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APPROXIMATIONS IN H_v-MODULES

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Abstract. In this paper we consider the relation ε^* defined on an H_v-module M and we interprete the lower and upper approximations as subsets of the module M/ε^* and we give some results in this connection.

1. Introduction

The concept of hyperstructure first was introduced by Marty in [10] and has been studied in the following decades and nowadays by many mathematicians. Vougiouklis in the fourth A.H.A. congress (1990) [13], introduced the notion of H_v-structures. The concept of H_v-structures constitutes a generalization of the well-known algebraic hyperstructures. The principal notions of H_v-structures can be found in [16]. Since then many papers concerning various H_v-structures have appeared in the literature, for example [4, 5, 6, 12, 14, 15]. According to [16], H_v-modules are the largest class of multivalued systems that satisfy module-like axioms. In [15], Vougiouklis defined the concept of H_v-vector space which is a generalization of the concept of vector space. The notion of fuzzy H_v-submodules was introduced by Davvaz in [6]. And in [4], Davvaz introduced H_v-module of fractions of a hypermodule which is a generalization of the concept of module of fractions.

The notion of approximation spaces and rough sets were introduced by Pawlak in his paper [11], and since then has been the subject of many papers. Some authors for example, Bonikowaski [2], Iwinski [8], Biswas and Nanda [1], Comer [3], Kurouki and Wang [9], studied algebraic properties of rough sets.

In [5], [7], the present author applied the concept of approximation spaces in the theory of algebraic hyperstructures. In this paper we remark upon some relationships between approximation spaces and H_v-modules.

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The main tools in the theory of hyperstructures are the fundamental relations. These relations, on the one hand, connect this theory, in some way with the corresponding classical theory and on the other hand, introduce new important classes. In section 4 of this paper we consider the relation ε^* defined on an H_v-module M and we interprete the lower and upper approximations as subsets of the module M/ε^* and we give some results in this connection.

2. Basic Definitions

We first recall some basic definitions for the sake of completeness.

Definition 2.1. (Vougiouklis [16]). Let H be a non-empty set and $\cdot : H \times H \longrightarrow P^*(H)$ be a hyperoperation; where $P^*(H)$ is the set of all non-empty subsets of H. The $(\cdot)$ is called weak associative if $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset$ for all $x, y, z \in H$; where

$$A \cdot B = \bigcup_{\substack{a \in A \\ b \in B}} a \cdot b \quad \text{for all } A, B \subseteq H.$$

The $(\cdot)$ is called weak commutative if $x \cdot y \cap y \cdot x \neq \emptyset$ for all $x, y \in H$. $(H, \cdot)$ is called an H_v-group if, i) $(\cdot)$ is weak associative, ii) $a \cdot H = H \cdot a = H$ for all $a \in H$.

Definition 2.2. (Vougiouklis [16]). A multivalued system $(R, +, \cdot)$ is called an H_v-ring if following axioms hold: i) $(R, +)$ is a weak commutative H_v-group, ii) $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset$ for all $x, y, z \in R$, iii) $(\cdot)$ is weak distributive with respect to $(+)$, i.e., for all $x, y, z \in R$, $x \cdot (y + z) \cap (x \cdot y + x \cdot z) \neq \emptyset$, $(x + y) \cdot z \cap (x \cdot z + y \cdot z) \neq \emptyset$.

Definition 2.3. (Vougiouklis [15, 16]). A non-empty set M is a left H_v-module over an H_v-ring R if $(M, +)$ is a weak commutative H_v-group and there exists the map $\cdot : R \times M \longrightarrow P^*(M)$, $(r, x) \longrightarrow r \cdot x$ such that for all a, b in R and x, y in M, we have $a \cdot (x + y) \cap (a \cdot x + a \cdot y) \neq \emptyset$; $(a + b) \cdot x \cap (a \cdot x + b \cdot x) \neq \emptyset$; $(ab) \cdot x \cap a \cdot (b \cdot x) \neq \emptyset$.

