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SOME FAMILIES OF INFINITE SERIES SUMMABLE BY MEANS OF FRACTIONAL CALCULUS

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Abstract. In a Five-volume work published recently, K. Nishimoto [1] has presented a systematic account of the theory and applications of fractional calculus in a number of areas (such as ordinary and partial differential equations, special functions, and summation of series). In 2001, K. Nishimoto, D.-K. Chyan, S.-D. Lin and S.-T. Tu [11] derived the following interesting families of infinite series via fractional calculus,

$$\sum_{k=2}^{\infty} \frac{(-c)^k}{k(k-1)} \frac{(kz-c)}{(z-c)^{k-1}} = c^2 \left(\left| \frac{-c}{z-c} \right| < 1 \right):$$

The object of the present paper is to extend the above families of infinite series to more general closed form relations. Various numerical results are also provided.

1. Introduction and definitions

Some of the most recent developments on the use of fractional calculus in obtaining sums of infinite series are reported by Nishimoto and S.-T. Tu (cf. [3] and [4]), as well by Nishimoto and H. M. Srivastava [5], by Choi [6], and by B. N. Al-Saqabi et al. [7], by J. Aular de Durán et al. [8], by T.-C. Wu et al. [9]. With a view of recalling these works, we find it to be convenient to choose the following definition of a fractional differintegral (that is, fractional derivative and fractional integral of $f(z)$ of order °):

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(**I**.) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-;\ D_+\}$; $C = \{C_-;\ C_+\}$;

C_- be a curve along the cut joining two points z and $-\infty + i\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\mathrm{Im}(z)$,

D_- be a domain surrounded by C_-, D_+ be a domain surrounded by C_+.

(Here D contains the points over the curve C.)

Moreover, let $f = f(z)$ be a regular function in D ($z \in D$),

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}d\zeta \quad (\nu \notin Z^-); \tag{1.1}$$

$$(f)_{-m} = \lim_{\nu \to -m}(f)_\nu \quad (m \in Z^+); \tag{1.2}$$

where $-\pi \leq \arg(\zeta - z) \leq \pi$ for C_-; $0 \leq \arg(\zeta - z) \leq 2\pi$ for C_+;

$\zeta \neq z$; $z \in C$; $\nu \in R$; Γ : Gamma function;

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z, of the function f, if $|(f)_\nu| < \infty$.

(**II**.) On the fractional calculus operator N^ν [2]

Theroem A. *Let fractional culculus operator* (*Nishimoto's Operator*) N^ν *be*

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}}\right) \quad (\nu \notin Z^-) \quad [\text{Refer to } (1.1)]; \tag{1.3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \quad (m \in Z^+); \tag{1.4}$$

and define the binary operation $\circ$ *as*

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in R); \tag{1.5}$$

then the set

$$\{N^\nu\} = \{N^\nu \mid \nu \in R\} \tag{1.6}$$

is an Abelian product group (having continuous index ν*) which has the inverse transform operator* $(N^\nu)^{-1} = N^{-\nu}$ *to the fractional calculus operator* N^ν*, for the function* f *such that* $f \in F = \{f \mid 0 \neq |f_\nu| < \infty;\ \nu \in R\}$*, where* $f = f(z)$ *and* $z \in C$*. (viz.* $-\infty < \nu < \infty$*).*

(For our convenience, we call $N^{¯} \circ N^{®}$ as product of $N^{¯}$ and $N^{®}$.)

Theroem B. *The "F.O.G.* $\{N^{°}\}$*" is an "Action product group which has continuous index* $°$*" for the set* F. (*F.O.G.*; *Fractional calculus operator group*)

Making use of the above definition (given by Nishimoto in 1976), we have the following useful lemmas.

Lemma 1. (*Generalized Leibniz's Rule*) [1]

$$(U \cdot V)_{®} = \sum_{k=0}^{1} \frac{¡(®+1)}{k!\,¡(®+1-k)} \cdot U_{®¡k} \cdot V_k \quad \left(\left| \frac{¡(®+1)}{¡(®-k+1)} \right| < \infty \right);$$

where $U = U(z)$ *and* $V = V(z)$ *and* $® \in R$.

