Delta_\epsilon(n_1, \dots, n_k) = \frac{\delta(n_1, \dots, n_k) - b(n_1, \dots, n_k)\epsilon}{c(n_1, \dots, n_k)}. \tag{2.6}$$

We recall the following general result from [7, 8].

Theorem 2.1. *Let M be an n-dimensional submanifold of a real space form $R^m(\epsilon)$ of constant sectional curvature ϵ. Then for each $(n_1, \dots, n_k) \in \mathcal{S}(n)$ we have*

$$H^2 \geq \Delta_\epsilon(n_1, \dots, n_k), \tag{2.7}$$

where H^2 is the squared norm of the mean curvature vector.

The equality case of inequality (2.7) *holds at a point $p \in M$ if and only if, for each normal vector ξ at p, there exists an orthonormal basis $e_1, \dots, e_n$ at p, such that the shape operator A_ξ of M in $R^m(\epsilon)$ with respect to $e_1, \dots, e_n$ takes the following form:*

$$A_\xi = \begin{pmatrix} A_1^\xi & & & 0 \\ & \ddots & & \\ & & A_k^\xi & \\ 0 & & & \mu_\xi I \end{pmatrix} \tag{2.8}$$

where $\{A_j^\xi\}_{j=1}^k$ are symmetric $n_j \times n_j$ submatrices satisfy

$$\operatorname{trace}(A_1^\xi) = \cdots = \operatorname{trace}(A_k^\xi) = \mu_\xi. \tag{2.9}$$

For an isometric immersion $x : M \to R^m(\epsilon)$ of a Riemannian n-manifold into $R^m(\epsilon)$, this theorem implies that

$$H^2(p) \geq \hat{\Delta}_\epsilon(p), \tag{2.10}$$

where $\hat{¢}_\epsilon$ denotes the invariant on M defined by

$$\hat{¢}_\epsilon = \max\left\{ ¢_\epsilon(n_1, \ldots, n_k) \,|\, (n_1, \ldots, n_k) \in \mathcal{S}(n) \right\}. \tag{2.11}$$

Theorem 2.1 implies the the following result for hypersurfaces.

Theorem 2.2. *Let M be a hypersurface of a real space form $R^{n+1}(\epsilon)$ $(n \geq 2)$ of constant sectional curvature ϵ. Then we have*

$$H^2 \geq \hat{¢}_\epsilon. \tag{2.12}$$

The equality case of inequality (2.12) *holds at a point $p \in M$ if and only if there is a k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ such that, up to a suitable permutation of the principal curvatures $\kappa_1, \ldots, \kappa_n$ of M at p, we have*

$$\begin{aligned} &\kappa_1 + \ldots + \kappa_{n_1} = \kappa_{n_1+1} + \ldots + \kappa_{n_1+n_2} = \cdots = \\ &= \kappa_{n_1+\ldots+n_{k-1}+1} + \ldots + \kappa_{n+1+\ldots+n_k} = \kappa_{n+1+\ldots+n_k+1} = \cdots = \kappa_n. \end{aligned} \tag{2.13}$$

Remark 2.1. If $m = 0$, (2.13) reduces to $\kappa_1 = \cdots = \kappa_n$, i.e., M is totallly umbilical.

In general, there do not exist direct relations between these new invariants. On the other hand, Theorem 2.1 gives rise to the following maximum principle.

A Maximum Principle. Let M be an n-dimensional submanifold of $R^m(\epsilon)$. If it satisfies the equality case of (2.7) for some $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, then

$$¢_\epsilon(n_1, \ldots, n_k) \geq ¢_\epsilon(m_1, \ldots, m_j) \tag{2.14}$$

for every $(m_1, \ldots, m_j) \in \mathcal{S}(n)$.

This maximum principle yields the following important fact.

Fact. If an isometric immersion $x : M \to R^m(\epsilon)$ satisfies the equality case of (2.7) for a k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, then it is an ideal immersion automatically.

Applying inequality (2.10) the first author introduced in [7, 8] the notion of ideal immersions as follows.

Definition 2.1. An isometric immersion $x : M \to R^m(\epsilon)$ is called an *ideal immersion* if the equality case of (2.10) holds at every point $p \in M$. An isometric immersion is called $(n_1, \ldots, n_k)$-*ideal* if it satisfies $H^2 = ¢_\epsilon(n_1, \ldots, n_k)$ identically for $(n_1, \ldots, n_k) \in \mathcal{S}(n)$.

Physical Interpretation of Ideal Immersions. An isometric immersion $x : M \to R^m(\epsilon)$ is ideal means that M receives the least possible amount of tension (given by $\hat{¢}_\epsilon(p)$) at each point $p \in M$ from the ambient space. This is due to (2.10) and the well-known fact that the mean curvature vector field is exactly the tension field for isometric immersions. Therefore, the squared mean curvature $H^2(p)$ at a point $p \in M$ simply measures the amount of tension M receiving from the ambient space $R^m(\epsilon)$ at that point.

The following result provides a direct relationship between the first nonzero eigenvalue λ_1 of Laplacian and the invariants $\delta_0(n_1, \ldots, n_k)$.

Theorem 2.3. ([8]) *If M is a compact homogeneous Riemannian n-manifold with irreducible isotropy action, then the first nonzero eigenvalue λ_1 of the Laplacian on M satisfies*

$$\lambda_1 \geq n\, \delta(n_1, \ldots, n_k) \tag{2.15}$$

for every $(n_1, \ldots, n_k) \in \mathcal{S}(n)$. Therefore, we have

$$\lambda_1 \geq n\, \hat{\delta}_0. \tag{2.16}$$

The equality sign of (2.16) holds if and only if M admits a 1-type ideal immersion in a Euclidean space.

When $k = 0$, inequality (2.15) reduces to a well-known result of T. Nagano [21], namely, $\lambda_1 \geq n\rho$, where $\rho = 2\tau/n(n-1)$ is the normalized scalar curvature. Strict inequality $\hat{\delta}_0 > \rho$ holds for most Riemannian manifolds.

3. Some Special Functions

First, we review briefly some known facts on Jacobi's elliptic functions, theta function, zeta function and hypergeometric function for later use (for details, see, for instances, [2, 20]).

Let θ be the temperature at time t at any point in a solid material whose conducting properties are uniform and isotropic. If ρ is the material's density, s its specific heat, and k its thermal conductivity, θ satisfies the heat conduction equation: $\kappa\nabla^2\theta = \partial\theta/\partial t$, where $\kappa = k/s\rho$ is the diffusivity. In the special case where there is no variation of temperature in the x and y-directions, the heat flow is everywhere parallel to the z-axis and the heat equation reduced to

$$\kappa\frac{\partial^2\theta}{\partial z^2} = \frac{\partial\theta}{\partial t}, \quad \theta = \theta(z,t). \tag{3.1}$$

Consider the boundary conditions: $\theta(0,t) = \theta(\pi,t) = 0$ and $\theta(z,0) = \pi\delta(z - \pi/2)$ for $0 < z < \pi$, where $\delta(z)$ is Dirac's unit impulse function. Then the solution of the boundary value problem is given by

$$\theta(z,t) = 2\sum_{n=0} (-1)^n e^{-(2n+1)^2\kappa t}\sin(2n+1)z \tag{3.2}$$

By writing $e^{i\,4\kappa t} = q$, the solution (3.2) assume the form:

$$\theta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)z, \tag{3.3}$$

which is the first of the four theta functions. When the precise value of q is not important, we shall suppress the dependence upon q.

If one changes the boundary conditions to $\partial\theta/\partial z = 0$ on $z = 0$ and on $z = \pi$ with $\theta(z,0) = \pi\delta(z-\pi/2)$ for $0 < z < \pi$, then the corresponding solution of the boundary value problem of the heat equation (3.1) is given by

$$\theta_4(z) = \theta_4(z,q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz. \tag{3.4}$$

The theta function $\theta_1(z)$ is periodic with period 2π. Incrementing z by $\frac{1}{2}\pi$ yields the second theta function:

$$\theta_2(z) = \theta_2(z,q) = \theta_1(z+\tfrac{1}{2}\pi,q) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos(2n+1)z. \tag{3.5}$$

Similarly, incrementing z by $\frac{1}{2}\pi$ for θ_4 yields the third theta function:

$$\theta_3(z) = \theta_3(z,q) = \theta_4(z+\tfrac{1}{2}\pi,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz. \tag{3.6}$$

The four theta functions $\theta_1,\theta_2,\theta_3,\theta_4$ can be extended to complex values for z and q such that $|q| < 1$.

The elliptic functions $\mathrm{sn}\,u, \mathrm{cn}\,u$ and $\mathrm{dn}\,u$ are defined as ratios of theta functions:

$$\mathrm{sn}\,u = \frac{\theta_3(0)\theta_1(z)}{\theta_2(0)\theta_4(z)},\quad \mathrm{cn}\,u = \frac{\theta_4(0)\theta_2(z)}{\theta_2(0)\theta_4(z)},\quad \mathrm{dn}\,u = \frac{\theta_4(0)\theta_3(z)}{\theta_3(0)\theta_4(z)}, \tag{3.7}$$

where $u = \theta_3^2(0)z$. Define parameters k and k' by

$$k = \frac{\theta_2^2(0)}{\theta_3^2(0)},\quad k' = \frac{\theta_4^2(0)}{\theta_3^2(0)}$$

which are called the *modulus* and the *complementary modulus* of the elliptic functions; k and k' satisfy $k^2 + k'^2 = 1$. When it is required to state the modulus explicitly, the elliptic functions of Jacobi will be written $\mathrm{sn}(u,k), \mathrm{cn}(u,k), \mathrm{dn}(u,k)$.

The elliptic functions $\mathrm{sn}\,u, \mathrm{cn}\,u$ and $\mathrm{dn}\,u$ satisfy the following relations:

$$\begin{cases} \mathrm{sn}^2u + \mathrm{cn}^2u = 1, & \mathrm{dn}^2u + k^2\mathrm{sn}^2u = 1, \quad k^2\mathrm{cn}^2u + k'^2 = \mathrm{dn}^2u, \\ \mathrm{nd}^2u - 1 = k^2\mathrm{sd}^2u, & 1 - \mathrm{cd}^2u = k'^2\mathrm{sd}^2(u), \quad \mathrm{sc}^2u + 1 = \mathrm{nc}^2u, \\ \mathrm{cs}^2u + k'^2 = \mathrm{ds}^2u = \mathrm{ns}^2u - k^2, & k'^2\mathrm{nc}^2u = \mathrm{dc}^2u - k^2, \end{cases} \tag{3.8}$$

$$
\left\{
\begin{array}{ll}
\mathrm{sn}'(u) = \mathrm{cn}(u)\mathrm{dn}(u), & \mathrm{cn}'(u) = -\mathrm{sn}(u)\mathrm{dn}(u), \\
\mathrm{dn}'(u) = -k^2\mathrm{sn}(u)\mathrm{cn}(u), & \mathrm{cd}'(u) = -k'^2\mathrm{sd}(u)\mathrm{nd}(u), \\
\mathrm{sd}'(u) = \mathrm{cd}(u)\mathrm{nd}(u), & \mathrm{nd}'(u) = k^2\mathrm{sd}(u)\mathrm{cd}(u), \\
\mathrm{ns}'(u) = -\mathrm{ds}(u)\mathrm{cs}(u), & \mathrm{ds}'(u) = -\mathrm{cs}(u)\mathrm{ns}(u), \\
\mathrm{cs}'(u) = -\mathrm{ns}(u)\mathrm{ds}(u), & \mathrm{nc}'(u) = \mathrm{sc}(u)\mathrm{dc}(u).
\end{array}
\right.
\tag{3.9}
$$

The *theta function* $\Theta(u)$ and the *zeta function* $Z(u)$ are defined by

$$\Theta(u) = \theta_4\left(\frac{\pi u}{2K}\right), \quad Z(u) = \frac{d}{du}(\ln\theta_4), \quad K = \frac{\pi}{2}\theta_3^2(0). \tag{3.10}$$

Put

$$u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{3.11}$$

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{3.12}$$

where we first suppose that x and k satisfy $0 < k < 1,\ -1 \leq x \leq 1$. The number K is known as the quarter-period of the Jacobi's elliptic functions.

Equation (3.11) defines u as an odd function of x which is positive, increasing from 0 to K as x increases from 0 to 1. Inversely, the same equation defines x as an odd function of u which increases from 0 to 1 as u increase from 0 to K; this function is nothing but the Jacobi elliptic function $\mathrm{sn}(u,k)$, so we have

$$u = \mathrm{sn}^{-1}(x), \quad x = \mathrm{sn}(u). \tag{3.13}$$

Now, we review briefly the hypergeometric function. The second order differential equation:

$$z(z-1)\frac{d^2u}{dz^2} + \{c - (a+b+1)z\}\frac{du}{dz} - abu = 0 \tag{3.14}$$

is called the *hypergeometric equation*, where a, b and c are the parameters. Its solutions are called the *hypergeometric functions*. Many elementary and special functions can be obtained from the hypergeometric functions by choosing appropriate values for their parameters.

Equation (3.14) has three singular points at $z = 0, z = 1$ and $z = \infty$, respectively. Its solutions in the neighborhood of these three points can be obtained in the form of power series by using the method of Frobenius.

