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DIVISORS OF GENERIC HYPERSURFACES OF GENERAL TYPE

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Abstract. We prove that a generic hypersurface of general type in P^n for $n \geq 3$ does not contain a reduced irreducible divisor which admits a desingularization having nef anticanonical bundle.

1. Introduction

In this paper we shall generalize a theorem [2] of Mei-chu Chang and Ziv Ran to the higher dimensional case. They proved the following.

Theorem 1.1. *A generic hypersurface of degree* ≥ 5 *in* P^3 *or* P^4 *does not contain a reduced irreducible divisor which admits a desingularization having numerically effective anticanonical bundle.*

The P^3 case is a conjecture of Harris which is first proven by G. Xu with a different method. The natural generalization of Theorem 1 is the nonexistence of a divisor with numerically effective anticanonical bundle on a generic hypersurface of degree $\geq r + 1$ in P^r for $r \geq 5$. In this paper we are going to prove the case of general type only. Our main theorem is the following.

Theorem 1.2. *A generic hypersurface of degree* $\geq r + 2$ *in* P^r *for* $r \geq 3$ *does not contain a reduced irreducible divisor which admits a desingularization having numerically effective anticanonical bundle.*

In [12], the author gave another proof of the above theorem. In fact, the result of [12] is stronger, which allow the generic hypersurface to be Calabi-Yau, i.e., degree $= r + 1$. Nevertheless, the method given here has two advantages. First,

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this method can obtain some genus bound for the divisor (at least for the filling-up case). Second, it seems to me that this method has chance to generalize to higher codimension subvarieties. We hope to return to this matter elsewhere.

Let us fix some notations in this paper. Thus let $X^{r-1} \subset \mathbb{P}^r$ be a generic hypersurface of degree $d \geq r+2$, and suppose $\tilde{D}^{r-2} \subset X^{r-1}$ is an irreducible and reduced divisor. Let $f : D \longrightarrow \tilde{D} \subset X$ be a desingularization, with $-K_D$ nef.

Following the idea of [2], we view X^{r-1} as $X^r \cap H$, where $X^r \subset \mathbb{P}^{r+1}$ is a generic but fixed hypersurface of degree d, and H is a hyperplane which we view as the variable. As H moves, X^{r-1} and $\tilde{D}^{r-2}$ must move along and there are the following two possibilities:

(a) $\tilde{D}^{r-2}$ fills up X^r; or

(b) $\tilde{D}^{r-2}$ extends to an irreducible variety $\tilde{D}^{r-1} \subset X^r$.

We rule out these two cases in the next two sections.

2. Case of Filling Up

Proposition 2.1. *Let $\mathcal{M}$ be a torsion-free sheaf on a smooth variety D of dimension n. Assume $\mathcal{M}$ is generically generated by global sections while for some ample divisor H on D; $c_1(\mathcal{M}).H^{n-1} \leq 0$. Then $\mathcal{M}$ is trivial.*

Proof. Let $\mathcal{M}^{\vee\vee}$ be the double dual of $\mathcal{M}$, which contains $\mathcal{M}$ as a subsheaf. Let ρ be the rank of $(\mathcal{M})$. Choosing ρ general sections of $\mathcal{M}$, we get the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{\rho \text{ copies}} \mathcal{O} & \longrightarrow & \mathcal{M}^{\vee\vee} & \longrightarrow & \tau & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\| & & & & \\ 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}^{\vee\vee} & & & & \end{array}.$$

Then

$$c_1(\tau).H^{n-1} = c_1(\mathcal{M}^{\vee\vee}).H^{n-1} = c_1(\mathcal{M}).H^{n-1} \leq 0.$$

It follows that $c_1(\tau) = 0$ since $c_1(\tau)$ is an effective divisor. Hence the support of τ is contained in the singularity set of $\mathcal{M}^{\vee\vee}$, which is at least 3-codimensional. Let V be the support of τ. Then

$$\mathcal{M}^{\vee\vee} \simeq i_*\mathcal{M}_{\overline{D-V}} \simeq i_*(\oplus\mathcal{O}_{D-V}) \simeq \oplus\mathcal{O},$$

where $i : D - V \longrightarrow D$ is the inclusion map. Consider the square of the above diagram. We obtain

$$\mathcal{M} = \mathcal{M}^{\vee\vee} \simeq \oplus\mathcal{O}.$$

■

Assume we are in the case that $\tilde{D}^{r-2}$ moves with H, filling up X^r and degrees of X^{r-1} and X^r are greater than or equal to $n+2$. Let $\tilde{f}: D \longrightarrow X^r$ be the obvious map. Then we have the following diagram.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & T_D \arrow[r] \arrow[d, equal] & f^*T_{X^{r-1}} \arrow[r] \arrow[d] & N_f \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & T_D \arrow[r] & \tilde{f}^*T_{X^r} \arrow[r] & N_{\tilde{f}} \arrow[r] \arrow[d] & 0, \\
 & & & L \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