Lemma 2. [10]

(i) $((z-c)^{¯})_{®} = e^{¡i¼®} \dfrac{¡(®-¯)}{¡(-¯)} (z-c)^{¯¡®} \quad \left(\left| \dfrac{¡(®-¯)}{¡(-¯)} \right| < \infty \right).$

(ii) $((z-c)^{¡®})_{¡®} = -e^{i¼®} \dfrac{1}{¡(®)} \log(z-c) \quad (|¡(®)| < \infty).$

Lemma 3. [10]

(i) $(\log(z-c))_{®} = -e^{¡i¼®}\, ¡(®)(z-c)^{¡®} \quad (|¡(®)| < \infty).$

(ii) $(\log(z-c))_{¡n} = \dfrac{(z-c)^n}{n!} \{\log(z-c) - H_n\}$

where $H_n = \sum_{k=1}^{n} \frac{1}{k};\ H_0 = 0;\ n \in Z^{+}.$

2. Main Generalization Theorem

In 2001, Nishimoto et al. [11] obtained the following infinite sums. For $\left|\frac{¡c}{z¡c}\right| < 1$, we have

$$\sum_{k=2}^{1} \frac{(-c)^k}{k(k-1)} \cdot \frac{(kz-c)}{(z-c)^{k¡1}} = c^2: \tag{2.1}$$

In this paper, we are interested to investigate the above families of infinite sums of the form (2.1) in more general closed form relations. With the aid of Lemmas 1, 2 and 3, we have

Theroem: *For* $\left|\frac{-c}{z-c}\right| < 1$, *we have*

$$\sum_{k=n+1}^{\infty} \frac{(-1)^n \Gamma(k-n)}{k!}\left(\frac{-c}{z-c}\right)^k$$

$$(2.2) \qquad = \frac{1}{n!}\left[\{\log(z-c) - H_n\} - \left(\frac{z}{z-c}\right)^n \{\log z - H_n\}\right]$$

$$+ \sum_{k=1}^{n} \frac{(-1)^k}{k!(n-k)!}\left(\frac{-c}{z-c}\right)^k \{\log(z-c) - H_{(n-k)}\}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$, $H_0 = 0$, $n \in Z^+$.

Proof. By using the well-known relation, for $\left|\frac{-c}{z-c}\right| < 1$, we have

$$(2.3) \qquad \sum_{k=1}^{\infty} \frac{(-c)^k}{k}(z-c)^{-k} = \log(z-c) - \log z.$$

Operating N^{-n} $(n \in Z^+)$ to the both sides of (2.3), we obtain

$$(2.4) \qquad \sum_{k=1}^{n} \frac{(-c)^k}{k}((z-c)^{-k})_{-n} + \sum_{k=n+1}^{\infty} \frac{(-c)^k}{k}((z-c)^{-k})_{-n}$$

$$= (\log(z-c))_{-n} - (\log z)_{-n}.$$

Since

$$((z-c)^{-k})_{-n} = e^{-i\pi n}\frac{\Gamma(k-n)}{\Gamma(k)}(z-c)^{n-k} \quad (k \geq n+1) \quad \text{(Lemma 2)};$$

$$(\log(z-c))_{-n} = \frac{(z-c)^n}{n!}\{\log(z-c) - H_n\} \quad \text{(Lemma 3)}$$

and

$$((z-c)^{-k})_{-n} = \left(((z-c)^{-k})_{-k}\right)_{-(n-k)}$$

$$= -e^{-i\pi k}\frac{1}{\Gamma(k)}\,(\log(z-c))_{-(n-k)}$$

$$= \frac{(-1)^{k+1}}{\Gamma(k)}\,\frac{(z-c)^{n-k}}{(n-k)!}\,\{\log(z-c) - H_{n-k}\} \quad (n \geq k);$$

(2.4) becomes

$$\sum_{k=n+1}^{1} \frac{(-1)^n \text{¡} (k-n)}{k!}\left(\frac{-c}{z-c}\right)^k (z-c)^n$$
$$= \frac{(z-c)^n}{n!}\{\log(z-c) - H_n\} - \frac{z^n}{n!}\{\log z - H_n\}$$
$$+ \sum_{k=1}^{n} \frac{(-1)^k}{k!(n-k)!}\left(\frac{-c}{z-c}\right)^k (z-c)^n\{\log(z-c) - H_{(n \text{ ¡ } k)}\}:$$

Dividing by $(z-c)^n$, we prove the theorem.

Corollary 1. [2] *For* $\left|\frac{\text{¡} c}{z \text{ ¡ } c}\right| < 1$, *we have*

$$\sum_{k=2}^{1} \frac{(-c)^k}{k(k-1)} \frac{(kz-c)}{(z-c)^{k \text{ ¡ } 1}} = c^2:$$

Proof. Let $n = 1$ in Theorem, we obtain the previous result (2.1).