In the neighborhood of the point $z = 0$, a solution of (3.14) is given by

$$F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)}\frac{\Gamma(b+k)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+k)}\frac{z^k}{n!}, \tag{3.15}$$

where Γ is the Gamma function and the region of convergence is $|z| < 1$. A second independent solution of (3.14) is given by $z^{1-c}F(a+1-c, b+1-c, 2-c, z)$. The hypergeometric function $F(a,b,c,z)$ can also be expressed as

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt. \tag{3.16}$$

For simplicity we define the following special function:

$$B_{c,\delta}(a,\alpha,x) = \tag{3.17}$$

$$\begin{cases}
\displaystyle\int_0^x \frac{at^\alpha dt}{(\delta - ct^2)\sqrt{\delta - ct^2 - a^2t^{2\alpha}}} & \text{if } \alpha > 2,\ \delta = 0,\ c < 0, \\
\displaystyle\int_x^{\left(\frac{\sqrt{-c}}{a}\right)^{\frac{1}{\alpha-1}}} \frac{-at^\alpha dt}{(\delta - ct^2)\sqrt{\delta - ct^2 - a^2t^{2\alpha}}} & \text{if } 1 < \alpha \le 2,\ \delta = 0,\ c < 0, \\
\displaystyle\int_x^1 \frac{-at^\alpha dt}{(\delta - ct^2)\sqrt{\delta - ct^2 - a^2t^{2\alpha}}} & \text{if } \alpha < 1,\ \delta = 0,\ c < 0, \\
\displaystyle\int_0^x \frac{at^\alpha dt}{(\delta - ct^2)\sqrt{\delta - ct^2 - a^2t^{2\alpha}}} & \text{if } \alpha > 0,\ \delta > 0 \text{ and } c \text{ arbitrary}, \\
\displaystyle\int_x^{(\sqrt{\delta}/a)^{1/\alpha}} \frac{-at^\alpha dt}{(\delta - ct^2)\sqrt{\delta - ct^2 - a^2t^{2\alpha}}} & \text{if } -1 < \alpha < 0,\ \delta > 0,\ c = 0, \\
\displaystyle\int_x^1 \frac{-at^\alpha dt}{(\delta - ct^2)\sqrt{\delta - ct^2 - a^2t^{2\alpha}}} & \text{if } \alpha < -1,\ \delta > 0,\ c = 0, \\
\displaystyle\int_x^1 \frac{-at^\alpha dt}{(\delta - ct^2)\sqrt{\delta - ct^2 - a^2t^{2\alpha}}} & \text{if } \alpha < 1,\ \delta \neq 0,\ c < 0, \\
\displaystyle\int_\beta^x \frac{at^\alpha dt}{(\delta - ct^2)\sqrt{\delta - ct^2 - a^2t^{2\alpha}}} & \left\{\begin{array}{l}\text{if } \alpha > 1;\ c, \delta < 0;\ \beta \text{ is the smallest} \\ \text{positive root of } \delta - ct^2 - a^2t^{2\alpha} \\ \text{whenever it exists,}\end{array}\right. \\
\displaystyle\int_\gamma^x \frac{at^\alpha dt}{(\delta - ct^2)\sqrt{\delta - ct^2 - a^2t^{2\alpha}}} & \left\{\begin{array}{l}\text{if } \alpha < 0;\ c, \delta > 0;\ \gamma \text{ is the smallest} \\ \text{positive root of } \delta t^{-2\alpha} - ct^{2-2\alpha} - a^2 \\ \text{whenever it exists.}\end{array}\right.
\end{cases}$$

By making the substitution $t = 1/u$, one obtain from (3.17) the following relationship:

$$B_{c,\delta}(a,\alpha,x) = -B_{-\delta,-c}\left(a, 1-\alpha, \frac{1}{x}\right). \tag{3.18}$$

Moreover, we have the following relations between the special functions $B_{c,\delta}$ and hypergeometric functions:

$$B_{0,\delta}(a,\alpha,x) = \frac{ax^{\alpha+1}}{(\alpha+1)\delta^{3/2}}\, F\left(\frac{1}{2}, \frac{\alpha+1}{2\alpha}, \frac{3\alpha+1}{2\alpha}, \frac{a^2}{\delta}x^{2\alpha}\right), \quad \delta > 0 \text{ and } \alpha > 0 \text{ or } \alpha < -1, \tag{3.19}$$

$$B_{-c,0}(a,\alpha,x) = \frac{ax^{\alpha-2}}{(2-\alpha)c^{3/2}}\, F\left(\frac{1}{2}, \frac{\alpha-2}{2\alpha-2}, \frac{3\alpha-4}{2\alpha-2}, \frac{a^2}{c}x^{2\alpha-2}\right), \quad c > 0 \text{ and } \alpha > 2 \text{ or } \alpha < 1. \tag{3.20}$$

4. Some Special Families of Riemannian Manifolds

We recall the following model for the hyperbolic space. Let $H^{n+1}(c)$, $c < 0$, be the hypersurface of $\mathbb{R}^{n+2}$ given by

$$H^{n+1}(c) = \left\{x \in \mathbb{R}^{n+2} \mid -x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+2}^2 = c^{-1},\ x_1 > 0\right\}.$$

If we endow $H^{n+1}(c)$ with the Riemannian metric induced by the Lorentzian metric $ds^2 = -dx_1^2 + dx_2^2 + \cdots + dx_{n+2}^2$ on $\mathbb{R}^{n+2}$, then $H^{n+1}(c)$ has constant negative curvature c. A totally umbilical hypersurfaces of $H^{n+1}(c)$ is given by the intersection of $H^{n+1}(c)$ with an affine hyperplane L. The hypersurface is totally geodesic if L goes through the origin. If L does not pass through the origin, then the hypersurface is hyperbolic, parabolic or elliptic if the angle between L and e_1 is smaller than, equal to, or greater than $\pi/4$.

Now, let $S^{n+1}(c)$, $c > 0$, be the hypersphere of curvature c in Euclidean $(n+2)$-space $\mathbb{E}^{n+2}$, centered at the origin, *i.e.*,

$$S^{n+1}(c) = \left\{x \in \mathbb{E}^{n+2} \mid x_1^2 + \cdots + x_{n+1}^2 + x_{n+2}^2 = c^{-1}\right\}.$$

Then any totally umbilical hypersurface of $S^{n+1}(c)$ is given by the intersection of $S^{n+1}(c)$ and an affine hyperplane L and it is totally geodesic if L goes through the origin.

We denote by $R^{n+1}(c)$ the complete, simply-connected Riemannian $(n+1)$-manifold of constant curvature c. So, we have $R^{n+1}(c) = S^{n+1}(c), \mathbb{E}^{n+1}$ or $H^{n+1}(c)$ according to $c > 0, c = 0$ or $c < 0$. Denote by g_1, g_0 and g_{-1} the standard metrics on $S^{n-1}(1)$, $\mathbb{E}^n$ and $H^{n-1}(-1)$, respectively.

For each $a > 1$ we denote by C_a^n (respectively, by P_a^n) the warped product n-manifold $I \times_{f(x)} S^{n-1}(1)$ with $f = (2ak/(a^2+1))\mathrm{cn}(ax,k)$, $k = \sqrt{a^2+1}/\sqrt{2}a$ (respectively, $f = (2ak/(a^2+1))\mathrm{cn}(ax,k)$, $k = \sqrt{a^2-1}/\sqrt{2}a$). Also for each a with $0 < a < 1$, we denote by D_a^n the warped product manifolds $\mathbb{R} \times_f H^{n-1}(-1)$ with $f = (2a/(1-a^2)k)\,\mathrm{dn}(ax/k,k)$, $k = \sqrt{2}a/\sqrt{a^2+1}$; by L^n the warped product n-manifold $\mathbb{R} \times_{\mathrm{sech}(x)} \mathbb{R}^{n-1}$; and by F_α^n the n-manifold $\mathbb{R} \times H^{n-1}(-1)$

with metric $g = ds^2 + \frac{\alpha}{\alpha-1} g_{-1}$ for $\alpha > 1$

The D^n_a, F^n, L^n are complete Riemannian n-manifolds, but P^n_a and C^n_a are not complete. Topologically, S^n is the two point compactification of both P^n_a and C^n_a. The Riemannian metrics defined on P^n_a and C^n_a can be extended smoothly to their two point compactification S^n. We denote by $\hat{P}^n_a$ and $\hat{C}^n_a$ the S^n together with the Riemannian metrics given by the extensions of the metrics on P^n_a and C^n_a to S^n, respectively (see [6, 10] for details).

Let $A^n_a\ (a > 1), B^n_a\ (0 < a < 1), G^n, H^n_a\ (a > 0)$ and $Y^n_a\ (0 < a < 1)$ denote the warped product manifolds:

$$\mathbb{R} \times_{\sqrt{a^2-1}\cosh x} S^{n-1}(1), \quad \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times_{\sqrt{1-a^2}\cos x} S^{n-1}(1),$$

$$\mathbb{R} \times_{\cosh x} \mathbb{E}^{n-1}, \quad \mathbb{R} \times_{\sqrt{a^2+1}\cosh x} H^{n-1}(-1), \quad (0, \infty) \times_{\sqrt{1-a^2}\sinh x} S^{n-1}(1),$$

respectively. A^n_a, G^n, H^n_a are complete Riemannian manifolds. S^n is the two point compactification of B^n_a. Similar to P^n_a and C^n_a, the warped metric on B^a_n can be extended smoothly to its two point compactification. Let $\hat{B}^n_a$ denote the S^n together with the Riemannian metrics on S^n extended from the metric on B^n_a (see [11] for details).

Beside the families defined above, we also need the following families of special Riemannian manifolds.

We denote by E^n_a $(0 < a < 1)$ the n-manifold $\mathbb{R} \times S^{n-1}$ equipped with the metric $g = (a^2 k'^2/k^2)\mathrm{nd}^2(ax/k, k)(dt^2 + g_1)$, $k = \sqrt{2}a/\sqrt{1+a^2}$, $k' = \sqrt{1-a^2}/\sqrt{1+a^2}$; by Q^n_a $(a > 1)$ the $I \times S^{n-1}$ with $g = a^2 k'^2\, \mathrm{nc}^2(ax, k)(dt^2 + g_1)$, $k = \sqrt{a^2+1}/\sqrt{2}a$ and $k' = \sqrt{a^2-1}/\sqrt{2}a$; by R^n_a $(a > 1)$ the n-manifold $I \times H^{n-1}$ equipped with $g = a^2 k'^2\, \mathrm{nc}^2(ax, k)(dt^2 + g_{-1})$, $k = \sqrt{a^2-1}/\sqrt{2}a$, $k' = \sqrt{a^2+1}/\sqrt{2}a$; by S^n_a $(a > 0)$ the n-manifold $I \times \mathbb{R}^{n-1}$ with $g = 2a^2\, \mathrm{ds}^2(\sqrt{2}ax, 1/\sqrt{2})\ (dt^2 + g_0)$; by K^n_a $(a > 0)$ the n-manifold $\mathbb{R} \times S^{n-1}$ with $g = \cosh^2(t/a)\ (dt^2 + a^2 g_1)$; by O^n the $\mathbb{R} \times S^{n-1}$ with $g = ds^2 + e^{2s} g_1$; and by I^n_a the n-manifold $\mathbb{R} \times S^{n-1}(1)$ with $g = ds^2 + \frac{a}{a-1} g_1$ for $a < 0$. Also, for any $a > 0$, we denote by $N^n_{a\alpha}$ the n-manifold $(-a^{-1/\alpha}, a^{-1/\alpha}) \times S^{n-1}$ equipped with $g = (1 - a^2 t^{2\alpha})^{-1} dt^2 + t^2 g_1$; by $J^n_{a\alpha}$ the n-manifold $I \times S^{n-1}$ with $g = (1 - t^2 - a^2 t^{2\alpha})^{-1} dt^2 + t^2 g_1$; by $U^n_{a\alpha}$ the n-manifold $I \times \mathbb{R}^{n-1}$ with $g = (1 + t^2 - a^2 t^{2\alpha})^{-1} dt^2 + t^2 g_1$; by $V^n_{a\alpha}$ the n-manifold $I \times H^{n-1}$ with $g = (t^2 - 1 - a^2 t^{2\alpha})^{-1} dt^2 + t^2 g_{-1}$. Finally, denote by $Z^n_{a\alpha}$ the n-manifold $I \times \mathbb{R}^{n-1}$ with $g = = \{(2\alpha - 2)^2 t^2 (1 - a^2 t^{(2\alpha-1)/(2\alpha-2)})\}^{-1} dt^2 + t^2 g_0$.

In the above the open intervals I's are the maximal open intervals on which the corresponding metrics g are Riemannian.

5. Rotation Hypersurfaces in Real Space Forms

We recall what is a rotation hypersurface of a real space form $R^{n+1}(c)$ with $c \neq 0$ following [3]. We always consider $R^{n+1}(c), c \neq 0$ as a hypersurface in

(R^{n+2}, ds^2) as above. Let P^3 be a 3-dimensional linear subspace linear space of E^{n+2} that intersects $R^{n+1}(c)$. We denote the intersection by $R^2(c)$, if $c < 0$ we take only the upper part. Let P^2 be any linear subspace in P^3. We recall that any isometry of $R^{n+1}(c)$ is the restriction to $R^{n+1}(c)$ of an orthogonal transformation of (R^{n+2}, ds^2), and conversely. Let $O(P^2)$ be the group of orthogonal transformations (with positive determinant) that leave P^2 pointwise fixed. We take any curve γ in $R^2(c)$ which does not intersect P^2. The orbit of γ under $O(P^2)$ is called the *rotation hypersurface* with profile curve γ and axis P^2. The orbit of $\gamma(s)$ for a fixed s is a sphere, and if $c < 0$, then this sphere is elliptic, hyperbolic, or parabolic according to P^2 respectively being Lorentzian, Riemannian, or degenerate.

In order to give a parametrization of a rotation hypersurface of the different types, we introduce the vector $\mathsf{u} \in P^3$ such that P^2 coincides with $\mathsf{u}^\perp = \{\mathsf{v} \in P^3 \mid \langle \mathsf{v}, \mathsf{u}\rangle = 0\}$. We can always assume that u has length 1, -1, or 0, according to P^2 respectively being Lorentzian, Riemannian, or degenerate, and that $\langle \mathsf{u}, \gamma' \rangle > 0$. Let $\delta = \langle \mathsf{u}, \mathsf{u}\rangle$. We define the map Q as the orthogonal projection of P^3 on $\mathsf{u}^\perp$ if $\delta \neq 0$ and as the identity map of P^3 if $\delta = 0$. Further, let P^{n-1} be the orthogonal complement of P^3 in R^{n+2} and let P^n be the linear space, spanned by P^{n-1} and u. If $\delta = 1$ (resp. $\delta = -1$), then P^n is Riemannian (resp. Lorentzian) and we can define a mapping ϕ of $R^{n-1}(\delta)$ into P^n by considering $R^{n-1}(\delta)$ as a unit hypersphere in P^n. If $\delta = 0$, then we can define a mapping ϕ of $R^{n-1}(0)$ into P^n by identifying $R^{n-1}(0)$ and P^{n-1} and defining

$$\phi(p) = p - \frac{1}{2}\langle p, p\rangle\, \mathsf{u}. \tag{5.1}$$

Then a parametrization of the rotation hypersurface of γ around the axis P^2 is

$$f(s,p) = Q(\gamma(s)) + \langle \gamma(s), \mathsf{u}\rangle\, \phi(p). \tag{5.2}$$

6. Conformally Flat Ideal Hypersurfaces in Real Space Forms

The main purpose of this section is to prove the following basic properties of conformally flat ideal hypersurfaces in real space forms.

