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CONVERGENCE OF DOUBLE LAGUERRE SERIES

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Abstract. Let $s_{mn}(x,y)$ denote the rectangular partial sums of the double Laguerre series with the coefficients $\{c_{jk}\}$. We give sufficient conditions on $\{c_{jk}\}$ to obtain the regular convergence and weighted L^r-convergence of $s_{mn}(x,y)$.

1. Introduction

For $a \geq 0$, let $L_n^a(t)$ denote the n-th Laguerre polynomial of order a defined by

$$L_n^a(t) = \frac{1}{n!} t^{-a} e^t \frac{d^n}{dt^n}(t^{n+a}e^{-t}), \qquad n = 0, 1, 2, \cdots .$$

Then $\{L_n^a(t)\}_{n=0}^{\infty}$ forms a complete orthogonal set in $L^2(\mathbb{R}^+, t^a e^{-t}dt)$. The problems of the mean convergence and the pointwise convergence of different types of Laguerre series (including those with respect to the systems $\{\mathcal{L}_n^a(t)\}_{n=0}^{\infty}$ and $\{l_n^a(t)\}_{n=0}^{\infty}$) have been studied by many authors in the last four decades, e.g., Askey-Wainger [1], Chen-Lin [3], Długosz [4], Muckenhoupt [6, 7, 8], and Stempak [9, 10, 11].

In this paper, we consider the following double Laguerre series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk} L_j^a(x) L_k^a(y), \qquad x, y \in \mathbb{R}^+, \tag{1.1}$$

where $\{c_{jk} : j, k \geq 0\}$ satisfies the following conditions for some $p \in \mathbb{N}$:

$$|c_{jk}|\,(\bar{j}\bar{k})^{p/2-1/4}\,\mathcal{L}(\bar{j}, \bar{k}) \longrightarrow 0 \quad \text{as} \quad \max\{j,k\} \to \infty, \tag{1.2}$$

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$$\lim_{n\to\infty}\sum_{j=0}^{\infty}|\Delta_{p0}c_{jn}|(\hat{j}\hat{n})^{p/2-1/4}\Phi(\hat{j},\hat{n})=0, \tag{1.3}$$

$$\lim_{m\to\infty}\sum_{k=0}^{\infty}|\Delta_{0p}c_{mk}|(\hat{m}\hat{k})^{p/2-1/4}\Phi(\hat{m},\hat{k})=0, \tag{1.4}$$

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{pp}c_{jk}|(\hat{j}\hat{k})^{p/2-1/4}\Phi(\hat{j},\hat{k})<\infty. \tag{1.5}$$

Here $\hat{\xi}\equiv\max\{\xi,1\}$, Φ is a suitable positive function on $[1,\infty)\times[1,\infty)$, and the finite-order differences $\Delta_{pq}c_{jk}$ are defined by

$$\Delta_{pq}c_{jk}=\sum_{s=0}^{p}\sum_{t=0}^{q}(-1)^{s+t}\binom{p}{s}\binom{q}{t}c_{j+s,k+t}.$$

The conditions $(1.2)-(1.5)$ describe certain concept of bounded variation, which are closely related to those in [3, Theorem 2]. We pay attention to the following two cases:

(i) $\Phi(x,y)=(xy)^{a/2}$,

(ii) $\Phi(x,y)=(xy)^{a/2}\theta(x)\vartheta(y)$,

where θ,ϑ are two positive increasing functions defined on $[1,\infty)$.

Let $s_{mn}(x,y)$ denote the rectangular partial sums of series (1.1) defined by

$$s_{mn}(x,y)=\sum_{j=0}^{m}\sum_{k=0}^{n}c_{jk}L_j^a(x)L_k^a(y).$$

We say that the series (1.1) *converges regularly* to $f(x,y)$ if $s_{mn}(x,y)\to f(x,y)$ as $\min\{m,n\}\to\infty$, the row series $\sum_{j=0}^{\infty}c_{jk}L_j^a(x)L_k^a(y)$ converges for each fixed value of k, and the column series $\sum_{k=0}^{\infty}c_{jk}L_j^a(x)L_k^a(y)$ converges for each fixed value of j (*cf.* [5]). For $E\subseteq\mathbb{R}^+\times\mathbb{R}^+$, the series (1.1) is said to *converge uniformly* on E to $f(x,y)$ if $s_{mn}(x,y)\to f(x,y)$ uniformly on E as $\min\{m,n\}\to\infty$. Set

$$\|f\|_{r,\phi}=\left(\int_0^{\infty}\int_0^{\infty}|f(x,y)|^r|\phi(x,y)|\,dxdy\right)^{1/r}.$$