Definition 2.4. Let M and N be two H_v-modules over an H_v-ring R. A mapping $f : M \longrightarrow N$ is called an H_v-homomorphism if, for all $x, y \in M$ and for all $r \in R$; the following relations hold: $f(x + y) \cap (f(x) + f(y)) \neq \emptyset$; $f(r \cdot x) \cap r \cdot f(x) \neq \emptyset$. f is called an inclusion homomorphism if; for all $x, y \in M$ and for all $r \in R$; the following relations hold: $f(x + y) \subseteq f(x) + f(y)$; $f(r \cdot x) \subseteq r \cdot f(x)$. Finally, f is called a strong homomorphism if for all $x, y \in M$ and for all $r \in R$, we have $f(x + y) = f(x) + f(y)$; $f(r \cdot x) = r \cdot f(x)$. If there exists a strong one to one homomorphism from M onto N, then M and N are called isomorphic.

3. Approximations in H_v-Modules

Let ρ be an equivalence relation defined on the H_v-module M and $\rho(x)$ be the equivalence class of the relation ρ generated by an element $x \in M$. Any finite union of equivalence classes of M is called a definable set in M. Let A be any subset of U. In general, A is not a definable set in M. However, the set A may be approximated by two definable set in M. The first one is called a ρ-lower approximation of A in M, denoted by $\underline{\rho}(A)$ and defined as follows: $\underline{\rho}(A) = \{x \in M \mid \rho(x) \subseteq A\}$. The second set is called a ρ-upper approximation of A in M, denoted by $\overline{\rho}(A)$ and defined as follows: $\overline{\rho}(A) = \{x \in M \mid \rho(x) \cap A \neq \emptyset\}$. The ρ-lower approximation of A in M is the greatest definable set in M contained in A. The ρ-upper approximation of A in M is the least definable set in M containing A. The difference $\overline{\underline{\rho}}(A) = \overline{\rho}(A) - \underline{\rho}(A)$ is called the ρ-boundary region of A. In the case when $\overline{\underline{\rho}}(A) = \emptyset$ the set A is said to be ρ-exact.

Let M_1 and M_2 be H_v-modules over an H_v-ring R and T be a strong homomorphism from M_1 into M_2. The relation $T \circ T^{-1}$ is an equivalence relation ρ on M_1 ($a\rho b$ if and only if $T(a) = T(b)$) known as the kernel of T.

Theorem 3.1. *Let M_1 and M_2 be H_v-modules over an H_v-ring R and T be a strong homomorphism from M_1 into M_2. If A is a non-empty subset of M_1, then*

$$T(\overline{\rho}(A)) = T(A).$$

Proof. Since $A \subseteq \overline{\rho}(A)$, it follows that $T(A) \subseteq T(\overline{\rho}(A))$. To see that the converse inclution holds, let y be any element of $T(\overline{\rho}(A))$. Then there exists an element $x \in \overline{\rho}(A)$ such that $T(x) = y$. Therefore there exists $a \in M_1$ such that $a \in \rho(x) \cap A$, and so $T(a) = T(x)$ and $a \in A$. Then we obtain $y = T(x) = T(a) \in T(A)$, and so $T(\overline{\rho}(A)) \subseteq T(A)$. ■

Theorem 3.2. *Let ρ, θ be equivalence relations on an H_v-module M. If A is a non-empty subset of M, then $\overline{(\rho \cap \theta)}(A) \subseteq \overline{\rho}(A) \cap \overline{\theta}(A)$.*

Proof. Note that $\rho \cap \theta$ is also an equivalence relation on M. Let $x \in \overline{(\rho \cap \theta)}(A)$. Then $(\rho \cap \theta)(x) \cap A \neq \emptyset$, and so there exists $a \in (\rho \cap \theta)(x) \cap A$. Since $(a, x) \in \rho \cap \theta$, we have $(a, x) \in \rho$ and $(a, x) \in \theta$. Therefore we have $a \in \rho(x)$ and $a \in \theta(x)$. Since $a \in A$, then $\rho(x) \cap A \neq \emptyset$ and $\theta(x) \cap A \neq \emptyset$. Thus $x \in \overline{\rho}(A)$ and $x \in \overline{\theta}(A)$. Thus we obtain that $\overline{(\rho \cap \theta)}(A) \subseteq \overline{\rho}(A) \cap \overline{\theta}(A)$. This complete the proof. ■

Theorem 3.3. *Let ρ, θ be equivalence relations on an H_v-module M. If A is a non-empty subset of M, then $\underline{(\rho \cap \theta)}(A) = \underline{\rho}(A) \cap \underline{\theta}(A)$.*

