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VALUE DISTRIBUTION FOR p-ADIC HYPERSURFACES

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Abstract. The purpose of this paper is to give a p-adic version of value distribution theory for hypersurfaces.

1. Introduction

Nevanlinna theory is a far-reaching generalization of Picard's theorem. There are two main theorems and defect relations which occupy a central place in Nevanlinna theory. Recently, Nevanlinna theory was extended ([1], [6], [7], [10]) to p-adic meromorphic functions on $\mathbb{C}_p$. Khoai ([8]), Khoai - Tu ([11]), and Cherry - Ye ([4]) began to study several variable p-adic Nevanlinna theory, in particular, they established p-adic value distribution theory for the case of hyperplanes. In [8] Khoai gives a p-adic version of the Poisson-Jensen formula for several variable functions. His method is based on the higher dimensional analogs of the valuation polygon. However, the formula obtained in [8] is hard to compute. In [4] Cherry and Ye consider a meromorphic function in several variables and restrict it to a generic line through the origin, and prove that the counting function for this one variable function does not depend on the choice of line through the origin. They use this observation to define counting functions as in the one variable theory, and then a several variable Poisson-Jensen formula follows. Their formula gives the relation between the modulus of a function on the boundary of a ball and the zero set in the ball, while the formula in [8] deals with the zero set on the boundary of a parallelepiped.

In this paper by using the ideas in [8] and some arguments in [4], [11], [12], we give a p-adic version of the Poisson-Jensen formula for several variable functions. Our formula permits to compute the modulus of a function on the boundary of a parallelepied by using information about the zero set. This

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formula lets us to give some results on the value distribution for the case of hypersurfaces. Notice that, in a recent paper ([12] Min Ru also obtained similar results, but for the case of holomorphic curves.

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2. Height of p-Adic Holomorphic Functions of Several Variables

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers and $\mathbb{C}_p$ the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{Q}_p$ is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on $\mathbb{C}_p$ which extends ord_p.

We use the notations

$b_{(m)} = (b_1, ..., b_m)$,

$\widehat{(b_i)} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_m)$,

$D_r = \{z \in \mathbb{C}_p : |z| \le r, r > 0\}$, $\quad D_{<r>} = \{z \in \mathbb{C}_p : |z| = r, r > 0\}$,

$D = \{z \in \mathbb{C}_p : |z| \le 1\}$,

$D_{r_{(m)}} = D_{r_1} \times \cdots \times D_{r_m}$, where $r_{(m)} = (r_1, \ldots, r_m)$ for $r_i \in \mathbb{R}_+$,

$D_{<r_{(m)}>} = D_{<r_1>} \times \cdots \times D_{<r_m>}$,

$D^m = D \times \cdots \times D$ the unit polydisc in $\mathbb{C}_p^m$, $\quad |f|_{r_{(m)}} = |f|_{(r_1,...,r_m)}$,

$\gamma_i \in \mathbb{N}, \gamma = (\gamma_1, ..., \gamma_m)$,

$|\gamma| = \gamma_1 + \cdots + \gamma_m,\ z^\gamma = z_1^{\gamma_1} ... z_m^{\gamma_m},\ r^\gamma = r_1^{\gamma_1} ... r_m^{\gamma_m}$,

$\log = \log_p, \quad t_i = -\log r_i, \quad i = 1, ..., m.$

Notice that the set of $(r_1, ..., r_m) \in \mathbb{R}_+^m$ such that there exist $x_1, ..., x_m \in \mathbb{C}_p$ with $|x_i| = r_i, i = 1, ..., m$, is dense in $\mathbb{R}_+^m$. Therefore, without loss of generality one can assume that $D_{<r_{(m)}>} \neq \emptyset$.

Let f be a non-zero holomorphic function in $D_{r_{(m)}}$ represented by a convergent series

$$f = \sum_{|\gamma| \ge 0} a_\gamma z^\gamma, |z_i| \le r_i \text{ for } i = 1, \ldots, m.$$

We define

$$|f|_{r_{(m)}} = \max_{0 \le |\gamma| < \infty} |a_\gamma| r^\gamma.$$

Set $\gamma t = \gamma_1 t_1 + \cdots + \gamma_m t_m$.

Then we have

$$\lim_{|\gamma| \to \infty} (v(a_\gamma) + \gamma t) = +\infty.$$

Hence, there exists an $\vec{\gamma} \in \mathbb{N}^m$ such that $v(a_\gamma) + \gamma t$ is minimal.