Note that $\|\cdot\|_{r,\phi}^r$ defines a metric for $0<r<1$, and $\|\cdot\|_{r,\phi}$ is a norm for $r\geq 1$.

In this paper, we are concerned with the following convergence problems of series (1.1):

(i) the regular convergence and its mean convergence with respect to $\|\cdot\|_{r,\phi}$ for $0 < r < 1$ and £$(x,y) = (xy)^{a/2}$;

(ii) the regular convergence and its mean convergence with respect to $\|\cdot\|_{r,\phi}$ for $r \geq 1$ and £$(x,y) = (xy)^{a/2}\theta(x)\vartheta(y)$.

As a corollary of case (ii), we obtain a result which has the same format as [8, Theorem 12] (see Corollary 3.2). A detailed argument on these problems will be given in next two sections. Throughout this paper C, C_p, and C_{ap} denote constants, which are not necessarily the same at each occurrence.

2. Convergence for $0 < r < 1$ and £$(x,y) = (xy)^{a/2}$

Let $\{c_{jk} : j,k \geq 0\}$ satisfy conditions (1.2) – (1.5) for £$(x,y) = (xy)^{a/2}$ with $a \geq 0$; that is,

(1.2′) $$|c_{jk}|\,(\bar{j}\bar{k})^{(a+p)/2-1/4} \longrightarrow 0 \quad \text{as} \quad \max\{j,k\} \to \infty,$$

(1.3′) $$\lim_{n\to\infty} \sum_{j=0}^{\infty} |\Delta_{p0} c_{jn}|\,(\bar{j}\bar{n})^{(a+p)/2-1/4} = 0,$$

(1.4′) $$\lim_{m\to\infty} \sum_{k=0}^{\infty} |\Delta_{0p} c_{mk}|\,(\bar{m}\bar{k})^{(a+p)/2-1/4} = 0,$$

(1.5′) $$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\Delta_{pp} c_{jk}|\,(\bar{j}\bar{k})^{(a+p)/2-1/4} < \infty.$$

The main result in this section reads as follows.

Theorem 2.1. *Let $a \geq 0$ and $p \geq 1$. Assume that $\{c_{jk}\}$ satisfies conditions (1.2′) – (1.5′). Then the series (1.1) converges regularly to some function $f(x,y)$ for all $x,y > 0$, and the convergence is uniform on any rectangle $\{\epsilon \leq x \leq \alpha, \delta \leq y \leq \beta\}$, where $0 < \epsilon < \alpha < \infty$ and $0 < \delta < \beta < \infty$. Moreover, for $0 < r < 1/p$, $\lambda > r$, and $\tau \geq r(a/2 - 1/4)$, we have*

(i) $|f(x,y)|^r e^{-\lambda(x+y)/2}(xy)^\tau \in L^1(\mathbb{R}^+ \times \mathbb{R}^+)$;

(ii) $\|s_{mn} - f\|_{r,\phi} \to 0$ *as* $\min\{m,n\} \to \infty$, *where* $\phi(x,y) = O\left(e^{-\lambda(x+y)/2}(xy)^\tau\right)$.

The conclusion (i) of Theorem 2.1 displays the Lebesgue integrability of $|f(x,y)|^r$ with respect to the weight function $e^{-\lambda(x+y)/2}(xy)^\tau$, where $f(x,y)$ is the limiting function of series (1.1). The case $\lambda = 1$ and $\tau = a/2$ reduces the weight function into the form $e^{-(x+y)/2}(xy)^{a/2}$. This is the two-dimensional case of the weight function $e^{-t/2}t^{a/2}$, which appears in the definition of the Laguerre function $\mathcal{L}_n^a(t)$ and is involved in the inequality of the form

$$\|s_n(t)U(t)\|_p \le C\,\|f(t)V(t)\|_p$$

considered in [1, 8]. As for $\lambda = r$ and $\tau = r(a+p)/2$, it is of interest in connection with the weight function

$$\left(\frac{t}{1+t}\right)^{\kappa}(1+t)^{\mu}(1+\log^+ t)^{\nu}e^{-tr/2}t^{r(a+p)/2},$$

which plays the role of $[U(t)]^r$ considered in [8, Theorems 7-12] (see Corollary 3.2 for further comments regarding this).