Proof. We have

$$x \in \underline{(\rho \cap \sigma)}(A) \iff (\rho \cap \sigma)(x) \subseteq A \iff \rho(x) \subseteq A \text{ and } \sigma(x) \subseteq A$$
$$\iff x \in \underline{\rho}(A) \text{ and } x \in \underline{\sigma}(A) \iff \underline{\rho}(A) \cap \underline{\sigma}(A). \quad \blacksquare$$

4. On the Fundamental Relation ε^*

Consider the left H_v-module M over an H_v-ring R. The relation γ^* is the smallest equivalence relation on R such that the quotient R/γ^* is a ring. γ^* is called the fundamental equivalence relation on R and R/γ^* is called the fundamental ring, see [12], [16]. The fundamental relation ε^* on M over R is the smallest equivalence relation such that M/ε^* is a module over the ring R/γ^*, see [15], [16].

According to [16] if U denotes the set of all expressions consisting of finite hyperoperations of either on R and M or the external hyperoperation applied on finite sets of R and M. Then a relation ε can be defined on M whose transitive closure is the fundamental relation ε^*. The relation ε is as follows: $x\varepsilon y$ iff $\{x, y\} \subseteq u$ for some $u \in U$. Suppose $\gamma^*(r)$ is the equivalence class containing $r \in R$ and $\varepsilon^*(x)$ is the equivalence class containing $x \in M$. On M/ε^* the sum $\oplus$ and the external product $\odot$ using the γ^* classes in R, are defined as follows:

$$\varepsilon^*(x) \oplus \varepsilon^*(y) = \varepsilon^*(c) \text{ for all } c \in \varepsilon^*(x) + \varepsilon^*(y);$$
$$\gamma^*(r) \odot \varepsilon^*(x) = \varepsilon^*(d) \text{ for all } d \in \gamma^*(r) \cdot \varepsilon^*(x).$$

The kernel of the canonical map $\varphi : M \longrightarrow M/\varepsilon^*$ is called the core of M and is denoted by ω_M. Here we also denote by ω_M the unit element of the group $(M/\varepsilon^*, \oplus)$.

For a subset $A \subseteq M$ we define the approximations of A relative to the fundamental equivalence relation ε^* as follows:

$$\underline{\varepsilon^*}(A) = \{x \in M \mid \varepsilon^*(x) \subseteq A\} \text{ and } \overline{\varepsilon^*}(A) = \{x \in M \mid \varepsilon^*(x) \cap A \neq \emptyset\}.$$

The proof of the following theorem is similar to the Theorem 1 of [5] and Proposition 3.1 of [7].

Proposition 4.1. *If* A *and* B *are non-empty subsets of* M, *then the following hold* :

1) $\underline{\varepsilon^*}(A) \subseteq A \subseteq \overline{\varepsilon^*}(A)$;
2) $\overline{\varepsilon^*}(A \cup B) = \overline{\varepsilon^*}(A) \cup \overline{\varepsilon^*}(B)$;
3) $\underline{\varepsilon^*}(A \cap B) = \underline{\varepsilon^*}(A) \cap \underline{\varepsilon^*}(B)$;
4) $A \subseteq B$ implies $\overline{\varepsilon^*}(A) \subseteq \overline{\varepsilon^*}(B)$;
5) $A \subseteq B$ implies $\underline{\varepsilon^*}(A) \subseteq \underline{\varepsilon^*}(B)$;
6) $\underline{\varepsilon^*}(A) \cup \underline{\varepsilon^*}(B) \subseteq \underline{\varepsilon^*}(A \cup B)$;
7) $\overline{\varepsilon^*}(A \cap B) \subseteq \overline{\varepsilon^*}(A) \cap \overline{\varepsilon^*}(B)$;
8) $\underline{\varepsilon^*}(\underline{\varepsilon^*}(A)) = \underline{\varepsilon^*}(A)$;
9) $\overline{\varepsilon^*}(\overline{\varepsilon^*}(A)) = \overline{\varepsilon^*}(A)$.