Proposition 6.1. *Let M be a conformally flat ideal hypersurface of a real space form $R^{n+1}(c)$ $(n \geq 2)$. Then M is a quasi-umbilical hypersurface such that, up to permutations, the principal curvatures are given by*

$$\kappa_1 = \alpha\mu, \quad \kappa_2 = \cdots = \kappa_n = \mu \tag{6.1}$$

at each point $p \in M$, where α is an integer given by one of the following:

1. $\alpha = 0$ *or* 1 *for* $n \geq 3$ *and* $\alpha = 1\,(\alpha \neq 0)$ *when* $n = 2$*;*
2. $\alpha = 2$ *and* $n + 1$ *is not a prime;*
3. $n \geq 4$ *and* α *is a positive integer satisfying*

$$\alpha = ms - n + 1 \quad \text{and} \quad n - 1 \geq \alpha \geq 3 \tag{6.2}$$

for some positive integers s, m *with* $m \geq 2$ *and* $(m-1)s < n$*;*

4. $n \geq 4$ *and* α *is a negative integer* $\geq 3 - n$*.*

Proof. Assume that M is a conformally flat ideal hypersurface of a real space form $R^{n+1}(c)$ with $n \geq 2$. It follows from Theorem 2.2 that M is ideal associated with $\emptyset \in \mathcal{S}(n)$ if and only if M is totally umbilical.

Case (a): $n = 2$. In this case, we have $k = 0$ and $\mathcal{S}(n) = \{\emptyset\}$. Thus, M is ideal and totally umbilical. So, we get $\alpha = 1$.

Case (b): $n = 3$. In this case, we have $\mathcal{S}(n) = \{\emptyset, (2)\}$. Hence, M is ideal if and only if M is totally umbilical or (2)-ideal. If M is (2)-ideal, Lemma 5 of [11] implies that M is quasi-umbilical. Thus, the three principal curvatures of M are given either by $0, \mu, \mu$ or by $2\mu, \mu, \mu$. Thus, we have $\alpha = 0$, 1 or 2.

Case (c): $n \geq 4$. In this case, by a well-known result of Cartan and Schouten, M is quasi-umbilical (see [4]). Hence, there exists a principal curvature, say μ, of multiplicity at least $n - 1$. If M is totally umbilical, we have $\alpha = 1$.

If M is non-totally umbilical, the multiplicity of μ is $n - 1$ on some nonempty open subset U. Let λ denote the other principal curvature with multiplicity one so that $\lambda \neq \mu$ on U and $\lambda = \mu$ on $M - U$. Without loss of generality, we may put

$$\kappa_1 = \lambda, \quad \kappa_2 = \cdots = \kappa_n = \mu. \tag{6.3}$$

Since M is ideal, Theorem 2.2 implies that there is $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ with $k \geq 2$ such that the condition (2.13) holds.

We put $m = k + n - (n_1 + \cdots + n_k)$ and $n_{k+1} = \cdots = n_m = 1$. By applying (2.13) and (6.3), we find

$$\lambda = \alpha\mu, \quad n_2 = \cdots = n_m, \tag{6.4}$$

where

$$\alpha = n_2 + 1 - n_1. \tag{6.5}$$

Since α is an integer by (6.5), the continuity of λ and μ imply that α is a global constant. Also from (6.4) and the definition of $(n_1, \dots, n_m)$, we find

$$n_1 + (m-1)n_2 = n, \tag{6.6}$$

If $\mu = 0$, then (6.3) and (6.4) imply that M is totally geodesic which is a contradiction. Hence, we have $\mu \neq 0$ and $\alpha \neq 1$.

Case (c.1): $\alpha \geq 3$. From (6.6) we have $n_1 = n - (m-1)s$, with $s = n_2$. Substituting this into (6.5) we obtain $\alpha = ms - n + 1$. Since $n_1 \geq 1$, we get $n > (m-1)s$. Clearly, we also have $\alpha \leq n - 1$ from (6.5), with the equality $\alpha = n - 1$ holding if and only if $n_2 = n - 1$ and $n_1 = 1$. This gives Case (3).

Case (c.2): $\alpha = 2$, In this case, (6.3) and (6.5) yield $n_2 = n_1 + 1$. Combining this with (6.6), we obtain $n + 1 = (n_1 + 1)m$. So, $n + 1$ is not a prime. This gives Case (2).

Case (c.3): $\alpha < 0$. The smallest possible integer of $\alpha = n_2 + 1 - n_1$ is obtained from $n_2 = 1$ and $n_1 = n - 1$. So, we get $\alpha \geq 3 - n$. This gives Case (4). ∎

Proposition 6.1 implies immediately the following.

Proposition 6.2. *The only minimal conformally flat ideal hypersurfaces in a real space form are totally geodesic hypersurfaces.*

Example 6.1. For $n = 4$, 5, 6, 7 or 8, Proposition 6.1 implies that the only values of α are the following:

$$\begin{array}{ll} n = 4, & \alpha = -1,\ 0,\ 1,\ 3, \\ n = 5, & \alpha = -2,\ -1,\ 0,\ 1,\ 2,\ 4, \\ n = 6, & \alpha = -3,\ -2,\ -1,\ 0,\ 1,\ 3,\ 5, \\ n = 7, & \alpha = -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 6, \\ n = 8, & \alpha = -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 5,\ 7. \end{array} \tag{6.7}$$

Example 6.2. When n is an even integer ≥ 4, every odd integer in $\{1, 3, 5, \dots, n - 1\}$ is a possible value for α. Similarly, when n is an odd integer ≥ 3, every even integer in $\{2, 4, \dots, n - 1\}$ is a possible value for α. In particular, for any $n \geq 2$, $n - 1$ always occurs as a possible value of α . Furthermore, when n is an odd integer ≥ 7, 3 is a possible value of α if and only if $n + 2$ is not a prime.

Let M be a rotation hypersurface with profile curve γ in a real space form $R^{n+1}(c)$. If we assume that s is the arc length of γ , then the rotation hypersurface M^n is intrinsically the warped product $I \times_\rho R^{n-1}(\delta)$, where I is an open interval of $\mathbb{R}$ and ρ is defined by $\rho(s) = \langle \gamma(s), \mathbf{u} \rangle$ (see [3, (3.9)]).

The second fundamental form of M is then given by

$$h\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right) = \frac{\rho''(s)+c\rho}{\sqrt{\delta - c\rho^2-\rho'^2}}\xi, \tag{6.8}$$

$$h(X,Y) = -\frac{\sqrt{\delta - c\rho^2-\rho'^2(s)}}{\rho}\langle X,Y\rangle\,\xi \tag{6.9}$$

for X and Y tangent to $R^{n-1}(\delta)$, where ξ is a unit normal vector field of M in $R^{n+1}(c)$ (see [3, (3.10)]).

Lemma 6.1. *Let M be a conformally flat ideal hypersurface in a real space form $R^{n+1}(c)$ with $n \geq 3$ so that M is a rotation hypersurface with profile curve γ. If $\rho'(s) \neq 0$, then, up to sign, the arc length function s of γ is given by*

$$s(x) = \int_a^x \frac{d\rho}{\sqrt{\delta - c\rho^2 + A\rho^{2\alpha}}}, \tag{6.10}$$

where a and A are constants and α is an integer listed in Proposition 6.1, and x is the height function of γ given by $\langle\gamma(s),\mathrm{u}\rangle$.

Proof. Under the hypothesis of the lemma, we know from (6.8) and (6.9) that the principal curvatures of M are given by

$$\kappa_1 = \frac{\rho''+c\rho}{\sqrt{\delta - c\rho^2-\rho'^2}},\quad \kappa_2=\cdots=\kappa_n = -\frac{\sqrt{\delta - c\rho^2-\rho'^2}}{\rho}. \tag{6.11}$$

Comparing this with (6.1), we find

$$\rho\rho'' + c(1-\alpha)\rho^2 + \alpha\delta - \alpha\rho'^2 = 0. \tag{6.12}$$

After solving (6.12), we find

$$\rho'^2(s) = \delta - c\rho^2 + A\rho^{2\alpha}, \tag{6.13}$$

where A is a constant. From (6.13) we obtain formula (6.10) for the arc length function of the profile curve γ in terms of x. ∎

7. Conformally Flat Ideal Hypersurfaces in Euclidean Spaces

Example 7.1. For each integer $n \geq 2$ and each positive real number a, there is a well-known Lagrangian immersion from the n-sphere S^n into the complex Euclidean n-space $\mathbf{C}^n$ defined by

$$w_a(y_0,y_1,\ldots,y_n) = \frac{a}{1+y_0^2}\left(y_1,\ldots,y_n,y_0y_1,\ldots,y_0y_n\right), \tag{7.1}$$

where $y_0^2 + y_1^2 + \cdots + y_n^2 = 1$. This immersion w_a has a unique self-intersection point at $w_a(-1, 0, \dots, 0) = w_a(1, 0, \dots, 0)$.

The S^n together with the metric induced from the Whitney immersion w_a is called a *Whitney n-sphere* which is denoted by W_a^n. The Whitney n-sphere W_a^n is a conformally flat n-manifold.

The following theorem completely classifies conformally flat ideal hypersurfaces in Euclidean spaces.

Theorem 7.1. *Let M be a conformally flat hypersurface in $\mathbb{E}^{n+1}$ $(n \geq 2)$. Then M is an ideal hypersurface if and only if, up to rigid motions of $\mathbb{E}^{n+1}$, one of the following eight cases occurs:*

(1) *$n \geq 2$ and M is immersed as an open portion of a hyperplane.*

(2) *$n \geq 2$ and M is immersed as an open portion of a hypersphere.*

(3) *$n \geq 3$ and M is immersed as an open portion of a spherical hypercylinder $S^{n-1} \times \mathbb{R}$.*

(4) *$n \geq 3$ and M is immersed as an open portion of a round hypercone.*

(5) *$n \geq 3$, $n + 1$ is not a prime, M is an open part of a Whitney n-sphere W_a^n for $a > 0$, and the immersion $\mathbf{x} : W_a^n \to \mathbb{E}^{n+1}$ is given by*

$$\mathbf{x} = \left(\frac{1}{2} \int_0^t \mathrm{sd}^2 \left(\frac{\sqrt{2}}{a} u \right) du, a y_1 \mathrm{sd} \left(\frac{\sqrt{2}}{a} t \right), \dots, a y_n \mathrm{sd} \left(\frac{\sqrt{2}}{a} t \right) \right), \tag{7.2}$$

for $y_1^2 + \ldots + y_n^2 = 1/2$, where $k = 1/\sqrt{2}$ is the modulus of the Jacobi's elliptic function.

(6) *$n \geq 4$, M is an open portion of K_a^n for some $a > 0$, and the immersion $\mathbf{x} : K_a^n \to \mathbb{E}^{n+1}$ is the hypercaternoid given by*

$$\mathbf{x}(t, y_1, \dots, y_n) = \left(t, a y_1 \cosh \left(\frac{t}{a} \right), \dots, a y_n \cosh \left(\frac{t}{a} \right) \right) \tag{7.3}$$

for $y_1^2 + \ldots + y_n^2 = 1$..

(7) *$n \geq 4$, M is an open portion of $N_{a\alpha}^n$ for $a > 0$, and*

$$\mathbf{x}(t, y_1, \dots, y_n) = \left(\frac{a t^{\alpha+1}}{\alpha+1} F \left(\frac{1}{2}, \frac{\alpha+1}{2\alpha}, \frac{3\alpha+1}{2\alpha}, a^2 t^{2\alpha} \right), t y_1, \dots, t y_n \right) \tag{7.4}$$

for $y_1^2 + \ldots + y_n^2 = 1$, *where* F *is the hypergeometric function defined by* (3.16) *and* α *is a positive integer in* $[3, n-1]$ *such that* $\alpha = ms - n + 1$ *for some positive integers* s, m *with* $m \geq 2$ *and* $(m-1)s < n$.

(8) $n \geq 5$, M *is an open portion of* $N^n_{a\alpha}$ *for some* $a > 0$ *and negative integer* $\alpha \in [3-n, -2]$, *and the immersion is given by*

$$\mathbf{x}(t, y_1, \ldots, y_n) = \left(\frac{at^{\alpha+1}}{\alpha+1} F\left(\frac{1}{2}, \frac{\alpha+1}{2\alpha}, \frac{3\alpha+1}{2\alpha}, a^2 t^{2\alpha}\right), ty_1 \ldots, ty_n \right) \tag{7.5}$$

for $y_1^2 + \ldots + y_n^2 = 1$.

Proof. Assume that M is a conformally flat ideal hypersurface of $\mathbb{E}^{n+1}$. If $n = 2$, we already know in the proof of Proposition 6.1 that the immersion is totally umbilical. So, we have either Case (1) or Case (2) of the theorem for $n = 2$.

Now, let us assume that $n \geq 3$. Then Proposition 6.1 implies that the principal curvatures of M satisfies $\kappa_1 = \alpha\mu, \kappa_2 = \cdots = \kappa_n = \mu$, where α is one of the integers listed in Proposition 6.1. By applying Theorem 4.2 of [3] or Proposition 3 of [17], we know that M is an open part of a hypersurface of revolution in $\mathbb{E}^{n+1}$ whose profile curve is congruent to the graph of a function $y = \phi(x)$ satisfying

$$\phi\phi'' + \alpha(1 + \phi'^2) = 0. \tag{7.6}$$

Without loss of generality, we may assume that $\phi \geq 0$. After solving (7.6) for $\phi'(x)$, we find

$$\phi^{-\alpha} = a^2 \sqrt{1 + (\phi')^2}, \tag{7.7}$$

for some non-negative constant a. By solving (7.7), we find

$$c + x = \begin{cases} \displaystyle\int_0^y \frac{at^\alpha dt}{\sqrt{1 - a^2 t^{2\alpha}}}, & \text{when } \alpha \geq 2 \text{ and } 0 \leq y < a^{-\alpha^{-1}}; \\ \displaystyle\pm \int_{a^{-\alpha^{-1}}}^y \frac{at^\alpha dt}{\sqrt{1 - a^2 t^{2\alpha}}}, & \text{when } \alpha \leq -1 \text{ and } y > a^{-\alpha^{-1}}, \end{cases} \tag{7.8}$$

where c is a constant. Thus, after a suitable choice of Euclidean coordinate system, we obtain

$$x = \begin{cases} \displaystyle\int_0^y \frac{at^\alpha dt}{\sqrt{1 - a^2 t^{2\alpha}}}, & \text{when } \alpha \geq 2 \text{ and } 0 \leq y < a^{-\alpha^{-1}}; \\ \displaystyle - \int_y^{\infty} \frac{at^\alpha dt}{\sqrt{1 - a^2 t^{2\alpha}}}, & \text{when } \alpha \leq -1 \text{ and } y > a^{-\alpha^{-1}}. \end{cases} \tag{7.9}$$

Case (a): $\alpha = 1$. In this case, M is totally umbilical. Thus, M is an open part of a hyperplane or of a hypersphere. Thus, we obtain Cases (1) or (2) of the theorem. Conversely, we already know that every totally umbilical hypersurface in a Euclidean space is a conformally flat hypersurface which is ideal.

Case (b): $\alpha = 0$. In this case, the profile curve is contained in a line. Thus, M is an open part of a hyperplane, a spherical hypercylinder, or a spherical hypercone. Thus, we obtain Cases (1), (3), or (4) of the theorem. Conversely, such hypersurfaces are conformally flat hypersurfaces which are (2)-ideal.