Proof of Theorem 2.1. From [1, 8] we can find an absolute constant C such that

$$\left|L_j^b(t)\right| \le Ce^{t/2}t^{-b/2-1/4}(1\vee j)^{b/2-1/4}(1\vee t)^{1/2},$$

for all j, all t, and all $b = a, a+1, \cdots, a+p$. For $x, y > 0$ and $m, n > 0$, the summation by parts and the equation $\sum_{k=0}^{n} L_k^a(t) = L_n^{a+1}(t)$ (cf. [12, Eq. (5.1.13)]) yield

$$\begin{aligned} s_{mn}(x,y) = {} & \sum_{j=0}^{m}\sum_{k=0}^{n}(\Delta_{pp}c_{jk})L_j^{a+p}(x)L_k^{a+p}(y) \\ & + \sum_{t=0}^{p-1}\sum_{j=0}^{m}(\Delta_{pt}c_{j,n+1})L_j^{a+p}(x)L_n^{a+t+1}(y) \\ & + \sum_{s=0}^{p-1}\sum_{k=0}^{n}(\Delta_{sp}c_{m+1,k})L_m^{a+s+1}(x)L_k^{a+p}(y) \\ & + \sum_{s=0}^{p-1}\sum_{t=0}^{p-1}(\Delta_{st}c_{m+1,n+1})L_m^{a+s+1}(x)L_n^{a+t+1}(y). \end{aligned} \tag{2.2}$$

Using (2.1), we get the following estimates:

$$\begin{aligned} & \sum_{j=0}^{m}\sum_{k=0}^{n}\left|(\Delta_{pp}c_{jk})L_j^{a+p}(x)L_k^{a+p}(y)\right| \\ & \le C\left(\sum_{j=0}^{m}\sum_{k=0}^{n}|\Delta_{pp}c_{jk}|\,((1\vee j)(1\vee k))^{(a+p)/2-1/4}\right) \\ & \quad \times e^{(x+y)/2}(xy)^{-(a+p)/2-1/4}((1\vee x)(1\vee y))^{1/2} \end{aligned} \tag{2.3}$$

and

$$
\begin{aligned}
&\sum_{t=0}^{p-1}\sum_{j=0}^{m}\left|(\Delta_{pt}c_{j,n+1})L_j^{a+p}(x)L_n^{a+t+1}(y)\right|\\
&\quad\le C\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\left(\sum_{j=0}^{m}|\Delta_{p0}c_{j,n+1+v}|\,(jn)^{(a+p)/2-1/4}\right)\\
&\qquad\times e^{(x+y)/2}x^{-(a+p)/2-1/4}y^{-(a+t+1)/2-1/4}(xy)^{1/2}\\
&\quad\le C\left(\sup_{k>n}\sum_{j=0}^{m}|\Delta_{p0}c_{jk}|\,(jk)^{(a+p)/2-1/4}\right)\\
&\qquad\times e^{(x+y)/2}x^{-(a+p)/2-1/4}(xy)^{1/2}\left(\sum_{t=0}^{p-1}2^t y^{-(a+t+1)/2-1/4}\right).
\end{aligned}
\tag{2.4}
$$