Definition 2.1. *The height of the function $f(z_{(m)})$ is defined by*

$$H_f(t_{(m)}) = \min_{0\le|\gamma|<\infty}(v(a_\gamma)+\gamma t).$$

We also use the notation

$$H_f^+(t_{(m)}) = -H_f(t_{(m)}).$$

Write

$$f(z_{(m)}) = \sum_{k=0}^{\infty} f_{i,k}\widehat{(z_i)}z_i^k, \quad i=1,2,\ldots,m.$$

Set

$$\begin{aligned}
I_f(t_{(m)}) &= \Big\{(\gamma_1,\ldots,\gamma_m)\in\mathbb{N}^m : v(a_\gamma)+\gamma t = H_f(t_{(m)})\Big\}\\
n_{i,f}^+(t_{(m)}) &= \min\Big\{\gamma_i : \exists\ (\gamma_1,\ldots,\gamma_i,\ldots,\gamma_m)\in I_f(t_{(m)})\Big\},\\
n_{i,f}^-(t_{(m)}) &= \max\Big\{\gamma_i : \exists\ (\gamma_1,\ldots,\gamma_i,\ldots,\gamma_m)\in I_f(t_{(m)})\Big\},\\
n_{i,f}(0,0) &= \min\Big\{k : f_{i,k}\widehat{(z_i)}\not\equiv 0\Big\},\\
\nu_f(t_{(m)}) &= \sum_{i=1}^{m}\big((n_{i,f}^-(t_{(m)}) - n_{i,f}^+(t_{(m)})\big).
\end{aligned}$$

Call $\vec{t}$ a *critical point* if $\nu_f(\vec{t})\ne 0$.

Theorem 2.2. *Let $f(z)$ be a holomorphic function on D_r. Assume that f is not identically zero. Then there exist a polynomial*

$$g(z) = b_0 + b_1 z + \cdots + b_v z^v,\ \ deg\ g = n_f^-(t), t = -\log_p r,$$

and a holomorphic function $h = 1 + \sum_{n=1}^{\infty} c_n z^n$ on D_r such that

1) $f(z) = g(z)h(z)$,
2) $f(z)$ just has $n_f^-(t)$ zeros in D_r,
3) $n_f^-(t) - n_f^+(t)$ *is equal to the number of zeros of f at $v(z)=t$,*
4) *h has no zeros in D_r.*

For the proof, see Weierstrass Preparation Theorem [6].

The set of z in $\mathbb{C}_p$ with $|z|\le 1$ forms a closed subring of $\mathbb{C}_p$. We denote this subring by $\mathcal{O}$ (called the ring of integers of $\mathbb{C}_p$), and the set of z with $|z|<1$ forms a maximal ideal I in $\mathcal{O}$. We denote the field $\mathcal{O}/\mathcal{I}$, which is called the residue class field, by $\widehat{\mathbb{C}_p}$. Note that since $\mathbb{C}_p$ is algebraically closed, so is $\widehat{\mathbb{C}_p}$, and in particular $\widehat{\mathbb{C}_p}$ cannot be a finite field. Given an element w in $\mathcal{O}$, denote its equivalence class in $\widehat{\mathbb{C}_p}$ by $\widehat{w}$.

Let $f = \sum_{|\gamma|=0}^{\infty} a_\gamma z^\gamma$ be a non-zero entire function on $\mathbb{C}_p^m$. Choose $y = y_{(m)}$ such that

$$|y| = \max\{|\gamma| : |a_\gamma| = |f|_{(1,\ldots,1)}\}.$$

Define $\widehat{f}$ by

$$\widehat{f} = \sum_{|\gamma|=0}^{\infty} \frac{\widehat{a_\gamma}}{a_y} z^\gamma.$$

Since f is entire, $\left|\frac{a_\gamma}{a_y}\right| < 1$ for all but finitely many γ, and thus $\widehat{f}$ is a polynomial in $m-$variables with coefficients in $\widehat{\mathbb{C}_p}$. Since

$$\left|\frac{a_y}{a_y}\right| = 1,$$

$\widehat{f}$ is not the zero polynomial.

Lemma 2.3. *Let $f_s(z_{(m)}) = \sum_{|\gamma|=0}^{\infty} a_\gamma^s z^\gamma, s = 1, \ldots, q,$ be q non-zero entire functions on $\mathbb{C}_p^m$. Then for any $D_{r_{(m)}}$ in $\mathbb{C}_p^m$ ($D_{<r_{(m)}>} \neq \emptyset$) there exists $u = u_{(m)} \in D_{r_{(m)}}$ such that*

$$\left|f_s(u_{(m)})\right| = \left|f_s\right|_{r_{(m)}}, s = 1, \ldots, q.$$

Proof. First we prove that if $r_{(m)} = (1, \ldots, 1)$, then there exists $w = w_{(m)} \in D^m$ such that

$$|f_s(w)| = \max_{0 \le |\gamma| < \infty} |a_\gamma^s|, s = 1, \ldots, q. \tag{2.1}$$

For each $s = 1, \ldots, q$, choose $y_s = (y_1^s, \ldots, y_m^s)$ such that

$$|y_s| = \max\{|\gamma| : |a_\gamma^s| = |f|_{(1,\ldots,1)}\}.$$

Set

$$\mathcal{M} = \{\widehat{f}_s, s = 1, \ldots, q\}.$$

Since $\widehat{f}_s$ is not the zero polynomial, so is $\prod_{s=1}^{q} \hat{f}_s$.

Let $w = w_{(m)} \in D^m$ be such that $\widehat{w}$ is not a solution of $\prod_{s=1}^{q} \hat{f}_s$.

