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ON $(d, 2)$-DOMINATING NUMBERS OF BUTTERFLY NETWORKS[*][†]

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Abstract. In this paper, we study $(d, 2)$-dominating numbers for an important class of parallel networks - butterfly networks $B(n)$. The main result of this paper is to determine their $(d, 2)$-dominating numbers for $2n-1 \leq d \leq 2n+1$.

1. INTRODUCTION

In this paper, we use graphs to represent networks. We use [1] for terminology and notation not defined here. In addition, the length of a path $P[v_1, v_{p+1}] := v_1 \to v_2 \to v_3 \to \cdots \to v_p \to v_{p+1}$ is the number p of edges of P and will be denoted by $| P |$, where v_1 and v_{p+1} are called end-vertices of P and $v_2, v_3, \cdots, v_p$ are called internal vertices. For a nonempty and proper subset S of the vertex set $V(G)$ and $x \in V(G - S)$, an (x, S)-path is a path in G connecting x to some vertex in S.

The *butterfly network* $B(n)$ is the graph whose vertices are $x = (x_0, x_1, \cdots, x_n)$ with $0 \leq x_0 \leq n$ and $x_i \in \{0, 1\}$ for $1 \leq i \leq n$, and two vertices $x = (x_0, x_1, \cdots, x_n)$ and $y = (y_0, y_1, \cdots, y_n)$ are adjacent if and only if $y_0 = x_0 + 1$ and $x_i = y_i$ for $1 \leq i \leq n$ with $i \neq y_0$. Note that $B(1)$ is a 4-cycle. For a vertex $x = (x_0, x_1, \cdots, x_n)$ in $B(n)$, we say that x is in *level* x_0 of $B(n)$ and call x_i the *ith coordinate* of x. Fig. 1 shows an example of $B(3)$, in which the top line indicates the level numbers and the left column indicates the names $(x_1, x_2, \cdots, x_n)$.

Let T denote the vertices in level 0, it is easy to know $B(n) - T$ is two disjoint butterfly networks $B(n-1)$, one denoted by $B(n-1)^1$ has all vertices x with $x_1 = 0$ and $x_0 \neq 0$; the other denoted by $B(n-1)^2$ has all vertices x with $x_1 = 1$ and $x_0 \neq 0$. Cao, Du, Hsu, and Wan [2] have shown that $B(n)$ is 2-connected and its diameter is equal to $2n$.

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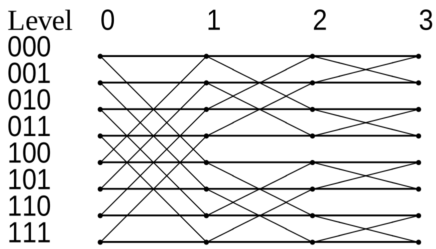

FIG. 1. The butterfly network $B(3)$

In order to characterize the reliability of transmission delay in a network, Hsu and Lyuu [5] introduce m-diameter (i.e. wide-diameter) as follows: For any pair $(x; y)$ of vertices in a graph G, the minimum integer d suth that there are at least m internally vertex-disjoint path of length at most d between x and y is called the m-distance of x and y and is denoted by $D_m(x; y)_G$. The m-diameter of G, denoted by $D_m(G)$, is the maximum of $D_m(x; y)_G$ over all pairs $(x; y)$ of vertices of G. General results on the m-diameters of m-connected graphs can be found in [4] and [5]. Results for some particular classes of graphs can be also found in [6], [7] and [9]. In particular, for a Butterfly network $B(n)$, its 2-diameter is $2n + 2$ for $n \geq 2$. (see [9]).

Recently, H. Li and J. M. Xu in [8] define a new parameter $(d; m)$-*dominating number* in m-connected graphs, in some sense, which can more accurately characterize the reliability of networks than the wide-diameter can.