Case (c): $\alpha = -1$. In this case, Proposition 6.1 yields $n \geq 4$. Moreover, (7.9) implies that the profile curve is a catenary, *i.e.*, $y = \phi(x) = a\cosh(x/a)$. Thus, M is an open portion of a hypercaternoid. It is easy to verify that the induced metric is given by $g = \cosh^2\left(\frac{t}{a}\right)(dt^2 + a^2 g_1)$. This gives Case (6) of the theorem.

Conversely, it is easy to see that a hypercaternoid defined by (7.3) is a conformally flat hypersurface which is (3)-ideal.

Case (d): $\alpha = 2$ *and* $n+1$ *is not a prime*. In this case, (7.9) reduces to

$$x = \phi(y) = \int_0^y \frac{at^2\,dt}{\sqrt{1-a^2t^4}}, \quad 0 \leq y < \frac{1}{\sqrt{a}}. \tag{7.10}$$

After making the substitution $t = (1/\sqrt{2a})\mathrm{sd}(\sqrt{2a}u)$ and applying (3.9) and the fourth and fifth equations in (3.8), we find

$$x = \phi(y) = \frac{1}{2}\int_0^{\frac{1}{\sqrt{2a}}\mathrm{sd}^{-1}\left(\sqrt{2a}y\right)} \mathrm{sd}^2(\sqrt{2a}u)\,du.$$

Consequently, the ideal hypersurface M is given by

$$\mathbf{x}(y, y_1, \ldots, y_n) = \left(\frac{1}{2}\int_0^{\frac{1}{\sqrt{2a}}\mathrm{sd}^{-1}(\sqrt{2a}y)} \mathrm{sd}^2\left(\sqrt{2a}u\right) du, yy_1, \ldots, yy_n\right), \tag{7.11}$$

where $y_1^2 + \ldots + y_n^2 = 1$. Therefore, we obtain (7.2) after making the substitution $y = \frac{1}{\sqrt{2a}}\mathrm{sd}\left(\sqrt{2at}\right)$ and replacing y_i and $\sqrt{a}$ by $\sqrt{2}y_i$ and a^{-1}, respectively. Hence, we obtain Case (5).

Since $n+1$ is not a prime, we may put $n+1 = qk$, $q, k \geq 2$. It is straightforward to verify that the hypersurface defined by (7.11) is a conformally flat hypersurface which is $(n_1, \ldots, n_k)$-ideal with $n_1 = q-1$, $n_2 = \cdots = n_k = q$.

Case (e): $n \geq 4$ *and* $\alpha \neq -1, 0, 1, 2$. In this case, either α is a positive integer in $[3, n-1]$ such that $\alpha = ks - n + 1$ for some positive integers s, k with $k \geq 2$ and $(k-1)s < n$ or α is a negative integer in $[3-n, -2]$.

We consider these two cases separately.

Case (e.1): $n \geq 4$ *and* $\alpha \in [3, n-1]$. This case occurs only when α is a positive integer in $[3, n-1]$ such that $\alpha = ms - n + 1$ for some positive integers s, m with $m \geq 2$ and $(m-1)s < n$. By making the substitution $t = yu$, we obtain

$$\int_0^y \frac{t^\alpha dt}{\sqrt{1-a^2t^{2\alpha}}} = \int_0^1 \frac{u^\alpha y^{\alpha+1} du}{\sqrt{1-a^2y^{2\alpha}u^{2\alpha}}}. \tag{7.12}$$

Next, by making the substitution $u = z^{1/2\alpha}$, we get

$$\begin{aligned}\int_0^1 \frac{u^\alpha y^{\alpha+1} du}{\sqrt{1-a^2y^{2\alpha}u^{2\alpha}}} &= \frac{y^{\alpha+1}}{2\alpha}\int_0^1 z^{\frac{\alpha+1}{2\alpha}-1}\left(1-a^2y^{2\alpha}z\right)^{-\frac{1}{2}}dz \\ &= \frac{y^{\alpha+1}}{\alpha+1}F\left(\frac{1}{2}, \frac{\alpha+1}{2\alpha}, \frac{3\alpha+1}{2\alpha}, a^2y^{2\alpha}\right),\end{aligned}$$

where we have applied formula (3.16) of hypergeometric function. Consequently, if $\alpha \geq 3$, we obtain from (7.9) that

$$x = \frac{ay^{\alpha+1}}{\alpha+1}F\left(\frac{1}{2}, \frac{\alpha+1}{2\alpha}, \frac{3\alpha+1}{2\alpha}, a^2y^{2\alpha}\right). \tag{7.13}$$

Consequently, after a suitable translation, the hypersurface is an open portion of a hypersurface of rotation defined by

$$\mathbf{x}(t, y_1, \ldots, y_n) = \left(\frac{at^{\alpha+1}}{\alpha+1}F\left(\frac{1}{2}, \frac{\alpha+1}{2\alpha}, \frac{3\alpha+1}{2\alpha}, a^2t^{2\alpha}\right), ty_1, \ldots, ty_n\right) \tag{7.14}$$

for $y_1^2 + \ldots + y_n^2 = 1$. So, we obtain Case (7). It is straightforward to verify that the hypersurface defined by (7.14) is $(n_1, \ldots, n_m)$-ideal with $n_1 = n - (m-1)s$ and $n_2 = \cdots = n_m = s$ when $s > 1$; and it is (n_1)-ideal when $s = 1$.

Case (e.2): $n \geq 5$ *and* $\alpha \in [3-n, -2]$. Making the substitution $t = y/u$ yields

$$\int_y^1 \frac{t^\alpha dt}{\sqrt{1-a^2t^{2\alpha}}} = y^{\alpha+1}\int_0^1 \frac{du}{u^{\alpha+2}\sqrt{1-a^2y^{2\alpha}u^{-2\alpha}}}. \tag{7.15}$$

Next, by making the substitution $u = z^{-1/2\alpha}$, we get

$$\int_0^1 \frac{du}{u^{\alpha+2}\sqrt{1-a^2y^{2\alpha}u^{-2\alpha}}} = -\frac{1}{2\alpha}\int_0^1 \frac{z^{\frac{1-\alpha}{2\alpha}}dz}{\sqrt{1-a^2y^{2\alpha}z}}.$$

Consequently, we obtain from (7.8) that

$$\pm(c+x) = \frac{ay^{\alpha+1}}{\alpha+1} F\left(\frac{1}{2}, \frac{\alpha+1}{2\alpha}, \frac{3\alpha+1}{2\alpha}, a^2 y^{2\alpha}\right).$$

Consequently, after a suitable translation, the hypersurface is an open portion of a hypersurface of rotation defined by

$$\mathbf{x}(t, y_1, \ldots, y_n) = \left(\frac{at^{\alpha+1}}{\alpha+1} F\left(\frac{1}{2}, \frac{\alpha+1}{2\alpha}, \frac{3\alpha+1}{2\alpha}, a^2 t^{2\alpha}\right), ty_1, \ldots, ty_n\right) \tag{7.16}$$

for $y_1^2 + \ldots + y_n^2 = 1$. It follows from (7.16) that the metric on M is given by $g = \left(1 - a^2 t^{2\alpha}\right)^{-1} dt^2 + t^2 g_0$. Hence, we obtain Case (8).

Conversely, it is straightforward to verify that the hypersurface defined by (7.16) is a $(2-\alpha)$-ideal conformally flat hypersurface. ∎

Remark 7.1. Theorem 7.1 shows that all values of α listed in Proposition 6.1 do occur.

Corollary 7.1. *Let M be a simply-connected open portion of a Whitney n-sphere W_a^n. Then M admits an ideal isometric immersion into $\mathbb{E}^{n+1}$ if and only if $n+1$ is not a prime. In particular, every open portion of an odd-dimensional Whitney sphere W_a^n can be isometric immersed in a Euclidean space as an ideal hypersurface.*

Since a rotation hypersurface in $\mathbb{E}^{n+1}$ is minimal if and only if the integer α given by (6.4) is $1-n$. Thus, from the proof of Theorem 7.1 we have the following.

Proposition 7.1. *A rotation hypersurface M in $\mathbb{E}^{n+1}$ is minimal if and only if, up to rigid motions of $\mathbb{E}^{n+1}$, the hypersurface M is an open portion of $N^n_{a\,1-n}$ for some $a > 0$ and the immersion is given by*

$$\mathbf{x}(t, y_1, \ldots, y_n) = \left(\frac{at^{2-n}}{2-n} F\left(\frac{1}{2}, \frac{2-n}{2-2n}, \frac{4-3n}{2-2n}, a^2 t^{2-2n}\right), ty_1, \ldots, ty_n\right), \quad y_1^2 + \ldots + y_n^2 = 1. \tag{7.17}$$

Similar results also hold for minimal rotation hypersurfaces in S^{n+1} and H^{n+1}.

8. Conformally Flat Ideal Hypersurfaces in Spheres

In this section we classify conformally flat ideal hypersurfaces in spheres.

Theorem 8.1. *Let $\mathrm{x} : M \to S^{n+1}(1) \subset \mathbb{E}^{n+2}$ be an isometric immersion of a conformally flat n-manifold with $n \geq 2$. Then x is ideal if and only if, up to rigid motions, one of the following seven cases occurs.*

(1) $n \geq 2$, *M is an open portion of* $S^n(a), a \geq 1$, *and* x *is totally umbilical.*

(2) $n \geq 3$, *M is an open portion of* $\hat{B}^n_a$, $a \in (0,1)$, *and*

$$\mathrm{x}(t, y_1, \ldots, y_n) = \cos t \left(\tan t, a, \sqrt{1-a^2}\, y_1, \ldots, \sqrt{1-a^2}\, y_n \right), \quad y_1^2 + y_2^2 + \cdots + y_n^2 = 1. \tag{8.1}$$

(3) $n \geq 3$, *$n+1$ is not a prime, M is an open portion of* $\hat{P}^n_b$ *for some* $b > 1$, *and the immersion is given by*

$$\begin{aligned} \mathbf{x}(t, y_1, \ldots, y_n) = \frac{1}{bk'} \Bigg(& y_1 \,\mathrm{cn}(bt), \ldots, y_n \,\mathrm{cn}(bt), \\ & \sqrt{b^2k'^2 - \mathrm{cn}^2(bt)} \cos\left(\frac{k'}{k} t + \frac{i}{2} \ln \frac{\Theta(bt-\gamma)}{\Theta(bt+\gamma)} + i\, bZ(\gamma) t \right), \\ & \sqrt{b^2k'^2 - \mathrm{cn}^2(bt)} \sin\left(\frac{k'}{k} t + \frac{i}{2} \ln \frac{\Theta(bx-\gamma)}{\Theta(bt+\gamma)} + i\, bZ(\gamma) t \right) \Bigg), \end{aligned} \tag{8.2}$$

for $y_1^2 + y_2^2 + \cdots + y_n^2 = 1$, *where* $k = \sqrt{b^2-1}/\sqrt{2}b$, $k' = \sqrt{b^2+1}/\sqrt{2}b$ *and* $\gamma = \mathrm{sn}^{-1}\left(i/bk^2\right)$.

(4) $n \geq 4$, *M is an open portion of* I^n_α *for some negative integer* $\alpha \in [3-n, -1]$. *Moreover, the immersion is given by*

$$\begin{aligned} \mathrm{x}(t, y_1, \ldots, y_n) = \frac{1}{\sqrt{1-\alpha}} \Big(& \cos\left(\sqrt{1-\alpha}\, s\right), \cos\left(\sqrt{1-\alpha}\, s\right), \\ & \sqrt{-\alpha}\, y_1, \ldots, \sqrt{-\alpha}\, y_n \Big), \quad y_1^2 + y_2^2 + \cdots + y_n^2 = 1. \end{aligned} \tag{8.3}$$

(5) $n \geq 4$, *M is an open portion of* $E^n_b, b \in (0,1)$, *and*

$$\begin{aligned} \mathbf{x}(t, y_1, \ldots, y_n) = \frac{1}{k} \Bigg(& bk' \,\mathrm{nd}\left(\frac{b}{k} t\right) y_1, \ldots, bk' \,\mathrm{nd}\left(\frac{b}{k} t\right) y_n, \\ & \sqrt{k^2 - b^2k'^2 \mathrm{nd}^2\left(\frac{b}{k} t\right)} \cos\left(k't - \frac{i}{2} \ln \frac{\Theta\left(\frac{b}{k}t - \gamma\right)}{\Theta\left(\frac{b}{k}t + \gamma\right)} - i \frac{b}{k} Z(\gamma) t \right), \\ & \sqrt{k^2 - b^2k'^2 \mathrm{nd}^2\left(\frac{b}{k} t\right)} \cos\left(k't - \frac{i}{2} \ln \frac{\Theta\left(\frac{b}{k}t - \gamma\right)}{\Theta\left(\frac{b}{k}t + \gamma\right)} - i \frac{b}{k} Z(\gamma) t \right) \Bigg), \end{aligned} \tag{8.4}$$

for $y_1^2 + y_2^2 + \cdots + y_n^2 = 1$, *where* $k = \sqrt{2}b/\sqrt{1+b^2}$, $k' = \sqrt{1-b^2}/\sqrt{1+b^2}$ *are the modulus and the complementary modulus of Jacobi's elliptic functions and* $\gamma = \mathrm{sn}^{-1}(k/b)$.

(6) *$n \geq 4$, M is an open portions of $J^n_{a\alpha}$ for some $a > 0$ and some integer $\alpha = ms - n + 1 \in [3, n-1]$, where s and m are positive integers satisfying $m \geq 2$ and $(m-1)s < n$. Moreover, the immersion is given by*

$$\mathbf{x} = \left(\sqrt{1-t^2}\cos\left(B_{1,1}(a,\alpha,t)\right), \sqrt{1-t^2}\sin\left(B_{1,1}(a,\alpha,t)\right), ty_1, \ldots, ty_n\right), \tag{8.5}$$

for $y_1^2 + y_2^2 + \cdots + y_n^2 = 1$, where $B_{1,1}$ is defined by (3.17).

(7) *$n \geq 5$, M is an open portions of $J^n_{a\alpha}$ for some integer $\alpha \in [3-n, -2]$ and for $a \in \left(0, \frac{(-\alpha)^{-\alpha/2}}{(1-\alpha)^{(1-\alpha)/2}}\right)$. Moreover, the immersion is given by*

$$\mathbf{x} = \left(\sqrt{1-t^2}\cos\left(B_{1,1}(a,\alpha,t)\right), \sqrt{1-t^2}\sin\left(B_{1,1}(a,\alpha,t)\right), ty_1, \ldots, ty_n\right) \tag{8.6}$$

for $y_1^2 + y_2^2 + \cdots + y_n^2 = 1$.