Set

$$\frac{f_s(w)}{a_{y_s}} = b_s, s = 1, \ldots, q.$$

We have

$$\widehat{b_s} = \widehat{f}_s(\widehat{w}).$$

Since $\widehat{w}$ is not a solution of all $\widehat{f}_s$,

$$b_s \notin I.$$

Thus

$$\Big|\frac{f_s(w)}{a_{y_s}}\Big| = 1.$$

Hence, $|f_s(w)| = |a_{y_s}|$.

Now let $x_1, \ldots, x_m \in \mathbb{C}_p$ such that $|x_i| = r_i$.
Consider the following transformations of $\mathbb{C}_p^m$

$$\varphi(z_{(m)}) = (x_1 z_1, \ldots, x_m z_m).$$

Set

$$x = (x_1, \ldots, x_m).$$

We have

$$\varphi(D^m) = D_{r_{(m)}},$$

and

$$f_s \circ \varphi(z_{(m)}) = \sum_{|\gamma|=0}^{\infty} \big(a_\gamma^s x^\gamma\big) z^\gamma$$

are non-zero entire functions on $\mathbb{C}_p^m$.

By (2.1) there exists $w = w_{(m)}$ such that

$$\begin{aligned}\big|f_s \circ \varphi(w)\big| &= \max_{0 \le |\gamma| < \infty} \big|a_\gamma^s x^\gamma\big| = \max_{0 \le |\gamma| < \infty} \big|a_\gamma^s\big|\big|x_1\big|^{\gamma_1} \cdots \big|x_m\big|^{\gamma_m} \\ &= \max_{0 \le |\gamma| < \infty} \big|a_\gamma^s\big| r^\gamma = \big|f_s\big|_{r_{(m)}}.\end{aligned}$$

Set $u = \varphi(w)$. Then $u \in D_{r_{(m)}}$ and $\big|f_s(u)\big| = \big|f_s\big|_{r_{(m)}},\ s = 1, \ldots, q.$ ∎

Lemma 2.4. *Let $f_s(z_{(m)})$, $s = 1, 2, \ldots, q$, be q non-zero holomorphic functions on $D_{r_{(m)}}$. Then there exists $u = u_{(m)} \in D_{r_{(m)}}$ such that*

$$|f_s(u)| = |f_s|_{r_{(m)}}, s = 1, 2, \ldots, q.$$

Proof. Let

$$f_s(z_{(m)}) = \sum_{|\gamma|=0}^{\infty} a_\gamma^s z^\gamma.$$

For each $s = 1, 2, \ldots, q$, we set

$$k_s = \max_{0 \le |\gamma| < \infty} \Big\{ |\gamma| : \big|a_\gamma^s\big| r^\gamma = \big|f_s\big|_{r_{(m)}} \Big\}.$$

Then

$$P_s = \sum_{0 \le |\gamma| \le k_s} a_\gamma^s z^\gamma, \ s = 1 \ldots, q$$

are non-zero entire functions on $\mathbb{C}_p^m$.

By Lemma 2.3, there exists $u_{(m)} = (u_1, \ldots, u_m) \in D_{r_{(m)}}$ with $|u_i| = r_i$ such that

$$|P_s(u_{(m)})| = |P_s|_{r_{(m)}}, \ s = 1, \ldots, q.$$

Moreover,

$$|P_s|_{r_{(m)}} = |f_s|_{r_{(m)}}, |P_s(u_{(m)})| = |f_s(u_{(m)})|, \ s = 1, \ldots, q.$$

Thus

$$|f_s(u_{(m)})| = |f_s|_{r_{(m)}}, \ s = 1, \ldots, q.$$

■

As an immediate consequence of Lemma 2.4 we have

Corollary 2.5. *Let $f(z_{(m)})$ be a non-zero holomorphic function on $D_{r_{(m)}}$. Then*

$$\max_{u \in D_{r_{(m)}}} |f(u)| = |f|_{r_{(m)}}.$$

3. p-Adic Poisson - Jensen Formula in Several Variables

Let f be a non-zero holomorphic function on $D_{r_{(m)}}$.

Write

$$f(z_{(m)}) = \sum_{k=0}^{\infty} f_{i,k}\widehat{(z_i)} z_i^k, \quad i = 1, 2, \ldots, m.$$

Let

$$n_{i,f}(0,0) = \min\{k : f_{i,k}\widehat{(z_i)} \not\equiv 0\}.$$

For a fixed i $\quad (i = 1, \ldots, m)$ we set for simplicity

$$n_{i,f}(0,0) = \ell, k_1 = n_{i,f}^-(t_{(m)}), \ k_2 = n_{i,f}^+(t_{(m)}).$$

Then there exist multi-indices $\gamma = (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m) \in I_f(t_{(m)})$ and $\mu = (\mu_1, \ldots, \mu_i, \ldots, \mu_m) \in I_f(t_{(m)})$ such that $\gamma_i = k_1$, $\mu_i = k_2$.

We consider the following holomorphic functions on $D_{r_{(m)}}$

$$f_\ell(z_{(m)}) = f_{i,\ell}(\widehat{z_i})z_i^\ell, f_{k_1}(z_{(m)}) = f_{i,k_1}(\widehat{z_i})z_i^{k_1}, f_{k_2}(z_{(m)}) = f_{i,k_2}(\widehat{z_i})z_i^{k_2}.$$

The functions are not identically zero.