Similarly, for a subset $X \subseteq R$ we can define two approximations of X relative to the fundamental relation γ^*:

$$\underline{\gamma^*}(X) = \{x \in R \mid \gamma^*(x) \subseteq X\} \text{ and } \overline{\gamma^*}(X) = \{x \in R \mid \gamma^*(x) \cap X \neq \emptyset\}.$$

Proposition 4.2. *If* A *and* B *are non-empty subsets of* M *and* X *a non-empty subset of* R, *then*

i) $\overline{\varepsilon^*}(A) + \overline{\varepsilon^*}(B) \subseteq \overline{\varepsilon^*}(A + B)$; ii) $\overline{\gamma^*}(X) \cdot \overline{\varepsilon^*}(A) \subseteq \overline{\varepsilon^*}(X \cdot A)$.

Proof. The proof of (i) is similar to the proof of Proposition 3.2 of [5]. We prove only (ii). Suppose c is any element of $\overline{\gamma^*}(X)\cdot\overline{\varepsilon^*}(A)$. Then $c \in x\cdot a$ with $x \in \overline{\gamma^*}(X)$ and $a \in \overline{\varepsilon^*}(A)$. Thus there exist $r \in R$ and $m \in M$ such that $r \in \gamma^*(x) \cap X$ and $m \in \varepsilon^*(a) \cap A$. Therefore $r \cdot m \subseteq \gamma^*(x) \cdot \varepsilon^*(a) \subseteq \varepsilon^*(x \cdot a)$. Since $r \cdot m \subseteq X \cdot A$, we get $r \cdot m \subseteq \varepsilon^*(x \cdot a) \cap X \cdot A$ and so $\varepsilon^*(x \cdot a) \cap X \cdot A \neq \emptyset$. Therefore for every $c \in x \cdot a$ we have $\varepsilon^*(c) \cap X \cdot A \neq \emptyset$, which implies $c \in \overline{\varepsilon^*}(X \cdot A)$. Thus we have $\overline{\gamma^*}(X) \cdot \overline{\varepsilon^*}(A) \subseteq \overline{\varepsilon^*}(X \cdot A)$. ■

The lower and upper approximations can be presented in an equivalent form as shown below. Let X be a non-empty set of R and A be a non-empty subset of M. Then

$$\underline{\gamma^*(X)} = \{\gamma^*(x) \in R/\gamma^* \mid \gamma^*(x) \subseteq X\} \text{ and } \overline{\gamma^*(X)} = \{\gamma^*(x) \in R/\gamma^* \mid \gamma^*(x) \cap X \neq \emptyset\},$$

and

$$\underline{\varepsilon^*(A)} = \{\varepsilon^*(a) \in M/\varepsilon^* \mid \varepsilon^*(x) \subseteq A\} \text{ and } \overline{\varepsilon^*(A)} = \{\varepsilon^*(a) \in M/\varepsilon^* \mid \varepsilon^*(x) \cap A \neq \emptyset\}.$$

Now, we discuss these sets as subsets of the fundamental ring R/γ^* and the fundamental module M/ε^*.

Theorem 4.3 [5]. *If* X *is an* H_v*-ideal of* R, *then* $\overline{\gamma^*(X)}$ *is an ideal of* R/γ^*.

Theorem 4.4. *If* A *is an* H_v*-subgroup of* $(M, +)$, *then* $\overline{\varepsilon^*(A)}$ *is a subgroup of* $(M/\varepsilon^*, \oplus)$.

Proof. The proof is similar to the proof of Theorem 7 of [7], by concerning the suitable modifications with using the definition of ε^*. ■

Lemma 4.5. *If* A *and* B *are non-empty subsets of* M, *then* $\overline{\varepsilon^*(A)} \oplus \overline{\varepsilon^*(B)} \subseteq \overline{\varepsilon^*(A + B)}$.