Similarly, we have

$$
\begin{aligned}
&\sum_{s=0}^{p-1}\sum_{k=0}^{n}\left|(\Delta_{sp}c_{m+1,k})L_m^{a+s+1}(x)L_k^{a+p}(y)\right|\\
&\quad\le C\left(\sup_{j>m}\sum_{k=0}^{n}|\Delta_{0p}c_{jk}|\,(jk)^{(a+p)/2-1/4}\right)\\
&\qquad\times e^{(x+y)/2}y^{-(a+p)/2-1/4}(xy)^{1/2}\left(\sum_{s=0}^{p-1}2^s x^{-(a+s+1)/2-1/4}\right)
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
&\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\left|(\Delta_{st}c_{m+1,n+1})L_m^{a+s+1}(x)L_n^{a+t+1}(y)\right|\\
&\quad\le C\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}\left|\Delta_{00}c_{m+1+u,n+1+v}\right|(mn)^{(a+p)/2-1/4}\\
&\qquad\times e^{(x+y)/2}(xy)^{1/2}x^{-(a+s+1)/2-1/4}y^{-(a+t+1)/2-1/4}\\
&\quad\le C\left(\sup_{j>m,k>n}|c_{jk}|\,(jk)^{(a+p)/2-1/4}\right)e^{(x+y)/2}(xy)^{1/2}\\
&\qquad\times\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}2^{s+t}x^{-(a+s+1)/2-1/4}y^{-(a+t+1)/2-1/4}.
\end{aligned}
\tag{2.6}
$$

Putting (1.2′) – (1.5′) and (2.2) – (2.6) together, we infer that $s_{mn}(x,y)$ converges to some function $f(x,y)$ for $x,y>0$, and the convergence is uniform on any rectangle $\{\epsilon \le x \le \alpha, \delta \le y \le \beta\}$, where $0 < \epsilon < \alpha < \infty$ and $0 < \delta < \beta < \infty$. The same argument as the above also shows that series (1.1) converges regularly to $f(x,y)$ for $x, y > 0$. Moreover,

$$f(x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\Delta_{pp}c_{jk})L_j^{a+p}(x)L_k^{a+p}(y). \tag{2.7}$$

Let $0 < r < 1/p$, $\lambda > r$, and $\tau \ge r(a/2 - 1/4)$. We have $r - \lambda < 0$ and $\tau - r\left(\frac{a+p}{2} + \frac{1}{4}\right) > -1$. This implies

$$\begin{aligned}\Psi(\alpha,\beta) &\equiv \int_0^\infty\int_0^\infty e^{(r-\lambda)(x+y)/2}x^{\tau - r(\alpha/2+1/4)}y^{\tau-r(\beta/2+1/4)}(\hat{x}\hat{y})^{r/2}\,dxdy,\\ &< \infty\end{aligned}$$

for all $\alpha, \beta = a, a+1, \cdots, a+p$. By (2.3) and (2.7), we infer that

$$\begin{aligned}&\int_0^\infty\int_0^\infty |f(x,y)|^r e^{-\lambda(x+y)/2}(xy)^\tau\,dxdy\\ &\le C\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{pp}c_{jk}|\,(\hat{j}\hat{k})^{(a+p)/2-1/4}\right\}^r \Psi(a+p, a+p)\\ &< \infty,\end{aligned}$$

which says that $|f(x,y)|^r e^{-\lambda(x+y)/2}(xy)^\tau \in L^1(\mathrm{R}^+ \times \mathrm{R}^+)$. Let $\phi(x,y) = O(e^{-\lambda(x+y)/2}(xy)^\tau)$. Set $\Lambda_{mn} \equiv \{(j,k) \in \mathrm{Z}^+ \times \mathrm{Z}^+ : j > m \text{ or } k > n\}$. We have $0 < r < 1$. Using $(1.2') - (1.5')$ and $(2.2) - (2.7)$, we obtain

$$\begin{aligned}&\|s_{mn} - f\|_{r,\phi}^r\\ &\le C\left(\sum_{\Lambda_{mn}}|\Delta_{pp}c_{jk}|\,(\hat{j}\hat{k})^{(a+p)/2-1/4}\right)^r \Psi(a+p, a+p)\\ &\quad + C\left(\sup_{k>n}\sum_{j=0}^{m}|\Delta_{p0}c_{jk}|\,(\hat{j}\hat{k})^{(a+p)/2-1/4}\right)^r \sum_{t=0}^{p-1}2^{tr}\,\Psi(a+p, a+t+1)\\ &\quad + C\left(\sup_{j>m}\sum_{k=0}^{n}|\Delta_{0p}c_{jk}|\,(\hat{j}\hat{k})^{(a+p)/2-1/4}\right)^r \sum_{s=0}^{p-1}2^{sr}\,\Psi(a+s+1, a+p)\\ &\quad + C\left(\sup_{j>m,k>n}|c_{jk}|\,(\hat{j}\hat{k})^{(a+p)/2-1/4}\right)^r \sum_{s=0}^{p-1}\sum_{t=0}^{p-1}2^{r(s+t)}\,\Psi(a+s+1, a+t+1)\\ &\longrightarrow 0 \quad \text{as} \quad \min\{m,n\} \to \infty.\end{aligned}$$