Set

$$U_i = \{u = u_{(m)} \in D_{r_{(m)}} : |f_\ell(u)| = |f_\ell|_{r_{(m)}}, |f(u)| = |f|_{r_{(m)}}, \\ |f_{k_1}(u)| = |f_{k_1}|_{r_{(m)}}, |f_{k_2}(u)| = |f_{k_2}|_{r_{(m)}}\},$$

where $i = 1, \ldots, m$.

By Lemma 2.4, U_i is a non-empty set. For each $u \in U_i$, set

$$f_{i,u}(z) = \sum_{k=0}^{\infty} f_{i,k}(\widehat{u_i})z_i^k, \; z = z_i \in D_{r_i}.$$

The following theorem shows that we can use the Weierstrass Preparation Theorem [6] to count zeros by slicing with a generic line through the point u :

Theorem 3.1. *Let $f(z_{(m)})$ be a holomorphic function on $D_{r_{(m)}}$. Assume that $f(z_{(m)})$ is not identically zero. Then for each $i = 1, \ldots, m$, and for all $u \in U_i$, we have*

1) $H_f(t_{(m)}) = H_{f_{i,u}}(t_i)$,
2) $n_{i,f}^-(t_{(m)})$ *is equal to the number of zeros of* $f_{i,u}$ *in* D_{r_i},
3) $n_{i,f}^-(t_{(m)}) - n_{i,f}^+(t_{(m)})$ *is equal to the number of zeros of* $f_{i,u}$ *at* $v(z) = t_i$.

Proof. Set $k_3 = n_{f_{i,u}}^-(t_i)$, $k_4 = n_{f_{i,u}}^+(t_i)$. Since

$$|f(u_{(m)})| = |f|_{r_{(m)}} = |a_\gamma| r_1^{\gamma_1} \ldots r_i^{k_1} \ldots r_m^{\gamma_m} = |a_\mu| r_1^{\mu_1} \ldots r_i^{k_2} \ldots r_m^{\mu_m} \\ = |f_{k_1}|_{r_{(m)}} = |f_{k_2}|_{r_{(m)}},$$

we obtain

$$|f_{i,k_1}(\widehat{u_i})| r_i^{k_1} = |f|_{r_{(m)}} = |f_{i,k_2}(\widehat{u_i})| r_i^{k_2} = |f(u_{(m)})|.$$

On the other hand, we have

$$|f_{i,k_2}(\widehat{u_i})| r_i^{k_2} = |f_{i,k_1}(\widehat{u_i})| r_i^{k_1} \leq |f_{i,u}|_{r_i} \leq |f|_{r_{(m)}}.$$

Hence

$$|f_{i,k_2}(\widehat{u_i})|r_i^{k_2} = |f_{i,u}|_{r_i} = |f_{i,k_1}(\widehat{u_i})|r_i^{k_1}.$$

From this it follows that $k_1 \leq k_3$ and $k_4 \leq k_2$. Now we consider j such that

$$|f_{i,j}(\widehat{u_i})|r_i^j = |f_{i,u}|_{r_i}.$$

Then there exists $\eta = (\eta_1, \ldots, \eta_i, \ldots, \eta_m)$ with $\eta_i = j$ such that

$$\begin{aligned}|f(u_{(m)})| &= |f_{i,u}(u_i)| \leq |f_{i,u}|_{r_i} = |f_{i,j}(\widehat{u_i})|r_i^j \\ &\leq |a_\eta|r^\eta \leq |f|_{r_{(m)}}.\end{aligned}$$

Since

$$|f(u_{(m)})| = |f|_{r_{(m)}},$$

we have

$$|a_\eta|r^\eta = |f|_{r_{(m)}}.$$

Hence $k_2 \leq j \leq k_1$. From this it follows that $k_4 \geq k_2$ and $k_3 \leq k_1$. Since $k_1 \leq k_3$ and $k_2 \geq k_4$, so $k_2 = k_4$ and $k_1 = k_3$. By Lemma 2.4 and Theorem 2.2, we see that $H_f(t_{(m)}) = H_{f_{i,u}}(t_i)$, and $n^-_{i,f}(t_{(m)})$ is equal to the number of zeros of $f_{i,u}$ in D_{r_i} and $n^-_{i,f}(t_{(m)}) - n^+_{i,f}(t_{(m)})$ is equal to the number of zeros of $f_{i,u}$ at t_i.

Theorem 3.1 is proved. ■

For each $i = 1, \ldots, m$, from Theorem 3.1 we see that $n_{i,f}(0,0) = n_{f_u}(0,0)$ for all $u \in U_i$.

Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Define $n_{i,f}(0, r_{(m)})$ to be the number of zeros with absolute value $\leq r_i$ of the one - variable function $f_{i,u}(z)$.