$$L \equiv f^*\mathcal{O}(1).$$

Note that the vertical sequence splits in a neighborhood of any fibre of f. Let τ be the torsion subsheaf of N_f. τ is also the torsion subsheaf of $N_{\tilde{f}}$. Note that τ is supported purely in codimension 1 because of the above horizontal sequence. Since $\tilde{D}^{r-2}$ fills up X^r, $N_{\tilde{f}}$ is generically generated by global sections. Hence $N_{\tilde{f}}/\tau$ is also generacally spaned and in particular $c_1(N_{\tilde{f}}) - c_1(\tau)$ is nef. By an easy calculation we have

$$c_1(N_{\tilde{f}}) = -(-K_D) - (d-r-2)L.$$

Since $-K_D$ is nef, it follows that $\tau = 0$, $d = r+2$ and K_D is numerically trivial. By the Proposition 4.3 we have $N_{\tilde{f}}$ is trivial. Because $h^0(N_{\tilde{f}}) \geq r+1$, we get a contradiction. Hence we rule out this case.

Note that,by Hopf's lemma, we can conclude that $h^0(c_1(N_{\tilde{f}})) \geq r-1$. Because of $c_1(N_{\tilde{f}}) = K_D - (d-r-2)L$, we obtain the following genus lower bound

$$\begin{aligned} h^0(K_D) &= h^0(c_1(N_{\tilde{f}}) + (d-r-2)L) \\ &\geq h^0(c_1(N_{\tilde{f}})) \\ &\geq r-1. \end{aligned}$$

If $d > r+2$, then

$$\begin{aligned} h^0(K_D - L) &= h^0(c_1(N_{\tilde{f}}) + (d-r-3)L) \\ &\geq h^0(c_1(N_{\tilde{f}})) \\ &\geq r-1. \end{aligned}$$

We conclude that D is a smooth variety of general type.

3. Case of Extension

Now we are in the case that $\bar{D}^{r-2}$ extends to a divisor

$$\bar{D}^{r-1} \subset X^r \subset P^{r+1};$$

such that

$$\bar{D}^{r-2} = \bar{D}^{r-1} \cap H.$$

We may ask whether the extension will keep going like:

$$\begin{array}{ccccc}
 & & \vdots & & \\
\cup & & \cup & & \cup \\
\bar{D}^{n-1} & \subset & X^n & \subset & P^{n+1}; \\
\cup & & \cup & & \cup \\
 & & \vdots & & \\
\cup & & \cup & & \cup \\
\bar{D}^{r-1} & \subset & X^r & \subset & P^{r+1}; \\
\cup & & \cup & & \cup \\
\bar{D}^{r-2} & \subset & X^{r-1} & \subset & P^r;
\end{array}$$

If it does not, then $\bar{D}^{r-2}$ moves in a family filling up X^n. By cuting X^n with linear spaces, we will get a contradiction. Hence we may assume $\bar{D}^{r-2}$ does extend to $\bar{D}^{n-1} \subset X^n$ for any n. Now let $g_n : D^n \longrightarrow X^{n+1}$ be a desingularization of $\bar{D}^n$.

As

$$K_{D^{r-2}} = (K_{D^{r-1}} + H)|_{D^{r-2}};$$

for every point $x \in D^{r-1}$ there is an irreducible curve C with $K_{D^{r-1}}.C < 0$ through x. By Mori-Miyaoka's uniruledness criterion [9], D^{r-1} is uniruled.

Let $h_{r-1} : P^1 \longrightarrow D^{r-1}$ be a rational curve through a general point of D^{r-1}, and put $h_n = i_n \circ h_{r-1}$ where $i_n : D^{r-1} \hookrightarrow D^n$ is the inclusion. We have the following exact sequence.

$$0 \longrightarrow N_{h_{r-1}} \longrightarrow N_{h_n} \longrightarrow (n-r+1)h^*_{r-1}L \longrightarrow 0;$$
$$L \equiv g^*_{r-1}\mathcal{O}(1).$$

Since $N_{h_{r-1}}$ is semipositive on P^1, it is nonspecial. It follows in particular that for large n a general deformation $\hat{h}_n$ of h_n will be linearly, hence projectively normal.

Noting the exact sequenc

$$0 \longrightarrow N_{\hat{h}_n} \longrightarrow N_{g_n \circ \hat{h}_n} \longrightarrow \hat{h}_n^* N_{g_n} \longrightarrow 0;$$

by an easy calculation and Riemann-Roch theorem we have

$$c_1(\hat{h}_n^* N_{g_n}(-1) = c_1(\hat{h}_{r-1}^* N_{g_{r-1}}(-1) = \deg(\hat{h}_{r-1}^* K_{D^{r-1}}) \leq -2.$$

By Riemann-Roch theorem we have that

$$H^1(\hat{h}_n^* N_{g_n}(-1) \neq 0;$$

and hence

$$H^1(N_{g_n \circ \hat{h}_n}(-1)) \neq 0.$$

In view of the construction of $\hat{h}_n$, the following lemma yields a contradiction.