Corollary 2. *For* $\left|\frac{\text{¡} c}{z \text{ ¡ } c}\right| < 1$, *we have*

$$\sum_{k=3}^{1} \frac{1}{k}\left(\frac{-c}{z-c}\right)^k \left[\frac{(z-c)^2}{(k-1)(k-2)} - \frac{1}{2}z^2\right] = \frac{c^4}{4(z-c)^2}: \tag{2.5}$$

Proof. Let $n = 2$ in Theorem, we have

$$\sum_{k=3}^{1} \frac{\text{¡} (k-2)}{k!}\left(\frac{-c}{z-c}\right)^k$$
$$= \frac{1}{2}\left[\log(z-c) - H_2 - \left(\frac{z}{z-c}\right)^2 (\log z - H_2)\right] \tag{2.6}$$
$$+ \sum_{k=1}^{2} \frac{(-1)^k}{k!(2-k)!}\left(\frac{-c}{z-c}\right)^k [\log(z-c) - H_{2 \text{ ¡ } k}]:$$

Since $H_2 = \frac{3}{2}$, $H_1 = 1$ and $H_0 = 0$, (2.6) becomes

$$\sum_{k=3}^{1} \frac{\text{¡} (k-2)}{k!}\left(\frac{-c}{z-c}\right)^k = \frac{1}{4(z-c)^2}[2z^2(\log(z-c) - \log z) + 2cz + c^2]: \tag{2.7}$$

By using the well known-relation (2.3) again, we obtain

$$\sum_{k=3}^{1} \frac{i\,(k-2)}{k!}\left(\frac{-c}{z-c}\right)^{k}(z-c)^2 = \frac{1}{2}z^2\sum_{k=3}^{1}\frac{1}{k}\left(\frac{-c}{z-c}\right)^{k} + \frac{1}{2}z^2\left(\frac{-c}{z-c}\right)$$
$$+\frac{1}{4}z^2\left(\frac{-c}{z-c}\right)^{2} + \frac{1}{2}cz + \frac{1}{4}c^2:$$

Then, by simplifying, we have (2.5).

Corollary 3. *For* $\left|\frac{i\,c}{z\,i\,c}\right| < 1$, *we have*

$$\sum_{k=4}^{1}\frac{1}{k}\left(\frac{-c}{z-c}\right)^{k}\left[\frac{(z-c)^3}{(k-1)(k-2)(k-3)} + \frac{1}{6}z^3\right] \tag{2.8}$$
$$= \frac{1}{36}\frac{c^4}{(z-c)^3}(3z^2-3cz+2c^2):$$

Proof. Similarly, let $n = 3$ in Theorem, we have

$$\sum_{k=4}^{1}\frac{(-1)^3\,i\,(k-3)}{k!}\left(\frac{-c}{z-c}\right)^{k}$$
$$= \frac{1}{6}\left[\log(z-c) - H_3 - \left(\frac{z}{z-c}\right)^{3}(\log z - H_3)\right] \tag{2.9}$$
$$+\sum_{k=1}^{3}\frac{(-1)^k}{k!(3-k)!}\left(\frac{-c}{z-c}\right)^{k}[\log(z-c) - H_{3\,i\,k}]:$$

Since $H_3 = \frac{11}{6}$, $H_2 = \frac{3}{2}$, $H_1 = 1$ and $H_0 = 0$, (2.9) becomes

$$\sum_{k=4}^{1}\frac{-\,i\,(k-3)}{k!}\left(\frac{-c}{z-c}\right)^{k} \tag{2.10}$$
$$= \frac{1}{36}\frac{1}{(z-c)^3}[6z^3(\log(z-c) - \log z) + 6cz^2 + 3c^2z + 2c^3]:$$

Making use of the well-known relation (2.3), we obtain

$$\sum_{k=4}^{1}\frac{-\,i\,(k-3)}{k!}\left(\frac{-c}{z-c}\right)^{k}(z-c)^3$$
$$= \frac{1}{36}\left[6z^3\sum_{k=1}^{1}\frac{1}{k}\left(\frac{-c}{z-c}\right)^{k} + 6cz^2 + 3c^2z + 2c^3\right]$$

or, equivalently,

$$\sum_{k=4}^{1} \frac{1}{k}\left(\frac{-c}{z-c}\right)^k \left[\frac{(z-c)^3}{(k-1)(k-2)(k-3)} + \frac{1}{6}z^3\right]$$
$$= \frac{1}{36}\frac{c^4}{(z-c)^3}[3z^2 - 3cz + 2c^2]$$

Thus, we prove (2.8).