Proof. We have

$$\begin{aligned}\overline{\varepsilon^*(A)} \oplus \overline{\varepsilon^*(B)} &= \{\varepsilon^*(a) \oplus \varepsilon^*(b) \mid \varepsilon^*(a) \in \overline{\varepsilon^*(A)},\ \varepsilon^*(b) \in \overline{\varepsilon^*(B)}\} \\ &= \{\varepsilon^*(a) \oplus \varepsilon^*(b) \mid \varepsilon^*(a) \cap A \neq \emptyset,\ \varepsilon^*(b) \cap B \neq \emptyset\}.\end{aligned}$$

Therefore $(\varepsilon^*(a) + \varepsilon^*(b)) \cap (A + B) \neq \emptyset$. Since $\varepsilon^*(a) + \varepsilon^*(b) \subseteq \varepsilon^*(a + b)$ we obtain $\varepsilon^*(a + b) \cap (A + B) \neq \emptyset$. Thus $\varepsilon^*(a + b) = \varepsilon^*(a) \oplus \varepsilon^*(b) \in \overline{\varepsilon^*(A + B)}$ and so $\overline{\varepsilon^*(A)} \oplus \overline{\varepsilon^*(B)} \subseteq \overline{\varepsilon^*(A + B)}$. ■

Lemma 4.6. *If X be a non-empty subset of R and A be an H_v-submodule of M, then $\overline{\gamma^*(X)} \odot \overline{\varepsilon^*(A)} \subseteq \overline{\varepsilon^*(A)}$.*

Proof. We have

$$\begin{aligned}\overline{\gamma^*(X)} \odot \overline{\varepsilon^*(A)} &= \Big\{ \bigoplus \sum_{finite} (\gamma^*(x) \odot \varepsilon^*(a)) \mid \gamma^*(x) \in \overline{\gamma^*(X)},\ \varepsilon^*(a) \in \overline{\varepsilon^*(A)} \Big\} \\ &= \Big\{ \bigoplus \sum_{finite} (\gamma^*(x) \odot \varepsilon^*(a)) \mid \gamma^*(x) \cap A \neq \emptyset,\ \varepsilon^*(a) \cap A \neq \emptyset \Big\}.\end{aligned}$$

Therefore $\gamma^*(x) \cdot \varepsilon^*(a) \cap X \cdot A \neq \emptyset$. Since $\gamma^*(x) \cdot \varepsilon^*(a) \subseteq \varepsilon^*(x \cdot a)$, we obtain $\varepsilon^*(x \cdot a) \cap X \cdot A \neq \emptyset$. Since A is an H_v-submudule of M, we have $X \cdot A \subseteq A$ and so $\varepsilon^*(x \cdot a) \cap A \neq \emptyset$. Thus $\varepsilon^*(x \cdot a) = \gamma^*(x) \odot \varepsilon^*(a) \in \varepsilon^*(A)$. Since A is an H_v-subgroup of $(M, +)$, by Theorem 4.4, $\overline{\varepsilon^*(A)}$ is a subgroup of $(M/\varepsilon^*, \oplus)$, therefore $\bigoplus \sum_{finite} \gamma^*(x) \odot \varepsilon^*(a) \in \overline{\varepsilon^*(A)}$ and so $\overline{\gamma^*(X)} \odot \overline{\varepsilon^*(A)} \subseteq \overline{\varepsilon^*(A)}$. ■

Theorem 4.7. *If A is an H_v-submudule of M, then $\overline{\varepsilon^*(A)}$ is a submodule of M/ε^*.*

Proof. Suppose that A is an H_v-submodule of M, then by Theorem 4.4, $(\overline{\varepsilon^*(A)}, \oplus)$ is a subgroup of $(M/\varepsilon^*, \oplus)$. Note that $\overline{\varepsilon^*(M)} = M/\varepsilon^*$. Then we have

$$M/\varepsilon^* \odot \overline{\varepsilon^*(A)} = \overline{\varepsilon^*(M)} \odot \overline{\varepsilon^*(A)} \subseteq \overline{\varepsilon^*(A)}.$$ ■

Definition 4.8. Let A, B and C be H_v-submodules of M. The sequence of strong homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be *exact* if for every $x \in A$, $gof(x) \in \omega_M$.

Theorem 4.9. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of H_v-submodules of M. Then the sequence $\overline{\varepsilon^*(A)} \xrightarrow{F} \overline{\varepsilon^*(B)} \xrightarrow{G} \overline{\varepsilon^*(C)}$ is an exact sequence of submodules of M/ε^* where $F(\varepsilon^*(a)) = \varepsilon^*(f(a))$, $G(\varepsilon^*(b)) = \varepsilon^*(g(b))$ for all $a \in A$, $b \in B$.*

Proof. The proof is similar to the proof of Theorem 4.8 of [5]. ■

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