Proof. Assume that $n \geq 2$ and $x : M \to S^{n+1}(1) \subset E^{n+2}$ is an ideal isometric immersion of a conformally flat n-manifold M into $S^{n+1}(1)$

When $n = 2$, the immersion $\mathbf{x}$ is totally umbilical. Thus, we obtain Case (1).

When $n = 3$, we have $\mathcal{S}(n) = \{\emptyset, (2)\}$. Thus, the immersion is either totally umbilical or (2)-ideal. If it is totally umbilical, we obtain Case (1). If it is (2)-ideal and non-totally umbilical, then, by Theorem 5 of [11], we get Cases (2) and (5).

Next, assume that $n \geq 4$. From Proposition 6.1, we have

$$\kappa_1 = \alpha\mu, \quad \kappa_2 = \cdots = \kappa_n = \mu \tag{8.7}$$

for some function μ and an integer α given in the list of Proposition 6.1.

If we have $\alpha = 1$ or $\mu = 0$, (8.7) implies that the immersion is totally umbilical. So, we obtain Case (1).

Now, assume that $\alpha \neq 1$ and $\mu \neq 0$. Then Theorem 4.2 of [3] implies that M is a rotation hypersurface in $S^{n+1}(1)$. In this case, the function δ defined in Section 4 is equal to 1 and the profile curve $\gamma(s) = \left(x(s), y(s), z(s)\right)$ of M satisfies

$$x^2(s) + y(s)^2 + z(s)^2 = 1. \tag{8.8}$$

Up to rigid motions, the rotation hypersurface M in $S^{n+1}(1) \subset E^{n+2}$ is given by

$$x(s, y_1, \ldots, y_n) = \left(y(s), z(s), x(s)y_1, \ldots, x(s)y_n\right) \tag{8.9}$$

with $y_1^2 + \cdots + y_n^2 = 1$. From (8.8), we also have $x^2 \leq 1$. If x is constant, say b, then the principal curvature of the hypersurface M in $S^{n+1}(1)$ are given by

$$\kappa_1 = \frac{b}{\sqrt{1-b^2}}, \quad \kappa_2 = \cdots = \kappa_n = -\frac{\sqrt{1-b^2}}{b} \tag{8.10}$$

which implies $b \neq 0, 1$. Thus, by applying condition $\kappa_1 = \alpha\kappa_2$ from Proposition 6.1, we get $b^2 = \alpha/(\alpha - 1)$. So, α is negative since $b^2 < 1$. Thus, by applying Proposition 6.1 again, we see that α is a negative integer $\geq 3 - n$. Hence, by (8.8), we may put $y = \cos t/\sqrt{1-\alpha}$, $z = \sin t/\sqrt{1-\alpha}$. So, we obtain from (8.9) that

$$\mathrm{x}(t, y_1, \ldots, y_n) = \left(\frac{\cos t}{\sqrt{1-\alpha}}, \frac{\sin t}{\sqrt{1-\alpha}}, \sqrt{\frac{\alpha}{\alpha-1}}\, y_1, \ldots, \sqrt{\frac{\alpha}{\alpha-1}}\, y_n \right) \tag{8.11}$$

for $y_1^2 + \cdots + y_n^2 = 1$. Thus, we obtain Case (4) after the substitution: $t = \sqrt{1-\alpha}\, s$.

Next, assume that x is non-constant. Because x is the height function of the profile curve, we may assume $x \geq 0$ and put $y = \sqrt{1-x^2}\cos\phi(x)$, $z = \sqrt{1-x^2}\sin\phi(x)$ for some function $\phi(x)$. From (8.8) we see that the arc length function s of the profile curve γ satisfies

$$\left(\frac{ds}{dx}\right)^2 = 1 + y'(x)^2 + z'(x)^2. \tag{8.12}$$

By applying Lemma 6.1 and (8.12), we find

$$\phi'^2 = \frac{-Ax^{2\alpha}}{(1-x^2)^2(1-x^2+Ax^{2\alpha})} \tag{8.13}$$

which implies $A \leq 0$, since $0 \leq x \leq 1$. So, we may put $A = -a^2$ with $a \geq 0$. From (8.13) we obtain

$$\phi(x) = \pm \int^x \frac{at^\alpha dt}{(1-t^2)\sqrt{1-t^2-a^2t^{2\alpha}}}.$$

When $\alpha \geq 0$, without loss of generality we may assume that

$$\phi(x) = \int_0^x \frac{at^\alpha dt}{(1-t^2)\sqrt{1-t^2-a^2t^{2\alpha}}}. \tag{8.14}$$

Since $\alpha \neq 1$, Proposition 6.1 implies that α is one of the following integers:

(a) $\alpha = 0$;

(b) $\alpha = 2$ and $n + 1$ is not a prime;

(c) $n \geq 4$ and α is a positive integer in $[3, n-1]$ satisfying $\alpha = ms - n + 1$ for some positive integers s, m with $k \geq 2$ and $(m-1)s < n$;

(d) $n \geq 4$ and α is a negative integer in $[3-n, -1]$.

We consider these four cases separately.

Case (a): $\alpha = 0$. In this case, we have

$$\phi(x) = \int_0^x \frac{a\,dt}{(1-t^2)\sqrt{1-a^2-t^2}} = -\tan^{-1}\left(\frac{\sqrt{1-a^2-x^2}}{ax}\right). \tag{8.15}$$

From $y = \sqrt{1-x^2}\cos\phi(x), z = \sqrt{1-x^2}\sin\phi(x)$ and (8.15), we find

$$y = \frac{ax}{\sqrt{1-a^2}}, \quad z = -\frac{\sqrt{1-a^2-x^2}}{\sqrt{1-a^2}}. \tag{8.16}$$

After making the substitution: $x = \sqrt{1-a^2}\cos t$, (8.16) becomes $x = \sqrt{1-a^2}\cos t$, $y = a\cos t, z = -\sin t$. Thus, we see from (8.9) that, up to rigid motions, M is given by

$$\mathbf{x}(t, y_1, \ldots, y_n) = \cos t\left(\tan t, a, \sqrt{1-a^2}\,y_1, \ldots, \sqrt{1-a^2}\,y_n\right) \tag{8.17}$$

for $y_1^2 + y_2^2 + \cdots + y_n^2 = 1$, and $a \in [0, 1)$. This gives Case (2) for $a \in (0, 1)$.

When $a = 0$, M is totally geodesic in $S^{n+1}(1)$ which is in Case (1).

Case (b): $\alpha = 2$ *and* $n + 1$ *is not a prime*. In this case, (8.14) reduces to

$$\phi(x) = \int_0^x \frac{at^2dt}{\left(1-t^2\right)\sqrt{1-t^2-a^2t^4}}, \tag{8.18}$$

which implies $0 < x < \sqrt{\sqrt{1+4a^2}-1}/\sqrt{2}a$. If we put

$$b = \left(1+4a^2\right)^{1/4}, \quad k = \sqrt{b^2-1}/\sqrt{2}b, \quad k' = \sqrt{b^2+1}/\sqrt{2}b, \tag{8.19}$$

then (8.18) becomes

$$\phi(x) = \int_0^x \frac{at^2dt}{\left(1-t^2\right)\sqrt{\left(1-b^2k'^2t^2\right)\left(1+b^2k^2t^2\right)}}. \tag{8.20}$$

If we make the substitution: $t = \mathrm{cn}(bu)/bk'$, we obtain from (8.20) that

$$\phi(x) = -\int_{K/b}^{(1/b)\mathrm{cn}^{-1}(bk'x)} \frac{a\,\mathrm{cn}^2(bu)}{b^2\left(k'^2 - b^{-2}\mathrm{cn}^2(bu)\right)}du, \tag{8.21}$$

where k is the the modulus and K is the quarter-period of the Jacobi's function $\mathrm{cn}(bu)$. (8.21) can be put in the form:

$$\phi(x) = -\int_{K/b}^{(1/b)\mathrm{cn}^{-1}(bk'x)} \frac{b\beta\sqrt{(k^2+\beta^2)(k'^2-\beta^2)}\,\mathrm{cn}^2(bu)}{k'^2-\beta^2\mathrm{cn}^2(bu)}du, \tag{8.22}$$

where $\beta = 1/b$. Hence, by applying formulas (6.15)-(6.18) of [11, page 485], we obtain, up to constants, that

$$\phi(x) = -\frac{k'}{bk}\mathrm{cn}^{-1}(bk'x) - \frac{i}{2}\left(\ln\frac{\Theta(\mathrm{cn}^{-1}(bk'x) - \gamma)}{\Theta(\mathrm{cn}^{-1}(bk'x) + \gamma)} + 2\,\mathrm{Z}(\gamma)\mathrm{cn}^{-1}(bk'x)\right)$$

where $\Theta(u) = \Theta(u, k)$ is the theta function and $\mathrm{Z}(u) = \mathrm{Z}(u, k)$ the zeta function and $\gamma = \mathrm{sn}^{-1}\left(i/bk^2\right)$. (Notice that α in (8.23) and (8.25) of [11] shall read a.) Thus, after making the substitution $t = (1/b)\mathrm{cn}^{-1}(bk'x)$ and choosing a suitable Euclidean coordinate system, we obtain Case (3) from (8.9).

Case (c): $n \geq 4$, $3 \leq \alpha \leq n-1$ *and* $\alpha = ms - n + 1$ *for some positive integers* s, m *with* $k \geq 2$ *and* $n > (m-1)s$. We obtain from (3.17), (8.9), and (8.14) that

$$\mathbf{x} = \left(\sqrt{1-t^2}\cos\left(B_{1,1}(a,\alpha,t)\right), \sqrt{1-t^2}\sin\left(B_{1,1}(a,\alpha,t)\right), ty_1, \ldots, ty_n\right)$$

for $y_1^2 + y_2^2 + \cdots + y_n^2 = 1$. This gives Case (6).

Case (d.1): $n \geq 4$ *and* $\alpha = -1$. From (8.14) we get

$$\phi(x) = \int_{\sqrt{\frac{1-\sqrt{1-4a^2}}{2}}}^{x} \frac{adt}{(1-t^2)\sqrt{t^2 - t^4 - a^2}}$$

for $\sqrt{\frac{1-\sqrt{1-4a^2}}{2}} < x < \sqrt{\frac{1+\sqrt{1-4a^2}}{2}}$. If we put $k = \sqrt{2}b/\sqrt{1+b^2}$, $k' = \sqrt{1-b^2}/\sqrt{1+b^2}$ and $b = \left(1-4a^2\right)^{1/4}$, then we obtain after making the substitution: $t = 1/u$ that

$$\begin{aligned}\phi(x) &= \int_{\frac{k}{bk'}}^{1/x} \frac{-au^2du}{\left(u^2-1\right)\sqrt{\left(1-\frac{1-b^2}{2}u^2\right)\left(\frac{1+b^2}{2}u^2-1\right)}}, \\ &= \int_{\frac{k}{bk'}}^{1/x} \frac{-ak'u^2du}{\left(u^2-1\right)\sqrt{\left(1-\frac{b^2k'^2}{k^2}u^2\right)\left(\frac{b^2k'^2}{k^2}u^2-k'^2\right)}}.\end{aligned} \tag{8.23}$$

Hence, after making the second substitution: $v = bk'u/k$, we get

$$\phi(x) = -\int_1^{bk'/kx} \frac{ak^3v^2dv}{b^3(k'^2 - b^{-2}k^2v^2)\sqrt{(1-v^2)(v^2-k'^2)}}. \tag{8.24}$$

Therefore, after making the third substitution: $v = \mathrm{dn}(bu/k)$ and applying formulas (3.8) and (3.9), we obtain

$$\phi(x) = -\int_0^{(k/b)\,\mathrm{dn}^{-1}(bk'/kx)} \frac{a\beta^2\,\mathrm{dn}^2\left(\frac{bu}{k}\right)}{\beta^2\mathrm{dn}^2\left(\frac{bu}{k}\right) - k'^2}du, \tag{8.25}$$

where $\beta = k/b$. (8.25) can be put in the form:

$$\phi(x) = -\int_0^{(k/b)\,\mathrm{dn}^{-1}(bk'/kx)} \frac{b\beta\sqrt{(\beta^2-1)(\beta^2-k'^2)}\,\mathrm{dn}^2\left(\frac{bu}{k}\right)}{k\left(\beta^2\mathrm{dn}^2\left(\frac{bu}{k}\right)-k'^2\right)}du. \tag{8.26}$$

Hence, by applying formulas (6.34)-(6.36) of [11, page 485], we find

$$\phi(x) = -\frac{k'k}{b}\mathrm{dn}^{-1}\left(\frac{bk'}{kx}\right) + \frac{i}{2}\ln\frac{\Theta\left(\mathrm{dn}^{-1}\left(\frac{bk'}{kx}\right)-\gamma\right)}{\Theta\left(\mathrm{dn}^{-1}\left(\frac{bk'}{kx}\right)+\gamma\right)} + i\,Z(\gamma)\mathrm{dn}^{-1}\left(\frac{bk'}{kx}\right),$$

where $\gamma = \mathrm{sn}^{-1}(k/b)$. So, after the final substitution: $t = (k/b)\mathrm{dn}^{-1}(bk'/kx)$, we obtain $\phi(x(t)) = -k't + \frac{i}{2}\ln\left(\Theta\left(\frac{b}{k}t-\gamma\right)/\Theta\left(\frac{b}{k}t+\gamma\right)\right) + i\frac{b}{k}Z(\gamma)t$. Therefore, up to rigid motions, we may obtain Case (5) from (8.9).

Case (d.2): $n \geq 5$ *and* $\alpha \in [3-n,-2]$. We see that from Remark 8.1 that a satisfies $0 < a < \frac{(-\alpha)^{-\alpha/2}}{(1-\alpha)^{(1-\alpha)/2}}$. Also from (3.17), (8.9) and (8.14) we know that the hypersurface is given by

$$\mathbf{x} = \left(\sqrt{1-t^2}\cos\left(B_{1,1}(a,\alpha,t)\right), \sqrt{1-t^2}\sin\left(B_{1,1}(a,\alpha,t)\right), ty_1,\ldots,ty_n\right) \tag{8.27}$$

for $y_1^2 + y_2^2 + \cdots + y_n^2 = 1$. The induced metric on the hypersurface is the the one given by $J^n_{a\alpha}$. This gives Case (7).