Theorem 3.1 tells us that

$$n_{i,f}(0, r_{(m)}) = n^-_{i,f}(t_{(m)}).$$

For a an element of $\mathbb{C}_p$ and f a holomorphic function on $D_{r_{(m)}}$, which is not identically equal to a, define

$$n_{i,f}(a, r_{(m)}) = n_{i,f-a}(0, r_{(m)}),\ n_{i,f}(a, 0) = n_{i,f-a}(0,0), \quad i = 1, \ldots, m.$$

Fix real numbers $\rho_1, \ldots, \rho_m$ with $0 < \rho_i \leq r_i,\ i = 1, \ldots, m$.

For each $x \in \mathbb{R}$, set

$$A_i(x) = (\rho_1, \ldots, \rho_{i-1}, x, r_{i+1}, \ldots, r_m),\ i = 1, \ldots, m.$$

Define *the counting function* $N_f(a, t_{(m)})$ by

$$N_f(a, t_{(m)}) = \frac{1}{\ln p} \sum_{k=1}^{m} \int_{\rho_k}^{r_k} \frac{n_{k,f}(a, A_k(x))}{x} dx.$$

If $a = 0$, then set $N_f(t_{(m)}) = N_f(0, t_{(m)})$.

For each $t \in \mathbb{R}$, set

$$T_i(t) = (c_1, \ldots, c_{i-1}, t, t_{i+1}, \ldots, t_m),$$

where

$$c_i = -\log \rho_i, i = 1, \ldots, m.$$

Theorem 3.2. (*P-adic Poisson - Jensen Formula in several variables*) *Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Then*

$$H_f^+(t_{(m)}) - H_f^+(c_{(m)}) = N_f(t_{(m)}).$$

Proof. Let

$$f = \sum_{k=0}^{\infty} f_{1,k}(\widehat{z_1}) z_1^k.$$

Set

$$\ell = n_{1,f}(0,0), \quad a = \log \left| f_{1,\ell}(\widehat{z_1}) \right|_{\widehat{r_1}},$$

$$M = \frac{1}{\ln p} \int_{0}^{r_1} n_{1,f}(0, A_1(x)) - \ell x dx + \ell \log r_1,$$

$$M_1 = \frac{1}{\ln p} \int_{0}^{\rho_1} n_{1,f}(0, A_1(x)) - \ell x dx + \ell \log \rho_1,$$

$$M_2 = \frac{1}{\ln p} \int_{\rho_1}^{r_1} n_{1,f}(0, A_1(x)) - \ell x dx + \ell \log \frac{r_1}{\rho_1},$$

$$M_3 = \frac{1}{\ln p} \int_{\rho_1}^{r_1} n_{1,f}(0, A_1(x)) x dx,$$

$$\Gamma = \{T_1(t) : \big(n_{1,f}^- \circ T_1(t) - n_{1,f}^+ \circ T_1(t)\big) \neq 0,\ t \geq t_1\}.$$

We will prove

$$H_f^+(t_{(m)}) - H_f^+ \circ T_1(c_1) = M_3. \tag{3.1}$$

To show (3.1) first prove the following

$$H_f^+(t_{(m)}) - a = M. \tag{3.2}$$

Case 1. $\ell = 0$.

Then

$$M = \frac{1}{\ln p} \int_0^{r_1} \frac{n_{1,f}(0, A_1(x))}{x} dx.$$

If $\Gamma = \emptyset$, then $H_f^+(t_{(m)}) = a$ and $M = 0$. Therefore

$$H_f^+(t_{(m)}) - a = M.$$

If $\Gamma \neq \emptyset$, then Γ is a finite set. Suppose that Γ contains n elements

$$y^{(1)} = T_1(t^{(1)}),$$
$$\vdots$$
$$y^{(n)} = T_1(t^{(n)}),$$

where $t_1 \leq t^{(1)} < t^{(2)} < \cdots < t^{(n)}$.

Set $b_i = p^{-t^{(i)}}$, $i = 1, 2, \ldots, n$, $s = n_{1,f}(0, r_{(m)})$, $s_1 = n_{1,f}(0, A_1(b_2))$, $a_1 = |f_{1,s}(\widehat{z_1})|_{\widehat{r_1}}$, $a_2 = H_f^+ \circ T_1(t^{(1)})$, $a_3 = H_f^+ \circ T_1(t^{(2)})$, $a_4 = |f_{1,s_1}(\widehat{z_1})|_{\widehat{r_1}}$. Then $0 < b_n < b_{n-1} < \cdots < b_1 \leq r_1$. We will prove (3.2) by induction on n.

Case $n = 1$.

If $b_1 = r_1$, then $n_{1,f}(0, A_1(x)) = 0$, $0 < x < r_1$. Moreover, by the continuity of $H_f^+ \circ T_1(t)$, we obtain (3.2). Consider $b_1 < r_1$. We have

$$M = s\big(\log r_1 - \log b_1\big) = \log(a_1 r_1^s) - \log(a_1 b_1^s).$$

Since $b_1 < r_1$ and $n = 1$,

$$H_f^+(t_{(m)}) = \log(a_1 r_1^s).$$

Furthermore, $T_1(t) \notin \Gamma$ with $t > t^{(1)}$ and $H_f^+ \circ T_1(t)$ is continuous.

Thus

$$\log\left(a_1 b_1^s\right) = a.$$

Hence (3.2) follows. So (3.2) is proved in this case.