3. Convergence for $r \geq 1$ and $\Phi(x,y) = (xy)^{a/2}\theta(x)\vartheta(y)$

Theorem 2.1 deals with the case $0 < r < 1$ only. In this section we investigate the validity of the following statements for $r \geq 1$:

$$|f(x,y)|^r \phi(x)\psi(y) \in L^1(\mathbb{R}^+ \times \mathbb{R}^+), \tag{3.1}$$

$$\int_0^\infty \int_0^\infty |s_{mn}(x,y) - f(x,y)|^r |\phi(x)\psi(y)|\, dxdy \longrightarrow 0 \quad \text{as} \quad \min\{m,n\} \to \infty, \tag{3.2}$$

where ϕ and ψ are two measurable functions on $\mathbb{R}^+$. This corresponds to the case $\phi(x,y) = \phi(x)\psi(y)$ in Theorem 2.1. To ensure the truth of (3.1) and (3.2), we shall replace $(1.2') - (1.5')$ by the following stronger conditions:

$$|c_{jk}|\,(\hat{j}\hat{k})^{(a+p)/2-1/4}\,\theta(\hat{j})\vartheta(\hat{k}) \longrightarrow 0 \quad \text{as} \quad \max\{j,k\} \to \infty, \tag{1.2''}$$

$$\lim_{n\to\infty} \sum_{j=0}^{\infty} |\Delta_{p0} c_{jn}|\,(\hat{j}\hat{n})^{(a+p)/2-1/4}\,\theta(\hat{j})\vartheta(\hat{n}) = 0, \tag{1.3''}$$

$$\lim_{m\to\infty} \sum_{k=0}^{\infty} |\Delta_{0p} c_{mk}|\,(\hat{m}\hat{k})^{(a+p)/2-1/4}\,\theta(\hat{m})\vartheta(\hat{k}) = 0, \tag{1.4''}$$

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\Delta_{pp} c_{jk}|\,(\hat{j}\hat{k})^{(a+p)/2-1/4}\,\theta(\hat{j})\vartheta(\hat{k}) < \infty, \tag{1.5''}$$

where θ and ϑ are two positive increasing functions defined on $[1,\infty)$. For this purpose, we introduce the concept of "type I_α^r" below, which is an analogue of "type I" in [2]. We say that (ϕ,θ) is a pair of *type* I_α^r if there is an absolute constant C such that

$$\rho^{1/4}\left(\int_0^{1/\rho} e^{tr/2} t^{-\alpha r/2} |\phi(t)|\, dt\right)^{1/r} + \rho^{1/6}\left(\int_{1/\rho}^\infty e^{tr/2} t^{-\alpha r/2 - r/4} |\phi(t)|\, dt\right)^{1/r} \leq C\theta(\rho) \quad \text{for all} \quad \rho \geq 1.$$

Theorem 3.1. *Let $a \geq 0$ and $p, r \geq 1$. Assume that θ and ϑ are two positive increasing functions defined on $[1,\infty)$ and $\{c_{jk}\}$ satisfies $(1.2'') - (1.5'')$. Then series (1.1) converges regularly to some function $f(x,y)$ for all $x, y > 0$, and the convergence is uniform on any rectangle $\{\epsilon \leq x \leq \alpha, \delta \leq y \leq \beta\}$, where $0 < \epsilon < \alpha < \infty$ and $0 < \delta < \beta < \infty$. Moreover, if (ϕ,θ) and (ψ,ϑ) are of type I_{a+p}^r, then (3.1) and (3.2) hold.*