Definition. Let G be a m-connected graphs, S a nonempty and proper subset of $V(G)$, y a vertex in $G - S$. For a given positive integer d, y is $(d; m)$-dominated by S in the graph if there are at least m internally vertex-disjoint $(y; S)$-paths in G such that each of which is of length at most d. S is said to be a $(d; m)$-dominating set of G, denoted by $S_{d;m}(G)$ if either $S = V(G)$ or S can $(d; m)$-dominate every vertex in $G - S$. The parameter

$$s_{d;m}(G) = \min\{|S_{d;m}(G)| : S_{d;m}(G) \text{ is a } (d; m) - \text{dominating set of } G\}$$

will be called the $(d; m)$-dominating number of G.

Remark 1. $(d; m)$-dominating number can be used to explain such a quesition: Let G be a communication network of the Department of National Defence. Given integer $d > 0$ and $m > 0$. How many command centers are necessary and sufficient such that there exist at least m internally disjoint paths of length at most d between each fight unit and these command centers? And how to select these vertices in G

as command centers? Results on (d, m)-dominating number can be found in [8], [10], [11].

In this paper, we will prove for $n \geq 3$, the $(d, 2)$-dominating number of $B(n)$ is 2 for $2n - 1 \leq d \leq 2n + 1$.

2. Preliminary Results

In order to prove the theorem, we first give some lemmas.

Lemma 1. *Let* G *be an* m*-connected* $(m \geq 2)$ *graph of order* n *and* d *a positive integer*, *then*

(a) *If* $d = D_m(G)$, *then* $s_{d,m}(G) = 1$;

(b) *If* $d' > d''$, *then* $s_{d',m}(G) \leq s_{d'',m}(G)$.

Proof. (a) and (b) can be obtained directly by the definitions. ∎

Lemma 2. *For butterfly networks* $B(n)$, $(n \geq 2)$, $s_{2n+2,2}(B(n)) = 1$.

Proof. Since 2-diameter of $B(n)$ is $2n + 2$, it is easy to prove $s_{2n+2,2}(B(n)) = 1$. ∎

Lemma 2 shows that it is interesting to determine $(d, 2)$-dominating numbers of $B(n)$ when $d \leq 2n + 1$, and lemma 1 shows that it is sufficient to prove $s_{2n-1,2}(B(n)) = 2$ and $s_{2n+1,2}(B(n)) > 1$ in order to prove the main results.

Lemma 3. *For any* $x = (0, x_1, x_2, \cdots, x_n)$ *in* $V(B(n) - S)$, *where* $S = \{(0, 0, \cdots, 0), (0, 1, \cdots, 1)\}$. *If* $x_n = 1$, *there exists a path of length no more than* $2n-2$ *between* x *and* $(0, 1, \cdots, 1)$, *otherwise*, *there exists a path of length no more than* $2n - 2$ *between* x *and* $(0, 0, \cdots, 0)$.

Proof. Without loss of generality, we assume that $x_n = 1$ and let w denote the binary string $(x_1, x_2, \cdots, x_n)$. Let $w^{i_1 i_2 \cdots i_g} = (x_1, x_2, \cdots, x_{i_1 - 1}, \overline{x}_{i_1}, x_{i_1+1}, \cdots, x_{i_2 - 1}, \overline{x}_{i_2}, x_{i_2+1}, \cdots, x_{i_g - 1}, \overline{x}_{i_g}, x_{i_g+1}, \cdots, x_n)$, where $\overline{x}_i = 1 - x_i$. Suppose that $x_{i_1} = x_{i_2} = \cdots = x_{i_g} = 0$ and $x_{i_{g+1}} = x_{i_{g+2}} = \cdots = x_{i_n} = 1$ where $\{i_1, i_2, \cdots, i_g, i_{g+1}, \cdots, i_n\} = \{1, 2, \cdots, n\}$ and $i_1 < i_2 < \cdots < i_g \neq n$, We construct the path between x and $(0, 1, \cdots, 1)$ as follows:

$$\begin{aligned} P : x &\to (1, w) \to \cdots \to (i_1 - 1, w) \to (i_1, w^{i_1}) \to (i_1 + 1, w^{i_1}) \to \cdots \\ &\to (i_2 - 1, w^{i_1}) \to (i_2, w^{i_1 i_2}) \to (i_2 + 1, w^{i_1 i_2}) \to \cdots \to (i_g, w^{i_1 i_2 \cdots i_g}) \\ &\to (i_g - 1, w^{i_1 i_2 \cdots i_g}) \to (i_g - 2, w^{i_1 i_2 \cdots i_g}) \to \cdots \to (0, w^{i_1 i_2 \cdots i_g}) \\ &= (0, 1, \cdots, 1). \end{aligned}$$

We easily know that $|P| = 2i_g \leq 2n - 2$. ■

Lemma 4. *For* $u = (k-1; 0; \cdots; 0)$ *and* $v = (k+1; 1; \cdots; 1)$ *in* $B(n)$ $(n \geq 3; 1 \leq k \leq \frac{n}{2})$; $P[u; v]$ *must pass the vertex* $w = (w_0; w_1; \cdots; w_n)$ *if* $|P[u; v]| \leq 2n+1$; *where* $w_0 = k$; $w_1 = \cdots = w_k = 1$ *and* $w_{k+1} = \cdots = w_n = 0$.

Proof. First $P[u; v]$ must pass some vertex x in level 0 of $B(n)$ since the first coordinates of u and v are distinct; Similarly, $P[u; v]$ must pass some vertex y in level n of $B(n)$ since the last coordinates of u and v are distinct. If $P[u; v]$ first pass y then x, we easily know $|P[u; v]| = |P[u; y]| + |P[y; x]| + |P[x; v]| \geq (n-k+1)+n+(k+1) = 2n+2$, a contradiction. So, $P[u; v]$ must first pass x then y and $|P[u; v]|$ is no less than $2n-2$ for $|P[u; v]| = |P[u; x]| + |P[x; y]| + |P[y; v]| \geq (k-1) + n + (n-k-1) = 2n-2$. If $P[u; v]$ has only one vertex t in level k, then $t_{k+1} = \cdots = t_n = 0$ since all vertices of $P[u; t]$ are in level less than $k+1$ in $B(n)$ and $u_{k+1} = \cdots = u_n = 0$. We also know $t_1 = \cdots = t_k = 1$ since all vertices of $P[v; t]$ are in level no less than k and $v_1 = \cdots = v_k = 1$. i.e., $t = w$. Note that it is impossible that $P[u; v]$ has more than two vertices in level k. (If not, we easily find $|P[u; v]|$ is more than $2n+1$.) We assume $P[u; v]$ has just two vertices t and z in level k of $B(n)$. Without loss of generality, we say t is the first vertex in level k which is in $P[u; v]$. Obviously, z is the last vertex in level k which is in $P[u; v]$. If all vertices of $P[v; t]$ are in level no less than k, then we know $t = w$ as above. If all vertices of $P[u; z]$ are in level no more than k, then we also know $z = w$. ■

Remark 2. We can easily find the following result from the proof of Lemma 4. For $u = (k-1; u_1; \cdots; u_n)$ and $v = (k+1; \overline{u_1}; \cdots; \overline{u_n})$ in $B(n)$ $(n \geq 3, 1 \leq k \leq n-1)$, $P[u; v]$ must pass the vertex $w = (k; \overline{u_1}; \cdots; \overline{u_k}; u_{k+1}; \cdots; u_n)$ if $|P[u; v]| \leq 2n+1$.

We can easily find the following mappings are automorphisms of $B(n)$:

$$\alpha_i : (x_0; x_1; \cdots; x_n) \to (x_0; x_1; \cdots; x_{i-1}; \overline{x_i}; x_{i+1}; \cdots; x_n) \quad (1 \leq i \leq n)$$

$$\beta : (x_0; x_1; \cdots; x_n) \to (n - x_0; x_n; \cdots; x_1)$$

These are useful in the proof of our main results.