Now we will prove (3.2) for any n.

Case $b_1 < r_1$.

Then $0 < b_n < b_{n-1} \cdots < b_1 < r_1$ and $t_1 < t^{(1)} < \cdots < t^{(n)}$. Apply the induction hypothesis,

$$\frac{1}{\ln p}\int_0^{b_1} \frac{n_{1,f}(0, A_1(x))}{x} dx = a_2 - a.$$

Thus

$$M = a_2 - a + \frac{1}{\ln p}\int_{b_1}^{r_1} \frac{n_{1,f}(0, A_1(x))}{x} dx.$$

On the other hand,

$$\begin{aligned} \frac{1}{\ln p}\int_{b_1}^{r_1} \frac{n_{1,f}(0, A_1(x))}{x} dx &= s\big(\log r_1 - \log b_1\big) \\ &= \log(a_1 r_1^s) - \log(a_1 b_1^s), \\ a_2 &= \log(a_1 b_1^s). \end{aligned}$$

Since $T_1(t) \notin \Gamma$ with $t_1 \le t < t^{(1)}$,

$$H_f^+(t_{(m)}) = \log(a_1 r_1^s). \tag{3.4}$$

By (3.3) and (3.4),

$$M = H_f^+(t_{(m)}) - a.$$

Case $b_1 = r_1$.

Then $0 < b_n < \ldots < b_2 < b_1 = r_1$ and $t_1 = t^{(1)} < \ldots < t^{(n)}$. Apply the induction hypothesis,

$$\frac{1}{\ln p}\int_0^{b_2} \frac{n_{1,f}(0, A_1(x))}{x} dx = a_3 - a.$$

Thus

$$M = a_3 - a + \frac{1}{\ln p}\int_{b_2}^{b_1} \frac{n_{1,f}(0, A_1(x))}{x} dx. \tag{3.5}$$

Moreover, $n_{1,f}(0, A_1(x)) = s_1$ with $b_2 \le x < b_1$, and

$$\frac{1}{\ln p}\int_{b_2}^{b_1} \frac{n_{1,f}(0, A_1(x))}{x} dx = s_1(\log b_1 - \log b_2) = \log\left(a_4 b_1^{s_1}\right) - \log\left(a_4 b_2^{s_1}\right),$$

$$a_3 = \log(a_4 b_2^{s_1}).$$

Since $T_1(t) \notin \Gamma$ with $t^{(1)} < t < t^{(2)}$, and by the continuity of $H_f^+ \circ T_1(t)$,

$$H_f^+(t_m) = \log(a_4 b_1^{s_1}).$$

Since (3.5) and (3.6), we obtain

$$M = H_f^+(t_m) - a.$$

Case $\ell \neq 0$.

Then $f = f_1 f_2$ with $f_1 = z_1^{\ell}$. We have

$$n_{1,f_2}(0,0) = 0,$$
$$n_{1,f}(0,0) = \ell, n_{1,f}(0, A_1(x)) = n_{1,f_2}(0, A_1(x)) + \ell,$$

$$H_f^+(t_{(m)}) = H_{f_1}^+(t_{(m)}) + H_{f_2}^+(t_{(m)}) = \ell \log r_1 + H_{f_2}^+(t_{(m)}).$$

By case $\ell = 0$,

$$\frac{1}{\ln p} \int_0^{r_1} \frac{n_{1,f}(0, A_1(x))}{x} dx = H_{f_2}^+(t_{(m)}) - a.$$

Thus

$$M = H_{f_2}^+(t_{(m)}) - a + \ell \log r_1 = H_f^+(t_{(m)}) - a.$$

Similarly we obtain

$$M_1 = H_f^+ \circ T_1(c_1) - a. \tag{3.7}$$

We have

$$M = M_1 + M_2, \quad M_3 = M_2.$$

Apply (3.2) and (3.7),

$$M_3 = M - M_1 = H_f^+(t_{(m)}) - H_f^+ \circ T_1(c_1).$$

Similarly we have

(3.8)

$$H_f^+ \circ T_{i-1}(c_{i-1}) - H_f^+ \circ T_i(c_i) = \frac{1}{\ln p} \int_{\rho_i}^{r_i} \frac{n_{1,f}(0, A_1(x))}{x} dx \quad \text{for } i = 2, \ldots, m.$$

Apply (3.8),

$$H_f^+(t_{(m)}) - H_f^+ \circ T_m(c_m) = H_f^+(t_{(m)}) - H_f^+ \circ T_1(c_1) + H_f^+ \circ T_1(c_1) - \dots$$
$$+ H_f^+ \circ T_{m-1}(c_{m-1}) - H_f^+ \circ T_m(c_m),$$

we obtain

$$H_f^+(t_{(m)}) - H_f^+(c_{(m)}) = N_f(t_{(m)}).$$ ∎

4. Value Distribution on p-Adic Hypersufaces

We say that an entire function g *divides* an entire function f if $f = gh$ for some entire function h, and we say that g is a *greatest common divisor* of n entire functions $f_1, \dots, f_n$ if whenever an entire function h divides each of non-zero f_i then h also divides g. We say that n entire functions $f_1, \dots, f_n$ are *without common factors* if 1 is a greatest common divisor.