An elementary calculation says that the type I_α^r pair (ϕ, θ) can be chosen from the following type of functions mentioned in [8]:

$$\phi(t) = O\left(\left(\frac{t}{1+t}\right)^{\kappa}(1+t)^{\mu}(1+\log^+ t)^{\nu} e^{-tr/2} t^{\alpha r/2}\right),$$

where $\kappa > -1$, $\mu \le r/4 - 1$, $\nu < -1$, $\log^+ t = \max\{\log t, 0\}$, and

$$\theta(\rho) = \begin{cases} \rho^{-(\kappa+1)/r+5/12} & \text{for } \; -1 < \kappa < r/4 - 1 \\ \rho^{1/6}\max\{(\log\rho)^{1/r}, 1\} & \text{for } \; \kappa = r/4 - 1 \\ \rho^{1/6} & \text{for } \; \kappa > r/4 - 1 \end{cases}.$$

In particular, (ϕ, θ) and (ψ, ϑ) are of type I_{a+p}^r, where $\theta(\rho) = \vartheta(\rho) = \rho^{1/6}$, $\phi(x) = [U(x)]^r$, $\psi(y) = [V(y)]^r$, and

$$U(x) = \left(\frac{x}{1+x}\right)^{\kappa_1}(1+x)^{\mu_1}(1+\log^+ x)^{\nu_1} e^{-x/2} x^{(a+p)/2},$$

$$V(y) = \left(\frac{y}{1+y}\right)^{\kappa_2}(1+y)^{\mu_2}(1+\log^+ y)^{\nu_2} e^{-y/2} y^{(a+p)/2}.$$

Here we assume that $\kappa_j > 1/4 - 1/r$, $\mu_j \le 1/4 - 1/r$, and $\nu_j < -1/r$ for $j = 1$ and 2. In this case, conditions $(1.2'') - (1.5'')$ become

(1.2''') $$|c_{jk}|\left(\frac{1}{j}\frac{1}{k}\right)^{(a+p)/2-1/12} \longrightarrow 0 \quad \text{as} \quad \max\{j,k\} \to \infty,$$

(1.3''') $$\lim_{n\to\infty} \sum_{j=0}^{\infty} |\Delta_{p0} c_{jn}| \left(\frac{1}{j}\frac{1}{n}\right)^{(a+p)/2-1/12} = 0,$$

(1.4''') $$\lim_{m\to\infty} \sum_{k=0}^{\infty} |\Delta_{0p} c_{mk}| \left(\frac{1}{m}\frac{1}{k}\right)^{(a+p)/2-1/12} = 0,$$

(1.5''') $$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\Delta_{pp} c_{jk}| \left(\frac{1}{j}\frac{1}{k}\right)^{(a+p)/2-1/12} < \infty.$$

Hence Theorem 3.1 has the following consequence, which gives the same format as [8, Theorem 12].

Corollary 3.2. *Let $a \ge 0$, $p, r \ge 1$, $\{c_{jk}\}$ satisfy $(1.2''') - (1.5''')$, and U, V be given as above. Then series (1.1) converges regularly to some function $f(x,y)$ for all $x, y > 0$, and the convergence is uniform on any rectangle $\{\epsilon \le x \le$*

$\alpha, \delta \le y \le \beta\}$, *where* $0 < \epsilon < \alpha < \infty$ *and* $0 < \delta < \beta < \infty$. *Moreover*, $f(x,y)U(x)V(y) \in L^r(\mathbb{R}^+ \times \mathbb{R}^+)$ *and*

$$\int_0^\infty \int_0^\infty \Big(|s_{mn}(x,y) - f(x,y)||U(x)V(y)| \Big)^r dxdy \longrightarrow 0 \quad \text{as} \quad \min\{m,n\} \to \infty.$$

To prove Theorem 3.1, we need the following lemma.

Lemma 3.3. *Let* $a \ge 0$, $p, r \ge 1$, *and* (ϕ, θ) *be of type* I^r_{a+p}. *Then there exists an absolute constant* C *such that*

$$\left(\int_0^\infty |L_j^{a+k}(t)|^r |\phi(t)|\, dt \right)^{1/r} \le C\, \hat{j}^{(a+p)/2-1/4}\theta(\hat{j})$$

for all j *and for all* $0 \le k \le p$.