3. The Main Results

Theorem 1. *The* $(d; 2)$*-dominating number of* B_n $(n \geq 3)$ *is* 2 *for* $d = 2n-1$.

Proof. Now we prove $S = \{s_1 = (0; 0; \cdots; 0); s_2 = (0; 1; \cdots; 1)\}$ is a $(2n-1; 2)$-dominating set of $B(n)$ $(n \geq 3)$. For any $x \in V(B(n) - S)$, we shall construct two vertex-disjoint paths between x and S, each of which has length no more than $2n-1$.

Case 1. $x = (x_0, x_1, \cdots, x_n)$ with $x_0 \geq 1$.

Suppose that $x_{i_1} = x_{i_2} = \cdots = x_{i_g} = 0$ and $x_{i_{g+1}} = x_{i_{g+2}} = \cdots = x_{i_n} = 1$ where $\{i_1, i_2, \cdots, i_g, i_{g+1}, \cdots, i_n\} = \{1, 2, \cdots, n\}$ and $i_1 < i_2 < \cdots < i_g$, $i_{g+1} < i_{g+2} < \cdots < i_n$. Without loss of generality, we assume $i_{t-1} \leq x_0 \leq i_t$ where $i_t \in \{i_1, i_2, \cdots, i_g\}$.

$$\begin{aligned} P_1 : x &\to (x_0+1, w) \to \cdots \to (i_t-1, w) \to (i_t, w^{i_t}) \to (i_t+1, w^{i_t}) \to \cdots \\ &\to (i_{t+1}-1, w^{i_t}) \to (i_{t+1}, w^{i_t i_{t+1}}) \to (i_{t+1}+1, w^{i_t i_{t+1}}) \to \cdots \\ &\to (i_g, w^{i_t i_{t+1}\cdots i_g}) \to (i_g-1, w^{i_t i_{t+1}\cdots i_g}) \to (i_g-2, w^{i_t i_{t+1}\cdots i_g}) \to \cdots \\ &\to (i_{t-1}, w^{i_t i_{t+1}\cdots i_g}) \to (i_{t-1}-1, w^{i_{t-1} i_t\cdots i_g}) \to (i_{t-1}-2, w^{i_{t-1} i_t\cdots i_g}) \\ &\to \cdots \to (i_{t-2}, w^{i_{t-1} i_t\cdots i_g}) \to (i_{t-2}-1, w^{i_{t-2} i_{t-1}\cdots i_g}) \to \cdots \\ &\to (0, w^{i_1 i_2\cdots i_g}) = (0, 1, \cdots, 1). \end{aligned}$$

Similarly, let $i_{m-1} \leq x_0 \leq i_m$ where $i_m \in \{i_{g+1}, i_{g+2}, \cdots, i_n\}$, we can construct a path P_2 between x and s_1.

$$\begin{aligned} P_2 : x &\to (x_0+1, w) \to \cdots \to (i_m-1, w) \to (i_m, w^{i_m}) \to (i_m+1, w^{i_m}) \\ &\to \cdots \to (i_{m+1}-1, w^{i_m}) \to (i_{m+1}, w^{i_m i_{m+1}}) \to (i_{m+1}+1, w^{i_m i_{m+1}}) \to \cdots \\ &\to (i_n, w^{i_m i_{m+1}\cdots i_n}) \to (i_n-1, w^{i_m i_{m+1}\cdots i_n}) \to (i_n-2, w^{i_m i_{m+1}\cdots i_n}) \\ &\to \cdots \to (i_{m-1}, w^{i_m i_{m+1}\cdots i_n}) \to (i_{m-1}-1, w^{i_{m-1} i_m\cdots i_n}) \\ &\to (i_{m-1}-2, w^{i_{m-1} i_m\cdots i_n}) \to \cdots \to (i_{m-2}, w^{i_{m-1} i_m\cdots i_n}) \\ &\to (i_{m-2}-1, w^{i_{m-2} i_{m-1}\cdots i_n}) \to \cdots \to (0, w^{i_{g+1} i_{g+2}\cdots i_n}) = (0, 0, \cdots, 0) \end{aligned}$$