Note that greatest common divisors exist in the ring of entire functions on $\mathbb{C}_p^m$ (see [4]).

By a *holomorphic map*

$$f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n(\mathbb{C}_p) = \mathbb{P}^n,$$

we mean an equivalence class of $(n+1)-$tuples of entire functions $(f_1, \dots, f_{n+1})$ such that $f_1, \dots, f_{n+1}$ do not have any common factors in the ring of entire functions on $\mathbb{C}_p^m$, where two $(n+1)-$tuples $(f_1, \dots, f_{n+1})$ and $(g_1, \dots, g_{n+1})$ are equivalent if there exists a constant c such that $f_i = cg_i$ for all i. We identify f with its representation by a collection of entire functions on $\mathbb{C}_p^m$

$$f = (f_1, \dots, f_{n+1}).$$

Definition 4.1. *The height of a holomorphic map f is defined by*

$$H_f(t_{(m)}) = \min_{1 \le i \le n+1} H_{f_i}(t_{(m)}).$$

We also use the notation

$$H_f^+(t_{(m)}) = -H_f(t_{(m)}).$$

A family $Q_1, \dots, Q_q$ of polynomials in $n+1$ variables with coefficients in $\mathbb{C}_p$ $(q \ge n+1)$ is said to be *admissible* if any set of $n+1$ polynomials in this family has no common zeros in $\mathbb{C}_p^{n+1} - \{0\}$.

Let X_i be hypersurfaces in $\mathbb{P}^n$ defined by the equations $Q_i = 0$, $i = 1, \dots, q$, where Q_i are homogeneous polynomials of degree d_i. $X_1, \dots, X_q$, $q \geq n+1$ are said to be in *general position* if the family $Q_1, \dots, Q_q$ is admissible.

Let X be a hypersurface of $\mathbb{P}^n$ such that the image of f is not contained in X, and X is defined by the equation $Q = 0$.

We set

$$N_f(X, t_{(m)}) = N_{Q\circ f}(t_{(m)}), \quad m_f(X, t_{(m)}) = \max_{1\leq i\leq n+1} \big(H^+_{f_i^d}(t_{(m)}) - H^+_{Q\circ f}(t_{(m)})\big),$$

$$T_f(X, t_{(m)}) = N_f(X, t_{(m)}) + m_f(X, t_{(m)}), \quad H_f(X, t_{(m)}) = H_{Q\circ f}(t_{(m)}),$$

$$H^+_f(X, t_{(m)}) = -H_f(X, t_{(m)}).$$

Notice that $H_f(t_{(m)})$ and $H_f(X, t_{(m)})$ are well defined upto an additive constant.

Theorem 4.2. (*first main theorem*). *Let $f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ be a holomorphic map. Let X be a hypersurface of degree d in $\mathbb{P}^n$ such that the image of f is not contained in X. Then*

$$T_f(X, t_{(m)}) = dH^+(t_{(m)}) + 0(1),$$

where the $0(1)$ *is bounded when* $T = \max_{1\leq i\leq m} t_i \to -\infty$.

Proof. Let $f = (f_1, \dots, f_{n+1})$. By definition,

$$\begin{aligned} T_f(X, t_{(m)}) &= N_{Q\circ f}(t_{(m)}) + \max_{1\leq i\leq n+1} (H^+_{f_i^d}(t_{(m)}) - H^+_{Q\circ f}(t_{(m)})) \\ &= dH^+_f(t_{(m)}) + (N_{Q\circ f}(t_{(m)}) - H^+_{Q\circ f}(t_{(m)})). \end{aligned}$$

By Theorem 3.2,

$$N_{Q\circ f}(t_{(m)}) - H^+_{Q\circ f}(t_{(m)}) = O(1),$$

Therefore,

$$T_f(X, t_{(m)}) = dH^+_f(t_{(m)}) + O(1).$$

Theorem 4.2 is proved. ■

Theorem 4.3. (second main theorem). *Let $f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ be a non-constant holomorphic map, and let X_i be hypersurfaces of degree d in general position in $\mathbb{P}^n$, such that the image of f is not contained in $X_i, i = 1, \dots, q$. Then*

$$(q-n)H^+_f(t_{(m)}) \leq \sum_{i=1}^{q} \frac{N_f(X_i, t_{(m)})}{d_i} + 0(1),$$

where $0(1)$ *is bounded when* $T = \max_{1\le i\le m} t_i \to -\infty$.

Proof. To show, first suppose that $d_1 = d_2 = \cdots = d_q = d$, and X_i are defined by the equations

$$Q_i(x_1,\dots,x_{n+1}) = 0 \quad \text{with} \quad i = 1,\dots,q.$$

Now let, for a fixed $t_{(m)}$, the following inequalities hold

$$H_{Q_q\circ f}(t_{(m)}) \le H_{Q_{q-1}\circ f}(t_{(m)}) \le \cdots \le H_{Q_1\circ f}(t_{(m)}). \tag{4.1}$$

From the hypothesis of general position, the Hilbert's Nullstellensatz [13] implies that for any integer $k, 1 \le k \le n+1$, there is an integer $m_k \ge d$ such that

$$x_k^{m_k} = \sum_{i=1}^{n+1} a_{ik}(x_1,\dots,x_{n+1})Q_i(x_1,\dots,x_{n+1}),$$

where $a_{ik}(x_1,\dots,x_{n+1})$, $1 \le i \le n+1$, $1 \le k \le n+1$, are homogeneous polynomials with coefficients in $\mathbb{C}_p$ of degree $m_k - d$.