Proof of Lemma 3.3. From [1, 8] we can find two positive constants C and γ, independent of α, t, and j, such that

$$\left| L_j^\alpha(t) \right| \le \begin{cases} C\, e^{t/2}\, j^\alpha & \text{if } 0 \le t \le 1/\nu \\ C\, e^{t/2}\, t^{-\alpha/2-1/4}\, j^{\alpha/2-1/4} & \text{if } 1/\nu < t \le \nu/2 \\ C\, e^{t/2}\, t^{-\alpha/2}\, j^{\alpha/2-1/3} & \text{if } \nu/2 < t \le 3\nu/2 \\ C\, e^{t/2-\gamma t}\, t^{-\alpha/2}\, j^{\alpha/2} & \text{if } 3\nu/2 < t < \infty \end{cases},$$

where $j \ge 1, \nu = 4j + 2\alpha + 2$, and $\alpha = a, a+1, \cdots, a+p$. For $1/\nu \le t \le 1/j$, we have $|t^{-\alpha/2-1/4}\, j^{\alpha/2-1/4}| \le Cj^\alpha$, and so the inequality $|L_j^\alpha(t)| \le Ce^{t/2}j^\alpha$ can be extended from $[0, 1/\nu]$ to $[0, 1/j]$. Obviously, we have

$$\sup_{t \ge 0, 0 \le k \le p} |e^{-\gamma t} t^{(p-k+1)/2}| < \infty.$$

Hence, for $0 \le k \le p$ and $\rho = \hat{j}$,

$$\begin{aligned} &\left(\int_0^\infty |L_j^{a+k}(t)|^r |\phi(t)|\, dt \right)^{1/r} \\ &\le C \Bigg\{ \hat{j}^{a+k} \left(\int_0^{1/\rho} e^{tr/2} |\phi(t)|\, dt \right)^{1/r} \\ &\quad + \hat{j}^{(a+k)/2-1/4} \left(\int_{1/\rho}^{\nu/2} e^{tr/2} t^{-(a+k)r/2-r/4} |\phi(t)|\, dt \right)^{1/r} \\ &\quad + \hat{j}^{(a+k)/2-1/3} \left(\int_{\nu/2}^{3\nu/2} e^{tr/2} t^{-(a+k)r/2} |\phi(t)|\, dt \right)^{1/r} \\ &\quad + \hat{j}^{(a+k)/2} \left(\int_{3\nu/2}^{\infty} e^{tr/2-\gamma tr} t^{-(a+k)r/2} |\phi(t)|\, dt \right)^{1/r} \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&\le C\left(\tfrac{1}{j}\right)^{(a+p)/2-1/4}\left\{\rho^{1/4}\left(\int_0^{1/\rho} e^{tr/2}t^{-(a+p)r/2}|\phi(t)|\,dt\right)^{1/r}\right.\\
&\qquad+\left(\int_{1/\rho}^{\nu/2} e^{tr/2}t^{-(a+p)r/2-r/4}|\phi(t)|\,dt\right)^{1/r}\\
&\qquad+\rho^{1/6}\left(\int_{\nu/2}^{3\nu/2} e^{tr/2}t^{-(a+p)r/2-r/4}|\phi(t)|\,dt\right)^{1/r}\\
&\qquad\left.+\left(\int_{3\nu/2}^{\infty} e^{tr/2}t^{-(a+p)r/2-r/4}|\phi(t)|\,dt\right)^{1/r}\right\}\\
&\le C\left(\tfrac{1}{j}\right)^{(a+p)/2-1/4}\theta\left(\tfrac{1}{j}\right).
\end{aligned}
$$

Proof of Theorem 3.1. Obviously, conditions $(1.2'')-(1.5'')$ imply $(1.2')-(1.5')$. Hence, by Theorem 2.1, series (1.1) converges regularly to some function $f(x,y)$ for all $x, y > 0$, and the convergence is uniform on any rectangle $\{\epsilon \le x \le \alpha, \delta \le y \le \beta\}$, where $0 < \epsilon < \alpha < \infty$ and $0 < \delta < \beta < \infty$. As proved in Theorem 2.1, for $x, y > 0$, we have

$$
\sum_{j=0}^{m}\sum_{k=0}^{n}(\Delta_{pp}c_{jk})\,L_j^{a+p}(x)L_k^{a+p}(y) \longrightarrow f(x,y) \qquad \text{as } \min\{m,n\}\to\infty. \tag{3.3}
$$