Lemma 3.1. *Let* $r : P^1 \longrightarrow X^n \subset P^{n+1}$ *be a projectively normal rational curve on a smooth hypersurface. Then there exists an extension* X^m *of* X^n *to* P^{m+1} *such that the evident map* $r_m : P^1 \longrightarrow X^m$ *has* $h^1(N_{r_m}(-1)) = 0$.

Proof. Consider a potential extension X^m, and let F be its homogeneous equation and $j : X^m \longrightarrow P^{m+1}$ the inclusion. Then we have an exact sequence

$$(1) \quad 0 \longrightarrow N_{r_m}(-1) \longrightarrow N_{j \circ r_m}(-1) \longrightarrow r_m^* \mathcal{O}(d-1) \longrightarrow 0;$$

where $d = \deg X^n$, and the natural map

$$\delta : H^0(T_{P^{m+1}}(-1) \longrightarrow H^0(N_{j \circ r_m}(-1)) \longrightarrow H^0(r_m^* \mathcal{O}(d-1));$$

is just given by

$$\frac{\partial}{\partial X_i} \longmapsto \frac{\partial F}{\partial X_i}.$$

By prjective normality, clearly for large m and general F, δ will be surjective. As $N_{j \circ r_m}(-1)$ is semipositive, its H^1 vanishes, hence the cohomology sequence yields $H^1(N_{r_m}(-1)) = 0$.

4. Appendix

In this section we review some properties of reflexive sheaves. A coherent sheaf $\mathcal{F}$ is reflexive if the natural map $\mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee}$ is an isomorphism where $\mathcal{F}^{\vee\vee}$ is the double dual of $\mathcal{F}$. Define the singularity set of $\mathcal{F}$ to be the locus where the $\mathcal{F}$ is not free over the local ring. For the proof of the following two lemmas, see [6] and [10].

Lemma 4.1. *Assume that* D *is regular. The singularity set of a torsion-free sheaf on* D *is at least* 2*-codimensional. Moreover, the singularity set of a reflexive sheaf on* D *is at least* 3*-codimensional.*

Lemma 4.2. *Let* $\mathcal{F}$ *be a coherent sheaf on a normal integral scheme* D. *The following conditions are equivalent* :

(i) $\mathcal{F}$ *is reflexive*;

(ii) $\mathcal{F}$ *is torsion-free*; *and for each open* $U \subseteq D$ *and each closed subset* $V \subset U$ *of codimension* ≥ 2; $\mathcal{F}_U \simeq i_{¤}\mathcal{F}_{U\,¡\,V}$; *where* $i : U - V \longrightarrow U$ *is the inclusion map.*

Proposition 4.3. *Let* $\mathcal{M}$ *be a torsion-free sheaf on a smooth variety* D *of dimension* n. *Assume* $\mathcal{M}$ *is generically generated by global sections while for some ample divisor* H *on* D; $c_1(\mathcal{M}){:}H^{n\,¡\,1} \leq 0$. *Then* $\mathcal{M}$ *is trivial.*

Proof. Let $\mathcal{M}^{--}$ be the double dual of $\mathcal{M}$; which contains $\mathcal{M}$ as a subsheaf. Let ½ be the rank of $(\mathcal{M})$. Choosing ½ general sections of $\mathcal{M}$; we get the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{\text{½ copies}} \mathcal{O} & \longrightarrow & \mathcal{M}^{--} & \longrightarrow & ¿ & \longrightarrow & 0 \\ & & \Big\downarrow & & \Vert & & & & \\ 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}^{--} & & & & \end{array} :$$

Then

$$c_1(¿){:}H^{n\,¡\,1} = c_1(\mathcal{M}^{--}){:}H^{n\,¡\,1} = c_1(\mathcal{M}){:}H^{n\,¡\,1} \leq 0:$$

It follows that $c_1(¿) = 0$ since $c_1(¿)$ is an effective divisor. Hence the support of $¿$ is contained in the singularity set of $\mathcal{M}^{--}$, which is at least 3-codimensional. Let V be the support of $¿$. Then

$$\mathcal{M}^{--} \simeq i_{¤}\mathcal{M}^{--}_{D\,¡\,V} \simeq i_{¤}(\oplus\mathcal{O}_{D\,¡\,V}) \simeq \oplus\mathcal{O}$$

where $i : D - V \longrightarrow D$ is the inclusion map. Consider the square of the above diagram. We obtain

$$\mathcal{M} = \mathcal{M}^{--} \simeq \oplus\mathcal{O}:$$

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