Therefore

$$f_k^{m_k} = \sum_{i=1}^{n+1} a_{ik}(f_1,\dots,f_{n+1})Q_i(f_1,\dots,f_{n+1}),\ k = 1,\dots,n+1.$$

From this it follows that

$$\begin{aligned} H_{f_k^{m_k}}(t_{(m)}) = m_k H_{f_k}(t_{(m)}) &\ge (m_k - d)H_f(t_{(m)}) \\ &\qquad + \min_{1\le i\le n+1} H_{Q_i\circ f}(t_{(m)}) + 0(1) \\ &= (m_k - d)H_f(t_{(m)}) \\ &\qquad + H_{Q_{n+1}\circ f}(t_{(m)}) + 0(1), \end{aligned}$$

where $0(1)$ is bounded when $T = \max_{1\le i\le m} t_i \to -\infty$. So

$$dH_{f_k}(t_{(m)}) \ge H_{Q_i\circ f}(t_{(m)}) + 0(1) \quad \text{for} \quad i = n+1,\dots,q. \tag{4.2}$$

Notice that if $Q_i\circ f$ is not a constant, then $H_{Q_i\circ f}(t_{(m)}) \to -\infty$ when $T \to -\infty$, $i = 1,\dots,q$. Thus, by (4.1) and (4.2)

$$d(q-n)H_f(t_{(m)}) \ge \sum_{i=1}^{q} H_{Q_i\circ f}(t_{(m)}) + 0(1).$$

Hence

$$d(q-n)H_f^+(t_{(m)}) \le \sum_{i=1}^{q} H^+_{Q_i \circ f}(t_{(m)}) + 0(1).$$

By Theorem 3.2, we obtain

$$H^+_{Q_i \circ f}(t_{(m)}) = N_{Q_i \circ f}(t_{(m)}) + 0(1).$$

Thus

$$d(q-n)H_f^+(t_{(m)}) \le \sum_{i=1}^{q} N_f(X_i, t_{(m)}) + 0(1). \tag{4.3}$$

Now we can return to the proof of Theorem 4.3. We set

$$d = d_1 \dots d_q \text{ and write } d = d_i k_i, \ i = 1, \dots, q.$$

Let Y_i be the hypersurfaces in $\mathbb{P}^n$ defined by the equations $Q_i^{k_i} = 0, \ i = 1, \dots, q$. Then Y_i are hypersurfaces of degree d in general position in $\mathbb{P}^n$. On the other hand, $Q_i^{k_i} \circ f$ not identically zero.

Thus, by (4.3),

$$d(q-n)H_f^+(t_{(m)}) \le \sum_{i=1}^{q} N_f(Y_i, t_{(m)}) + 0(1).$$

Since

$$N_f(Y_i, t_{(m)}) = k_i N_f(X_i, t_{(m)}),$$

so

$$(q-n)H_f^+(t_{(m)}) \le \sum_{i=1}^{q} \frac{N_f(X_i, t_{(m)})}{d_i} + 0(1).$$

Theorem 4.3 is proved. ■

Let $f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ be a holomorphic map and let X be a hypersurface of degree d in $\mathbb{P}^n$ such that the image of f is not contained in X. Then we define the *defect* $\delta_f(X)$ of f for the hypersurface X to be

$$\delta_f(X) = \lim_{T \to -\infty} \inf \left\{ 1 - \frac{N_f(X, t_{(m)})}{dH_f^+(t_{(m)})} \right\},$$

where $T = \max\limits_{1 \le i \le m} t_i$.

Theorem 4.3 implies the following

Theorem 4.4. (defect relation) *Let $f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ be a non-constant holomorphic map and let X_i be hypersurfaces of degree d_i in general position in $\mathbb{P}^n$ such that the image of f is not contained in $X_i, i = 1, \ldots, q$. Then*

$$\sum_{i=1}^{q} \delta_f(X_i) \leq n.$$

In particular, we have the following

Theorem 4.5. *Let $f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ be a holomorphic map and let X_i be hypersurfaces of degree d_i in general position in $\mathbb{P}^n$ such that the image of f omits $X_i, i = 1, \ldots, q$. Then f must be constant.*

Remark: Theorems 4.3, 4.4 and 4.5 are sharp by the following example:

Let $X_1, \ldots, X_{n+1}$ be the coordinate hyperplanes in projective space $\mathbb{P}^n(\mathbb{C}_p)$ and let $f = (1, 2, \ldots, n, z) : \mathbb{C}_p^m \to \mathbb{P}^n(\mathbb{C}_p)$. Then $f(\mathbb{C}_p^m)$ omits the first n coordinate hyperplanes, but f is non-constant.

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