Set

$$
\alpha_j^k = \left(\int_0^\infty |L_j^{a+k}(x)|^r|\phi(x)|\,dx\right)^{1/r}, \quad \beta_j^k = \left(\int_0^\infty |L_j^{a+k}(y)|^r|\psi(y)|\,dy\right)^{1/r}.
$$

Lemma 3.3 tells us that $\alpha_j^k \le C\left(\frac{1}{j}\right)^{(a+p)/2-1/4}\theta\left(\frac{1}{j}\right)$ and $\beta_j^k \le C\left(\frac{1}{j}\right)^{(a+p)/2-1/4}\vartheta\left(\frac{1}{j}\right)$, where $0 \le k \le p$. By (3.3), Fatou's lemma, and Minkowski's inequality, we infer that

$$
\begin{aligned}
&\left(\int_0^\infty\int_0^\infty |f(x,y)|^r|\phi(x)\psi(y)|\,dxdy\right)^{1/r}\\
&\quad\le C\left\{\liminf_{m\to\infty}\sum_{j=0}^{m}\sum_{k=0}^{m}|\Delta_{pp}c_{jk}|\,\alpha_j^p\beta_k^p\right\}\\
&\quad\le C\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{pp}c_{jk}|\left(\tfrac{1}{j}\tfrac{1}{k}\right)^{(a+p)/2-1/4}\theta\left(\tfrac{1}{j}\right)\vartheta\left(\tfrac{1}{k}\right)\right\}\\
&\quad<\infty.
\end{aligned}
$$

Moreover, let Ω_{mn} consist of all (j,k) with $j > m$ or $k > n$. By (2.2), (3.3), and

(1.2″) − (1.5″), we get

$$
\begin{aligned}
&\left(\int_0^\infty\int_0^\infty |s_{mn}(x,y)-f(x,y)|^r|\phi(x)\psi(y)|\,dxdy\right)^{1/r}\\
&\le C\Bigg\{\sum_{\square_{mn}} |\Delta_{pp}c_{jk}|\,(jk)^{(a+p)/2-1/4}\,\theta(j)\vartheta(k)\\
&\qquad+\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\left(\sum_{j=0}^{m}|\Delta_{p0}c_{j,n+1+v}|\,(jn)^{(a+p)/2-1/4}\theta(j)\vartheta(n)\right)\\
&\qquad+\sum_{s=0}^{p-1}\sum_{u=0}^{s}\binom{s}{u}\left(\sum_{k=0}^{n}|\Delta_{0p}c_{m+1+u,k}|\,(mk)^{(a+p)/2-1/4}\theta(m)\vartheta(k)\right)\\
&\qquad+\sum_{s,t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}|\Delta_{00}c_{m+1+u,n+1+v}|\,(mn)^{(a+p)/2-1/4}\,\theta(m)\vartheta(n)\Bigg\}\\
&\le C\Bigg\{\sum_{\square_{mn}} |\Delta_{pp}c_{jk}|\,(jk)^{(a+p)/2-1/4}\,\theta(j)\vartheta(k)\\
&\qquad+2^p\ \sup_{k>n}\left(\sum_{j=0}^{m}|\Delta_{p0}c_{jk}|\,(jk)^{(a+p)/2-1/4}\,\theta(j)\vartheta(k)\right)\\
&\qquad+2^p\ \sup_{j>m}\left(\sum_{k=0}^{n}|\Delta_{0p}c_{jk}|\,(jk)^{(a+p)/2-1/4}\,\theta(j)\vartheta(k)\right)\\
&\qquad+2^{2p}\ \sup_{j>m,k>n}\left(|c_{jk}|\,(jk)^{(a+p)/2-1/4}\,\theta(j)\vartheta(k)\right)\Bigg\}\\
&\longrightarrow 0 \qquad \text{as}\quad \min\{m,n\}\to\infty.
\end{aligned}
$$

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