Note that if $x_0 > i_g$ or $x_0 > i_n$, we can construct P_1 and P_2 as above. We easily know that $|P_1| = 2(i_g - x_0) + x_0 \leq 2n-1$ and $|P_2| = 2(i_n - x_0) + x_0 \leq 2n-1$ since $x_0 \geq 1$. For any vertices $y = (y_0, y_1, \cdots, y_n) \in V(P_1)$ and $z = (z_0, z_1, \cdots, z_n) \in V(P_2)$, we have the fact that $\sum_{i=1}^{n} y_i > \sum_{i=1}^{n} z_i$ if $y \neq x$ or $z \neq x$. Thus, P_1 and P_2 are internally vertex-disjoint.

Case 2. $x = (x_0, x_1, \cdots, x_n)$ with $x_0 = 0$.

We consider two neighbors of x, $x' = (1, x_1, \cdots, x_n)$ and $x'' = (1, \overline{x}_1, x_2, \cdots, x_n)$. Without loss of generality, we assume $x_1 = 0$. Thus x' is in the level 0 of $B(n-1)^1$ and x'' is in the level 0 of $B(n-1)^2$. By lemma 3, there is a path P in $B(n-1)^1$ of length no more than $2(n-1)-2$ between x' and $(1,0,\cdots,0)$ or $(1,0,1,\cdots,1)$. Since $(1,0,\cdots,0)$ is a neighbor of $(0,0,\cdots,0)$ and $(1,0,1,\cdots,1)$ is a neighbor of $(0,1,\cdots,1)$, we easily find a path between x and $(0,0,\cdots,0)$ or $(0,1,\cdots,1)$, which includes P and has length no more than $2n-2$. Similarly, we have there exists a path P' in $B(n-1)^2$ of length no more than $2(n-1)-2$ between x'' and $(1,1,0\cdots,0)$ or $(1,1,1,\cdots,1)$ by lemma 3 and we also find a path between x and $(0,0,\cdots,0)$ or $(0,1,\cdots,1)$, which includes P' and has length no more than $2n-2$. It is obvious that the paths are internally vertex-disjoint.

Thus, $s_{2n-1,2}(B_n) \leq 2$. For any vertex $x = (x_0,x_1,\cdots,x_n)$, there exists a vertex $y = (x_0,\overline{x}_1,\overline{x}_2,\cdots,\overline{x}_n)$ such that $\mathrm{dist}(x,y) = 2n$. So, it is impossible that $s_{2n-1,2}(B_n) = 1$. Thus $s_{2n-1,2}(B_n) = 2$.

The proof of Theorem 1 is completed. ■

Theorem 2. *The* $(d,2)$*-dominating numbers of* $B(n)$ *(*$n \geq 3$*) are* 2 *for* $2n \leq d \leq 2n+1$.

Proof. By Theorem 1 and Lemma 1(b), we know $s_{d,2}(B(n)) \leq 2$ for $n = 2n$ or $2n+1$. Suppose that $s_{2n+1,2}(B(n)) = 1$. i.e., all vertices of $B(n)$ can be dominated by some vertex u. By some automorphisms of $\{\alpha_1,\cdots,\alpha_n,\beta\}$, we can assume $u = (k-1,0,\cdots,0)$ with $1 \leq k \leq \frac{n}{2}$. But $v = (k+1,1,\cdots,1)$ is can't dominating by u since any two paths between u and v with length no more than $2n+1$ must be intersecting in the vertex w with $w_0 = k$ and $w_1 = \cdots = w_k = 1$, $w_{k+1} = \cdots = w_n = 0$ by Lemma 4. This is a contradiction. Thus, $s_{2n+1,2}(B(n)) = 2$. ■

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