Conversely, it is straightforward to verify that all of the hypersurfaces given in this theorem are ideal conformally flat hypersurfaces. ∎

Remark 8.1 When $\alpha = -\beta < 0$, the function $f(t) = 1 - t^2 - a^2t^{2\alpha}$ is positive at some point happen only when $a < (\beta^\beta/(1+\beta)^{(1+\beta)})^{1/2}$. This is due to the facts: (i) $f(0) = -a^2 < 0$, (ii) $f'(t) = 0$ only at $t_0 = \pm\sqrt{\beta/(1+\beta)}$, and (iii) $f(t_0)$ is positive if and only if $a < (\beta^\beta/(1+\beta)^{(1+\beta)})^{1/2}$. So, when $\alpha < 0$ and $f(t) = 1 - t^2 - a^2t^{2\alpha}$ is positive on some points, we shall replace the lower limit of the integral (8.14) by the smallest positive root, say γ of $f(t)$.

9. Conformally Flat Ideal Hypersurfaces in Hyperbolic Spaces

Theorem 9.1. *Let* $\mathrm{x} : M \to H^{n+1}(-1) \subset \mathbb{E}_1^{n+2}$ *be an isometric immersion of a conformally flat* n*-manifold with* $n \geq 2$*. Then* x *is ideal if and only if, up to rigid motions, one of the following twenty cases occurs.*

(1) $n \geq 2$, *M is an open portion of* $H^n(c)$ *for* $c \in [-1, 0)$, *and the immersion is totally umbilical.*

(2) *$n \geq 2$, M is an open portion of E^n, and the immersion is totally umbilical.*

(3) *$n \geq 2$, M is an open portion of $S^n(c)$ for $c > 0$, and the immersion is totally umbilical.*

(4) *$n \geq 3$, M is an open portion of G^n, and the immersion is given by*

$$\begin{aligned}\mathbf{x}(t, u_1, \dots, u_{n-1}) = \cosh t \Big(1 + \tfrac{1}{2}\big(u_1^2 + \cdots + u_{n-1}^2\big),\\ -\tfrac{1}{2}\big(u_1^2 + \cdots + u_{n-1}^2\big), \tanh t, u_1, \dots, u_{n-1}\Big).\end{aligned} \tag{9.1}$$

(5) *$n \geq 3$, $n+1$ is not a prime, M is an open portion of L^n, and*

$$\begin{aligned}\mathbf{x}(t, u_1, \dots, u_{n-1}) = \operatorname{sech} t \Big(\tfrac{1}{4} + \cosh^2 t + t^2 + u_1^2 + \cdots + u_{n-1}^2,\\ \tfrac{1}{4} - \cosh^2 t - t^2 - u_1^2 - \cdots - u_{n-1}^2, -t, u_1, \dots, u_{n-1}\Big).\end{aligned} \tag{9.2}$$

(6) *$n \geq 3$, M is an open portion of O^n, and the immersion is given by*

$$\begin{aligned}&\mathbf{x}(s, y_1, \dots, y_n) = \left(e^s + \tfrac{e^{-s}}{2}, \tfrac{e^{-s}}{2}, e^s y_1, \dots, e^s y_n\right),\\ &y_1^2 + \cdots + y_n^2 = 1.\end{aligned} \tag{9.3}$$

(7) *$n \geq 3$, M is an open portion of A_a^n for $a > 1$, and*

$$\begin{aligned}&\mathbf{x}(t, y_1, \dots, y_n) = \cosh t \left(a, \tanh t, \sqrt{a^2 - 1}\,y_1, \dots, \sqrt{a^2 - 1}\,y_n\right),\\ &y_1^2 + \cdots + y_n^2 = 1.\end{aligned} \tag{9.4}$$

(8) *$n \geq 3$, M is an open portion of Y_a^n for $a \in (0, 1)$, and*

$$\begin{aligned}&\mathbf{x}(t, y_1, \dots, y_n) = \sinh t \left(\coth t, a, \sqrt{1 - a^2}\,y_1, \dots, \sqrt{1 - a^2}\,y_n\right),\\ &y_1^2 + \cdots + y_n^2 = 1.\end{aligned} \tag{9.5}$$

(9) *$n \geq 3$, M is an open portion of H_a^n for $a > 0$, and*

$$\begin{aligned}&\mathbf{x}(t, y_1, \dots, y_n) = \cosh t \left(\sqrt{1 + a^2}\,y_1, \dots, \sqrt{1 + a^2}\,y_n, a, \tanh t\right),\\ &y_1^2 - y_2^2 - \cdots - y_n^2 = 1.\end{aligned} \tag{9.6}$$

(10) *$n \geq 3$, $n+1$ is not a prime, M is an open portion of F_2^n, and*

$$\begin{aligned}&\mathrm{x}(s, y_1, \dots, y_n) = \left(\sqrt{2}\,y_1, \dots, \sqrt{2}\,y_n, \cos s, \sin s\right),\\ &y_1^2 - y_2^2 - \cdots - y_n^2 = 1,\end{aligned} \tag{9.7}$$

(11) *$n \geq 3$, $n+1$ is not a prime, M is an open portion of $\hat{C}_b^n$ for $b > 1$, and*

$$\begin{aligned}\mathbf{x} = \frac{1}{bk'}\Bigg(&\sqrt{b^2k'^2 + \mathrm{cn}^2(bt)}\cosh\left(\frac{k'}{k}t - \frac{1}{2}\ln\frac{\Theta(bt-\gamma)}{\Theta(bt+\gamma)} - b\,Z(\gamma)t\right),\\ &\sqrt{b^2k'^2 + \mathrm{cn}^2(bt)}\sinh\left(\frac{k'}{k}t - \frac{1}{2}\ln\frac{\Theta(bt-\gamma)}{\Theta(bt+\gamma)} - b\,Z(\gamma)t\right),\\ &y_1\,\mathrm{cn}(bt),\dots,y_n\,\mathrm{cn}(bt)\Bigg), \qquad y_1^2 + y_2^2 + \cdots + y_n^2 = 1.\end{aligned}$$

(12) $n \geq 3$, $n+1$ *is not a prime,* M *is an open portion of* D_b^n, $b \in (0,1)$, *and*

$$\begin{aligned}\mathrm{x}(t,y_1,\dots,y_n) = \tfrac{1}{bk'}\Bigg(&k\,\mathrm{dn}\left(\tfrac{kt}{b}\right)y_1,\dots,k\,\mathrm{dn}\left(\frac{kt}{b}\right)y_n,\\ &\sqrt{k^2\mathrm{dn}^2\left(\tfrac{kt}{b}\right) - b^2k'^2}\cos\left(k't - \frac{i}{2}\ln\frac{\Theta\left(\left(\frac{bt}{k}\right)-\gamma\right)}{\Theta\left(\left(\frac{bt}{k}\right)+\gamma\right)} - i\frac{b}{k}Z(\gamma)t\right),\\ &\sqrt{k^2\mathrm{dn}^2\left(\tfrac{kt}{b}\right) - b^2k'^2}\sin\left(k't - \tfrac{i}{2}\ln\tfrac{\Theta\left(\left(\frac{bt}{k}\right)-\gamma\right)}{\Theta\left(\left(\frac{bt}{k}\right)+\gamma\right)} - i\tfrac{b}{k}Z(\gamma)t\right)\Bigg),\\ &y_1^2 - y_2^2 - \cdots - y_n^2 = 1.\end{aligned} \tag{9.8}$$

(13) $n \geq 4$, M *is an open portion of* F_α^n, *and the immersion is given by*

$$\mathrm{x} = \frac{1}{\sqrt{\alpha-1}}\left(\sqrt{\alpha}\,y_1,\dots,\sqrt{\alpha}\,y_n,\cos\left(\sqrt{\alpha-1}\,s\right),\sin\left(\sqrt{\alpha-1}\,s\right)\right) \tag{9.9}$$

for $y_1^2 - y_2^2 - \cdots - y_n^2 = 1$, *where* α *is a positive integer in* $[3, n-1]$ *such that* $\alpha = ms - n + 1$ *for some positive integers* s, m *with* $m \geq 2$ *and* $(m-1)s < n$.

(14) $n \geq 4$, M *is an open portion of* Q_b^n *for* $b > 1$, *and*

$$\begin{aligned}\mathbf{x} = \Bigg(&\sqrt{1 + b^2k'^2\mathrm{nc}^2(bt)}\cosh\left(\frac{k'}{k}t - \frac{1}{2}\ln\frac{\Theta(bt-\gamma)}{\Theta(bt+\gamma)} - b\,Z(\gamma)t\right),\\ &\sqrt{1 + b^2k'^2\mathrm{nc}^2(bt)}\sinh\left(\frac{k'}{k}t - \frac{1}{2}\ln\frac{\Theta(bt-\gamma)}{\Theta(bt+\gamma)} - b\,Z(\gamma)t\right),\\ &y_1\,bk'\mathrm{nc}(bt),\dots,y_n\,bk'\mathrm{nc}(bt)\Bigg), \qquad y_1^2 + y_2^2 + \cdots + y_n^2 = 1.\end{aligned}$$

(15) $n \geq 4$, M *is an open portion of* S_b^n *for* $b > 0$, *and*

$$
\begin{aligned}
\mathbf{x} = \sqrt{2}b\,\mathrm{ds}(\sqrt{2}bt)\Bigg(&\frac{1}{4}+\frac{\mathrm{sd}^2(\sqrt{2}bt)}{2b^2}+\frac{1}{4}\left(\int_0^t \mathrm{sd}^2(\sqrt{2}bu)\,du\right)^2+\sum_{i=1}^{n-1}u_i^2,\\
&\frac{1}{4}-\frac{\mathrm{sd}^2(\sqrt{2}bt)}{2b^2}-\frac{1}{4}\left(\int_0^t \mathrm{sd}^2(\sqrt{2}bu)\,du\right)^2-\sum_{i=1}^{n-1}u_i^2,\\
&\frac{1}{2}\int_0^t \mathrm{sd}^2(\sqrt{2}bu)\,du, u_1,\dots,u_{n-1}\Bigg)
\end{aligned}
$$

with $k = 1/\sqrt{2}$ as the modulus of the Jacobi's elliptic functions.

(16) $n \geq 4$, *M is an open portion of R^n_b for $b > 1$, and*

$$
\begin{aligned}
\mathrm{x}(t,y_1,\dots,y_n) = \Bigg(& y_1 b k' \mathrm{nc}(bt),\dots,y_n b k' \mathrm{nc}(bt),\\
&\sqrt{b^2k'^2\mathrm{nc}^2(bt)-1}\cos\left(\frac{k'}{k}t+\frac{i}{2}\ln\frac{\Theta(bt-\gamma)}{\Theta(bt+\gamma)}+i\,b\,Z(\gamma)t\right),\\
&\sqrt{b^2k'^2\mathrm{nc}^2(bt)-1}\sin\left(\frac{k'}{k}t+\frac{i}{2}\ln\frac{\Theta(bt-\gamma)}{\Theta(bt+\gamma)}+i\,b\,Z(\gamma)t\right)\Bigg),\\
& y_1^2-y_2^2-\cdots-y_n^2 = 1.
\end{aligned}
\tag{9.10}
$$

(17) $n \geq 4$, *M is an open portion of $Z^n_{a\alpha}$, and the immersion is given by*

$$
\begin{aligned}
&\mathrm{x}(t,u_1,\dots,u_{n-1}) =\\
&\Bigg(t^{\frac{1}{2\alpha-2}}\left(\frac{1}{4}+\sum_{i=1}^{n-1}u_i^2+t^{\frac{1}{1-\alpha}}\left(1+\frac{t}{(2-\alpha)^2}F^2\left(\frac{1}{2},\frac{\alpha-2}{2\alpha-2},\frac{3\alpha-4}{2\alpha-2},t\right)\right)\right),\\
&\quad\frac{1}{4}-\sum_{i=1}^{n-1}u_i^2-t^{\frac{1}{1-\alpha}}\left(1+\frac{t}{(2-\alpha)^2}F^2\left(\frac{1}{2},\frac{\alpha-2}{2\alpha-2},\frac{3\alpha-4}{2\alpha-2},t\right)\right),\\
&\quad\frac{t^{(\alpha-2)/2(\alpha-1)}}{2-\alpha}F\left(\frac{1}{2},\frac{\alpha-2}{2\alpha-2},\frac{3\alpha-4}{2\alpha-2},t\right),u_1,\dots,u_{n-1}\Bigg),
\end{aligned}
\tag{9.11}
$$

where a is a positive number and α is either a negative integer in $[-2, 3-n]$ or a positive integer in $[3, n-1]$ such that $\alpha = ms-n+1$ for some positive integers s, m satisfying $m \geq 2$ and $(m-1)s < n$.

(18) $n \geq 4$, *M is an open portion of $U^n_{a\alpha}$ for $a > 0$, and α an integer given as in Case (17), and the immersion is given by*

$$\begin{aligned}\mathbf{x}(t,y_1,\ldots,y_n) = \Big(&\sqrt{1+t^2}\cosh B_{-1,1}(a,\alpha,t),\\ &\sqrt{1+t^2}\sinh B_{-1,1}(a,\alpha,t), ty_1,\ldots,ty_n\Big),\ y_1^2+y_2^2+\cdots+y_n^2=1.\end{aligned} \tag{9.12}$$

(19) *$n \geq 4$, M is an open portion of $V^n_{a\alpha}$, $a \in \Big(0, \frac{1}{\sqrt{\alpha}}\big((\alpha-1)/\alpha\big)^{(\alpha-1)/2}\Big)$ and integer $\alpha = ms - n + 1 \in [3, n-1]$, where s and m are positive integers satisfying $m \geq 2$ and $(m-1)s < n$. Moreover, the immersion is given by*

$$\begin{aligned}\mathbf{x}(t,y_1,\ldots,y_n) = \Big(&ty_1,\ldots,ty_n,\sqrt{t^2-1}\cos B_{-1,-1}(a,\alpha,t),\\ &\sqrt{t^2-1}\sin B_{-1,-1}(a,\alpha,t)\Big),\ y_1^2-y_2^2-\cdots-y_n^2=1.\end{aligned} \tag{9.13}$$

(20) *$n \geq 5$, M is an open portion of $V^n_{a\alpha}$ for some $a > 0$ and some integer $\alpha \in [3-n, -2]$, and the immersion is given by*

$$\begin{aligned}\mathbf{x}(t,y_1,\ldots,y_n) = \Big(&ty_1,\ldots,ty_n,\sqrt{t^2-1}\cos B_{-1,-1}(a,\alpha,t),\\ &\sqrt{t^2-1}\sin B_{-1,-1}(a,\alpha,t)\Big),\ y_1^2-y_2^2-\cdots-y_n^2=1.\end{aligned} \tag{9.14}$$

Proof. Assume that M is a conformally flat ideal hypersurface of $H^{n+1}(-1) \subset \mathbb{E}^{n+2}_1$. When $n = 2$, we have Cases (1), (2) or (3). When $n = 3$, the immersion is either totally umbilical or (2)-ideal. In the first case, we obtain Case (1) for $c > -1$, Case (2) or Case (3). If it is (2)-ideal and non-totally umbilical, then, by Theorem 6 of [11], we have Cases (4)-(12).

Next, suppose that $n \geq 4$. Then, Proposition 6.1 implies that

$$\kappa_1 = \alpha\mu, \quad \kappa_2 = \cdots = \kappa_n = \mu, \tag{9.15}$$

where α is an integer given in Proposition 6.1. If we have $\alpha = 1$ or $\mu = 0$, the immersion is totally umbilical. So, we obtain Cases (1), (2) or (3) for $n \geq 4$. Now, assume that $\alpha \neq 1$ and $\mu \neq 0$. Then, Theorem 4.2 of [3] implies that M is a rotation hypersurface in $H^{n+1}(-1)$. Moreover, from Proposition 6.1, we also know that α is one of the following integers:

(i) $\alpha = 0$;

(ii) $\alpha = 2$ and $n+1$ is not a prime;

(iii) $n \geq 4$ and α is a positive integer in $[3, n-1]$ satisfying $\alpha = ms - n + 1$ for some positive integers s, m with $m \geq 2$ and $n > (m-1)s$;

(iv) $n \geq 4$ and α is a negative integer in $[3-n, -1]$.

Let δ be defined as in Section 4. We have $\delta = 0, 1$ or -1.

Case (a): $\delta = 0$. In this case, the profile curve γ in the 3-dimensional Minkowski space-time endowed with metric

$$ds^2 = 2dxdy + dz^2. \tag{9.16}$$

is given by $\gamma(s) = (x(s), y(s), z(s))$ satisfying

$$2xy + z^2 = -1, \tag{9.17}$$

If x is constant, (5.8) and (5.9) imply that the principal curvatures are given by $\kappa_1 = \cdots = \kappa_n = -1$. Thus, M is an open portion of $\mathbb{E}^n$ and immersed as a totally umbilical hypersurface. Hence, we obtain Case (2) for $n \geq 4$. Next, assume that x is non-constant. Then, we may assume $y = y(x), z = z(x)$. From (9.17) we have

$$\left(\frac{ds}{dx}\right)^2 = 2y'(x) + z'(x)^2. \tag{9.18}$$

By applying Lemma 6.1 and (9.18), we find

$$\frac{1}{x^2 + Ax^{2\alpha}} = 2y'(x) + z'(x)^2. \tag{9.19}$$

As in Section 7, one may prove that A is non-positive. So we can put $A = -a^2$.

On the other hand, (9.17) implies

$$y + xy' + zz' = 0. \tag{9.20}$$

Computing $(z'x - z)^2$ and applying (9.19) and (9.20) yield

$$z'x - z = \frac{ax^\alpha}{\sqrt{x^2 - a^2x^{2\alpha}}}, \tag{9.21}$$

which implies

$$z = x \int^x \frac{at^{\alpha-2}dt}{\sqrt{t^2 - a^2t^{2\alpha}}}. \tag{9.22}$$

From (9.17) we get $y = -(1 + z^2)/2x$. Thus, up to rigid motions, M is given by

$$\mathbf{x}(x, u_1, \ldots, u_{n-1}) = \left(x, y - \frac{x}{2}\sum_{i=1}^{n-1} u_i^2, z, xu_1, \ldots, xu_{n-1}\right) \tag{9.23}$$

in the $(n + 2)$-dimensional Minkowski space-time equipped with the non-standard metric $\tilde{g}_0 = 2dxdy + du_1^2 + du_1^2 + \cdots + du_{n-1}^2$.

Case (a.1): $\alpha = -1$. In this case, we may put

$$z = x\int_x^{\infty}\frac{a\,dt}{t^2\sqrt{t^4-a^2}}. \tag{9.24}$$

On the other hand, we have

$$\begin{aligned}\int_x^{\infty}\frac{a\,dt}{t^2\sqrt{t^4-a^2}} &= \frac{1}{2}\int_{x^2}^{\infty}\frac{a\,du}{u^{3/2}\sqrt{u^2-a^2}} = \frac{1}{2\sqrt{a}}\int_{\sec^{-1}(x^2/a)}^{\frac{\pi}{2}}\sqrt{\cos v}\,dv \\ &= \frac{1}{\sqrt{a}}\left\{E\left(\frac{\pi}{4},\sqrt{2}\right) - E\left(\frac{1}{2}\sec^{-1}\left(\frac{x^2}{a}\right),\sqrt{2}\right)\right\},\end{aligned} \tag{9.25}$$

where the first and the second equalities are obtained by making the substitutions: $t = \sqrt{u}$ and $u = a\sec v$, respectively. Thus, from (9.24) and (9.25) we obtain

$$z = \frac{x}{2}\int_0^{\frac{1}{\sqrt{2b}}\mathrm{sd}^{-1}(\sqrt{2b/x})}\mathrm{sd}^2\left(\sqrt{2}bu\right)du, \quad b = \sqrt{a}. \tag{9.26}$$

By making the substitution: $x = \sqrt{2}b\,\mathrm{ds}\left(\sqrt{2}bt\right)$, we obtain from (9.22) and (9.26) that

$$\begin{aligned} y &= -\frac{b\,\mathrm{ds}\left(\sqrt{2}bt\right)}{\sqrt{2}}\left(\frac{\mathrm{sd}^2\left(\sqrt{2}bt\right)}{2b^2} + \frac{1}{4}\left(\int_0^t \mathrm{sd}^2\left(\sqrt{2}bu\right)du\right)^2\right), \\ z &= \frac{b\,\mathrm{ds}\left(\sqrt{2}bt\right)}{\sqrt{2}}\int_0^t \mathrm{sd}^2\left(\sqrt{2}bu\right)du. \end{aligned}$$

Thus, from (9.22) and $y = -(1+z^2)/2x$, we know that the hypersurface is given by

$$\begin{aligned}\mathbf{x} = \sqrt{2}b\,\mathrm{ds}\left(\sqrt{2}bt\right)\Bigg(1, -\frac{1}{2}\left(\frac{\mathrm{sd}^2\left(\sqrt{2}bt\right)}{2b^2} + \frac{1}{4}\left(\int_0^t \mathrm{sd}^2\left(\sqrt{2}bu\right)du\right)^2 + \sum_{i=1}^{n-1}u_i^2\right), \\ \frac{1}{2}\int_0^t \mathrm{sd}^2\left(\sqrt{2}bu\right)du, u_1, \ldots, u_{n-1}\Bigg).\end{aligned}$$

After applying the transformation: $x_1 = x/4 - 2y, x_2 = x/4 + 2y$, we get Case (15).

Case (a.2): $\alpha = 0$. In this case, (9.22) reduces to $z = ax\int_a^x 1/(t^2\sqrt{t^2-a^2})dt = \sqrt{x^2-a^2}/a$. If we make the substitution: $x = a\cosh t$, then $z = \sinh t$ and $y = -(1/2a)\cosh t$. Thus, fro, (9.23) we know that the hypersurface is represented by

$$\mathbf{x}(t, u_1, \ldots, u_{n-1}) = \cosh t\left(a, -\frac{1}{2a} - \frac{a}{2}\sum_{i=1}^{n-1}u_i^2, \tanh t, au_1, \ldots, au_{n-1}\right)$$

with respect to the nonstandard metric $\tilde{g}_0$. Hence, after applying the following coordinate transformation on the first two coordinates: $x_1 = x/2a - ay, x_2 = x/2a + ay$ and replacing au_i by u_i, we see that M is represented by

$$\text{(9.27)} \qquad \begin{aligned} \mathbf{x}(t, u_1, \ldots, u_{n-1}) = \cosh t \Big(1 + \frac{1}{2}\left(u_1^2 + \cdots + u_{n-1}^2\right), \\ -\frac{1}{2}\left(u_1^2 + \cdots + u_{n-1}^2\right), \tanh t, u_1, \ldots, u_{n-1} \Big) \end{aligned}$$

with respect to the standard Minkowski metric g_0. From (9.27) we know that M is an open portion of G^n. This gives Case (4).

Case (a.3): $\alpha = 2$ *and* $n + 1$ *is not a prime*. In this case, (9.22) reduces to $z = -ax \int_x^{\frac{1}{a}} (1/(t\sqrt{1 - a^2t^2})dt = -ax \cosh^{-1}(1/ax)$. If we make the substitution $x = a^{-1}\operatorname{sech} t$, then $y = -(a/2)(\cosh t + t^2 \operatorname{sech} t), z = -t \operatorname{sech} t$. Thus, from (9.22) and $y = -(1 + z^2)/2x$ we see that M is represented by

$$\mathbf{x}(t, u_1, \ldots, u_{n-1}) = \operatorname{sech} t \left(\frac{1}{a}, -\frac{a}{2}\left(\cosh^2 t + t^2 + \frac{1}{a^2} \sum_{i=1}^{n-1} u_i^2 \right), -t, \frac{u_1}{a}, \ldots, \frac{u_{n-1}}{a} \right)$$

with respect to the nonstandard metric $\tilde{g}_0$. Hence, after applying the following coordinate transformation: $x_1 = ax/4 - 2y/ra, x_2 = ax/4 + 2y/a$ and replacing u_i/a by u_i, we obtain Case (5).

Case (a.4): $3 \le \alpha \le n - 1$ *and* $n \ge 4$ *or* $3 - n \le \alpha \le -2$ *and* $n \ge 5$. If the first case occurs, we have $\alpha = ms - n + 1$ for some positive integers s, m with $m \ge 2$ and $(m - 1)s < n$. If $n - 1 \ge \alpha \ge 3$, we make the substitution $t = xu$ to obtain

$$\text{(9.28)} \qquad \int_0^x \frac{t^{\alpha-2}dt}{\sqrt{t^2 - a^2t^{2\alpha}}} = \int_0^1 \frac{u^{\alpha-3}x^{\alpha-2}du}{\sqrt{1 - a^2x^{2\alpha-2}u^{2\alpha-2}}}.$$

Next, by making the substitution $u = z^{1/(2\alpha-2)}$, we get

$$\begin{aligned} \int_0^1 \frac{u^{\alpha-3}x^{\alpha-2}du}{\sqrt{1 - a^2x^{2\alpha-2}u^{2\alpha-2}}} &= \frac{x^{\alpha-2}}{2\alpha - 2} \int_0^1 z^{-\frac{\alpha}{2\alpha-2}} \left(1 - a^2x^{2\alpha-2}z\right)^{-\frac{1}{2}} dz \\ &= \frac{x^{\alpha-2}}{\alpha - 2} F\left(\frac{1}{2}, \frac{\alpha-2}{2\alpha-2}, \frac{3\alpha-4}{2\alpha-2}, a^2x^{2\alpha-2} \right). \end{aligned}$$

where we apply formula (3.16) of the hypergeometric function.

When $-2 \ge \alpha \ge 3 - n$, we make the substitution $t = x/u$ to obtain

$$\int_x^1 \frac{t^{\alpha-3}dt}{\sqrt{1 - a^2t^{2\alpha-2}}} = \int_0^1 \frac{x^{\alpha-2}u^{1-\alpha}du}{\sqrt{1 - a^2x^{2\alpha-2}u^{2\alpha-2}}}.$$

Next, by making the substitution $u = z^{1/(2-2\alpha)}$, we get

$$\int_0^1 \frac{x^{\alpha-2}u^{1-\alpha}du}{\sqrt{1-a^2x^{2\alpha-2}u^{2\alpha-2}}} = \frac{x^{\alpha-2}}{2-\alpha}F\left(\frac{1}{2},\frac{\alpha-2}{2\alpha-2},\frac{3\alpha-4}{2\alpha-2},a^2x^{2\alpha-2}\right).$$

Hence, in both cases of $\alpha \geq 3$ and $\alpha \leq -2$, we have

$$z = \frac{ax^{\alpha-1}}{2-\alpha}F\left(\frac{1}{2},\frac{\alpha-2}{2\alpha-2},\frac{3\alpha-4}{2\alpha-2},a^2x^{2\alpha-2}\right). \tag{9.29}$$

If we make the substitution: $x = (t/a^2)^{1/(2\alpha-2)}$, then from $y = -(1+z^2)/2x$, (9.23) and (9.29) we know that M is represented by

$$\mathbf{x} = t^{\frac{1}{2\alpha-2}}\left(\frac{1}{\beta}, -\frac{\beta}{2}t^{\frac{1}{1-\alpha}}\left(1+\frac{t}{(2-\alpha)^2}F^2\left(\frac{1}{2},\frac{\alpha-2}{2\alpha-2},\frac{3\alpha-4}{2\alpha-2},t\right)\right)\right.$$
$$\left.-\frac{1}{2\beta}\sum_{i=1}^{n-1}u_i^2, \frac{\sqrt{t}}{2-\alpha}F\left(\frac{1}{2},\frac{\alpha-2}{2\alpha-2},\frac{3\alpha-4}{2\alpha-2},t\right), \frac{1}{\beta}u_1,\ldots,\frac{1}{\beta}u_{n-1}\right), \quad \beta = a^{\frac{1}{\alpha-1}}$$

with respect to the nonstandard metric $\tilde{g}_0$. Therefore, after applying $x_1 = \beta x/4 - 2y/\beta, x_2 = \beta x/4 + 2y/\beta$ and replacing u_i/β by u_i, we obtain Case (17).

Case (b): $\delta = 1$. In this case, the profile curve $\gamma(s) = (x(s), y(s), z(s))$ satisfies

$$x^2 + y^2 - z^2 = -1. \tag{9.30}$$

If x is constant, say b, then the principal curvature are given by $\kappa_1 = -b/\sqrt{1+b^2}$ and $\kappa_2 = \cdots = \kappa_n = -\sqrt{1+b^2}/b$. Thus, by using $\kappa_1 = \alpha\kappa_2$, we obtain $\alpha = b^2/(1+b^2) \in (0,1)$ which is a contradiction. Thus x is non-constant and we may put

$$y = \sqrt{1+x^2}\sinh\phi(x), \quad z = \sqrt{1+x^2}\cosh\phi(x), \tag{9.31}$$

for some function $\phi(x)$. From (9.30) and (9.31) we get

$$\left(\frac{ds}{dx}\right)^2 = 1 + (1+x^2)\phi'^2 - \frac{x^2}{1+x^2}. \tag{9.32}$$

From Lemma 6.1 and (9.32) we obtain

$$\phi(x) = \int \frac{at^\alpha dt}{(1+t^2)\sqrt{1+t^2-a^2t^{2\alpha}}}, \quad a \geq 0. \tag{9.33}$$

Case (b.1): $\alpha = 0$. In this case, (9.33) reduces to

$$\phi(x) = \int \frac{a\,dt}{(1+t^2)\sqrt{1-a^2+t^2}}, \quad a \geq 0. \tag{9.34}$$

We separate this case into three subcases: $a = 1, a > 1$ and $0 < a < 1$.

Case (b.1.1): $a = 1$. In this case, we have

$$\phi(x) = \int_x^1 \frac{dt}{(1+t^2)t} = -\ln\left(\frac{x}{\sqrt{1+x^2}}\right). \tag{9.35}$$

Since

$$\cosh\left(\ln\left(\frac{x}{\sqrt{1+x^2}}\right)\right) = \frac{1+2x^2}{2x\sqrt{1+x^2}}, \quad \sinh\left(\ln\left(\frac{x}{\sqrt{1+x^2}}\right)\right) = \frac{1}{2x\sqrt{1+x^2}},$$

we obtain from (9.31) that $y = 1/2x, z = x + 1/2x$. Therefore, after replacing x by e^s, we obtain Case (6).

Case (b.1.2): $a > 1$. In this case, we have

$$\begin{aligned} \phi(x) &= \int_{\sqrt{a^2-1}}^{x} \frac{a\,dt}{(1+t^2)\sqrt{1-a^2+t^2}} = \tanh^{-1}\left(\frac{\sqrt{1-a^2+x^2}}{ax}\right), \\ \cosh(\phi(x)) &= \frac{ax}{\sqrt{(1+x^2)(a^2-1)}}, \quad \sinh(\phi(x)) = \sqrt{\frac{1-a^2+x^2}{(1+x^2)(a^2-1)}}. \end{aligned} \tag{9.36}$$

Hence, we obtain from (9.31) that $y = \sqrt{(1-a^2+x^2)/(a^2-1)}$, $z = ax/\sqrt{a^2-1}$. If we make the substitution: $x = \sqrt{a^2-1}\cosh t$, then we obtain Case (7).

Case (b.1.3): $0 < a < 1$. In this case, by applying an argument similar to Case (b-1-2), we obtain Case (8).

Case (b.2): $\alpha = 2$ *and* $n+1$ *is not a prime*. In this case, (9.33) reduces to

$$\phi(x) = \int_0^x \frac{at^2\,dt}{(1+t^2)\sqrt{1+t^2-a^2t^4}}, \tag{9.37}$$

which implies $0 < x < \sqrt{2}/\sqrt{\sqrt{1+4a^2}-1}$. If we put $b = (1+4a^2)^{1/4}, k = \sqrt{b^2+1}/\sqrt{2}b$ and $k' = \sqrt{b^2-1}/\sqrt{2}b$, then (9.37) becomes

$$\phi(x) = \int_0^x \frac{at^2\,dt}{(1+t^2)\sqrt{(1+b^2k^2t^2)(1-b^2k'^2t^2)}}. \tag{9.38}$$

If we make the substitution: $t = \mathrm{cn}(bu)/bk'$, we obtain from (9.38) that

$$\phi(x) = -\int_{K/b}^{(1/b)\mathrm{cn}^{-1}(bk'x)} \frac{a\,\mathrm{cn}^2(bu)}{b^2(k'^2 + b^{-2}\mathrm{cn}^2(bu))}du, \tag{9.39}$$

where k is the the modulus and K is the quarter-period of the Jacobi's function. (9.39) can be put in the form:

$$\phi(x) = -\int_{K/b}^{(1/b)\mathrm{cn}^{-1}(bx)} \frac{b\beta\sqrt{(k^2-\beta^2)(k'^2+\beta^2)}\,\mathrm{cn}^2(bu)}{k'^2 + \beta^2\mathrm{cn}^2(bu)}du, \tag{9.40}$$

where $\beta = 1/b$. Hence, by applying formulas (6.24)-(6.27) of [11, page 485], we obtain, up to constants, that

$$\phi(x) = -\frac{k'}{bk}\mathrm{cn}^{-1}(bk'x) + \frac{1}{2}\ln\frac{\Theta(\mathrm{cn}^{-1}(bk'x)-\gamma)}{\Theta(\mathrm{cn}^{-1}(bk'x)+\gamma)} + \mathrm{Z}(\gamma)\mathrm{cn}^{-1}(bk'x)\,,$$

where $\gamma = \mathrm{sn}^{-1}\bigl(1/bk^2\bigr)$. Thus, after making the substitution $t = (1/b)\mathrm{cn}^{-1}(bk'x)$ and choosing a suitable Euclidean coordinate system we obtain Case (11).

Case (b.3): $n \geq 4$ *and* $\alpha = -1$. In this case (9.33) reduces to

$$\phi(x) = \int_x^{\infty} \frac{a\,dt}{(1+t^2)\sqrt{t^2+t^4-a^2}}$$

which implies $x > \sqrt{\sqrt{1+4a^2}-1}/\sqrt{2}$. If we make the substitution $t = 1/u$, we find

$$\phi(x) = \int_0^{1/x} \frac{au^2du}{(1+u^2)\sqrt{1+u^2-a^2u^4}}. \tag{9.41}$$

Hence, by applying the same argument as in Case (b.2), we have

$$\phi(x) = -\frac{k'}{bk}\mathrm{cn}^{-1}\left(\frac{bk'}{x}\right) + \frac{1}{2}\ln\frac{\Theta\left(\mathrm{cn}^{-1}\left(\frac{bk'}{x}\right)-\gamma\right)}{\Theta\left(\mathrm{cn}^{-1}\left(\frac{bk'}{x}\right)+\gamma\right)} + \mathrm{Z}(\gamma)\mathrm{cn}^{-1}\left(\frac{bk'}{x}\right),$$

where $b = (1+4a^2)^{1/4}$, $k = \sqrt{b^2+1}/\sqrt{2}b$, $k' = \sqrt{b^2-1}/\sqrt{2}b$, $\gamma = \mathrm{sn}^{-1}\bigl(1/bk^2\bigr)$. Thus, after making the substitution $t = b^{-1}\mathrm{cn}^{-1}(bk'/x)$ and choosing a suitable coordinate system, we obtain Case (14).

Case (b.4): $n \geq 4$, $\alpha \in [3, n-1]$, *and* $\alpha = ms - n + 1$ *for some positive integers* s, m *satisfying* $m \geq 2$ *and* $(m-1)s < n$. From (3.17), (9.31) and (9.33) we obtain

(9.42)
$$\begin{aligned}\mathbf{x}(t, y_1, \ldots, y_n) = \Big(&\sqrt{1+t^2}\cosh\big(B_{-1,1}(a,\alpha,t)\big),\\ &\sqrt{1+t^2}\sinh\big(B_{-1,1}(a,\alpha,t)\big), ty_1, \ldots, ty_n\Big), \quad y_1^2+y_2^2+\cdots+y_n^2 = 1.\end{aligned}$$

Thus M is an open portion of $U^n_{a\alpha}$ with $a > 0$. This gives Case (18) for positive α.

Case (b.5): $(b.5)$: $n \geq 5$ *and* $\alpha \in [3-n, -2]$. As in Case (b.4), we obtain Case (18) for negative α.

Case (c): $\delta = -1$. In this case, the profile curve $\gamma(s) = (x(s), y(s), z(s))$ satisfies

(9.43)
$$-x^2 + y^2 + z^2 = -1,$$

which implies $x^2 \geq 1$. Without loss of generality, we may assume $x \geq 1$. If x is constant, say b, then the principal curvature of M in $S^{n+1}(1)$ are given by $\kappa_1 = -b/\sqrt{b^2-1}, \kappa_2 = \cdots = \kappa_n = -\sqrt{b^2-1}/b$. Thus, we get $b > 1$. By using the condition $\kappa_1 = \alpha\kappa_2$ from Proposition 6.1, we find

(9.44)
$$b^2 = \frac{\alpha}{\alpha-1}, \quad \alpha = \frac{b^2}{b^2-1}.$$

From $b > 1$ and (9.44), we know that $\alpha \geq 2$ holds. Hence, Proposition 6.1 implies that α is one of the following integers:

(i) $\alpha = 2$ and $n+1$ is not a prime;

(ii) $n \geq 4$ and α is a positive integer in $[3, n-1]$ such that $\alpha = ms - n + 1$ for some positive integers s, m with $m \geq 2$ and $(m-1)s < n$.

From (9.43) and (9.44), we may put

(9.45)
$$y = \frac{\cos t}{\sqrt{\alpha-1}}, \quad z = \frac{\sin t}{\sqrt{\alpha-1}}.$$

Thus, the hypersurface is represented by

(9.46)
$$\mathrm{x}(t, y_1, \ldots, y_n) = \left(\sqrt{\frac{\alpha}{\alpha-1}}\, y_1, \ldots, \sqrt{\frac{\alpha}{\alpha-1}}\, y_n, \frac{\cos t}{\sqrt{\alpha-1}}, \frac{\sin t}{\sqrt{\alpha-1}}\right)$$

for $y_1^2 - y_2^2 - \cdots - y_n^2 = 1$. When $\alpha = 2$ and $n+1$ is not a prime, we obtain Case (10) after making the substitution: $t = \sqrt{\alpha-1}\, s$. When $n \geq 4$ and α is a positive integer in $[3, n-1]$, we obtain Case (13) after making the same substitution.

Next, let us assume that x is non-constant. From (9.43) we may put

$$y = \sqrt{x^2-1}\cos\phi(x), \quad z = \sqrt{x^2-1}\sin\phi(x), \tag{9.47}$$

for some function $\phi(x)$. From (9.43) we get

$$\left(\frac{ds}{dx}\right)^2 = -1 + y'(x)^2 + z'(x)^2. \tag{9.48}$$

From (5.5), (9.47), and (9.48) we obtain

$$\frac{1}{-1+x^2+Ax^{2\alpha}} = -1 + \frac{x^2}{x^2-1} + (x^2-1)\phi'^2 \tag{9.49}$$

which implies $A \leq 0$. So, after setting $A = -a^2$ as before, we obtain

$$\phi'(x) = \frac{ax^\alpha}{(1-x^2)\sqrt{-1+x^2-a^2x^{2\alpha}}}. \tag{9.50}$$

Case (c.1): $\alpha = 0$. In this case, we have

$$\phi(x) = \int_{\sqrt{1+a^2}}^{x} \frac{adt}{(1-t^2)\sqrt{t^2-1-a^2}} = -\tan^{-1}\left(\frac{\sqrt{x^2-1-a^2}}{a}\right). \tag{9.51}$$

Thus, we obtain $y = ax/\sqrt{1+a^2}, z = \sqrt{x^2-1-a^2}/\sqrt{1+a^2}$. Hence, if we put $x = \sqrt{1+a^2}\cosh t$, then we obtain Case (9) for $n \geq 4$.

Case (c.2): $n \geq 4$ *and* $\alpha = -1$. In this case, (9.50) reduces to

$$\phi(x) = \int_x^\infty \frac{adt}{(t^2-1)\sqrt{-t^2+t^4-a^2}},$$

which implies $x > \sqrt{\sqrt{1+4a^2}+1}/\sqrt{2}$. So, after making the substitution: $t = 1/u$, we obtain

$$\phi(x) = \int_0^{1/x} \frac{au^2du}{(1-u^2)\sqrt{1-u^2-a^2u^4}}.$$

Hence, by applying the same argument as Case (b) in Section 8, we obtain

$$\phi(x) = -\frac{k'}{bk}\mathrm{cn}^{-1}\left(\frac{bk'}{x}\right) - \frac{i}{2}\ln\frac{\Theta\left(\mathrm{cn}^{-1}\left(\frac{bk'}{x}\right)-\gamma\right)}{\Theta\left(\mathrm{cn}^{-1}\left(\frac{bk'}{x}\right)+\gamma\right)} - iZ(\gamma)\mathrm{cn}^{-1}\left(\frac{bk'}{x}\right), \tag{9.52}$$

where $b = \left(1+4a^2\right)^{1/4}, k = \sqrt{b^2-1}/\sqrt{2}b, k' = \sqrt{b^2+1}/\sqrt{2}b, \gamma = \mathrm{sn}^{-1}\left(ci/bk^2\right)$. Thus, after making the substitution $t = (1/b)\mathrm{cn}^{-1}(bk'/x)$ and choosing a suitable coordinate system we obtain Case (16).

Case (c.3): $\alpha = 2$ *and* $n+1$ *is not a prime*. In this case we have

$$\phi(x) = \int_{k/b}^{x} \frac{at^2dt}{(1-t^2)\sqrt{-1+t^2-a^2t^4}}, \tag{9.53}$$

where $b = (1-4a^2)^{1/4}$, $k = \sqrt{2}b/\sqrt{1+b^2}$ and $k' = \sqrt{1-b^2}/\sqrt{1+b^2}$. Thus, we find $a < 1/2$. If we make the substitution $t = 1/u$, then we obtain $\phi(x) = -\int_{b/k}^{1/x}(a/((u^2-1)\sqrt{u^2-u^4-a^2}))du$. Hence, by applying the same argument as Case (d.1) in the proof of Section 8, we obtain

$$\phi(x) = -\frac{kk'}{b}\mathrm{dn}^{-1}\left(\frac{bk'x}{k}\right) - \frac{i}{2}\ln\frac{\left(\mathrm{dn}^{-1}\left(\frac{bk'x}{k}\right)-\gamma\right)}{\left(\mathrm{dn}^{-1}\left(\frac{bk'x}{k}\right)+\gamma\right)} - i\,\mathrm{Z}(\gamma)\mathrm{dn}^{-1}\left(\frac{bk'x}{k}\right)$$

where $\gamma = \mathrm{sn}^{-1}(k/b)$. Thus, after making the substitution $t = (k/b)\mathrm{dn}^{-1}(bk'x/k)$ and choosing a suitable coordinate system, we obtain Case (12).

Case (c.4): $n \geq 4$ *and* $\alpha \in [3, n-1]$. This case occurs only when $\alpha = ms - n + 1$ for some positive integers s, m with $m \geq 2$ and $n > (m-1)s$. From (9.50) and using an argument similar to Remark 8.1, we have

$$0 < a < \frac{1}{\sqrt{\alpha}}\left((\alpha-1)/\alpha\right)^{(\alpha-1)/2}.$$

Moreover, from (3.17), (9.47) and (9.50) we know that the hypersurface is given by

$$\begin{aligned} \mathbf{x}(t, y_1, \dots, y_n) = \Big(&ty_1, \dots, ty_n, \sqrt{t^2-1}\cos\left(B_{-1,-1}(a,\alpha,t)\right), \\ &\sqrt{t^2-1}\sin\left(B_{-1,-1}(a,\alpha,t)\right)\Big), \quad y_1^2 - y_2^2 - \cdots - y_n^2 = 1 \end{aligned} \tag{9.54}$$

which implies that M is an open portion of $V^n_{a\alpha}$. This gives Case (19).

Case (c.5): $n \geq 5$ *and* $\alpha \in [3-n, -2]$. In this case, we obtain Case (20) from (3.17), (9.47) and (9.50).

Conversely, we can verify that all of the hypersurfaces listed in the theorem are ideal conformally flat hypersurfaces. ∎

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