With the similar ways, by using our main generalization theorem, we will obtain a lot of families of Infinite Sum. These works are left to the interested readers.

TABLE I.

	C = 1; z = 2	C = 1; z = 3	C = 1; z = 5	C = 2; z = 5
m = 10	0.1555555555555556	0.06236436631944445	0.01562478277418348	0.4363517304837226
m = 30	0.2172413793103448	0.06249999995450435	0.015625	0.4444435932450222
m = 50	0.230204081632653	0.06249999999999997	0.015625	0.4444444442895319
m = 100	0.240050505050505	0.06249999999999999	0.015625	0.4444444444444442
m = 300	0.2466722408026755	0.06249999999999999	0.015625	0.4444444444444442
m = 500	0.248002004008016	0.06249999999999999	0.015625	0.4444444444444442
m = 1000	0.2490005005005006	0.06249999999999999	0.015625	0.4444444444444442
X	0.25	0.0625	0.015625	0.4444444444444444
	C = 2; z = 6	C = 2; z = 10	C = 3; z = ¡ 2	C = 4; z = ¡ 4
m = 10	0.2494574652777778	0.6249913109673394	0.81131010048	1.000607638888889
m = 30	0.2499999998180174	0.625	0.8100000202015822	1.000000000231224
m = 50	0.2499999999999999	0.625	0.81000000000046	1.999999999999992
m = 100	0.25	0.625	0.8099999999999997	1.999999999999992
m = 300	0.25	0.625	0.8099999999999997	1.999999999999992
m = 500	0.25	0.625	0.8099999999999997	1.999999999999992
m = 1000	0.25	0.625	0.8099999999999997	1.999999999999992
X	0.25	0.625	0.8100000000000001	1
	C = ¡ 2; z = 1	C = ¡ 3; z = ¡ 7	C = ¡ 5; z = 5	C = 10; z = ¡ 5
m = 10	0.4456005464388332	1.210250043869018	1.5634494357763889	11.14001366097083
m = 30	0.4444446002978597	1.265563727869062	1.56250000036129	11.1111150074465
m = 50	0.4444444444738906	1.265624882459362	1.562500000000001	11.1111111184726
m = 100	0.4444444444444446	1.265624999999966	1.562500000000001	11.1111111111111
m = 300	0.4444444444444446	1.265624999999999	1.562500000000001	11.1111111111111
m = 500	0.4444444444444446	1.265624999999999	1.562500000000001	11.1111111111111
m = 1000	0.4444444444444446	1.265624999999999	1.562500000000001	11.1111111111111
X	0.4444444444444444	1.265625	1.5625	11.1111111111111

TABLE II.

	$C = 1; z = 3$	$C = 1; z = 5$	$C = 2; z = 5$	$C = 2; z = 7$
$m = 10$	0.06958188657407406	0.02691009132950394	0.8860603528895964	0.4019378797985187
$m = 20$	0.06944451364625033	0.02690972222240444	0.8725495939883736	0.4017777864463448
$m = 30$	0.06944444448999106	0.0269097222222222	0.8724294034893044	0.4017777777783912
$m = 50$	0.06944444444444449	0.0269097222222222	0.8724279837973621	0.4017777777777778
$m = 100$	0.0694444444444445	0.0269097222222222	0.8724279835390948	0.4017777777777778
$m = 300$	0.0694444444444445	0.0269097222222222	0.8724279835390948	0.4017777777777778
$m = 500$	0.0694444444444445	0.0269097222222222	0.8724279835390948	0.4017777777777778
Y	0.0694444444444445	0.0269097222222222	0.8724279835390946	0.4017777777777778
	$C = -1; z = 2$	$C = -1; z = 4$	$C = -2; z = -5$	$C = -2; z = -7$
$m = 10$	0.02057512028616535	0.01377775307851852	-0.8860603528895964	-0.4019378797985187
$m = 20$	0.02057613167831183	0.01377777777777646	-0.8725495939883736	-0.4017777864463448
$m = 30$	0.0205761316872427	0.01377777777777778	-0.8724294034893044	-0.4017777777783912
$m = 50$	0.0205761316872428	0.01377777777777778	-0.8724279837973621	-0.401777777777778
$m = 100$	0.0205761316872428	0.01377777777777778	-0.8724279835390948	-0.401777777777778
$m = 300$	0.0205761316872428	0.01377777777777778	-0.8724279835390948	-0.401777777777778
$m = 500$	0.0205761316872428	0.01377777777777778	-0.8724279835390948	-0.401777777777778
Y	0.0205761316872428	0.01377777777777778	-0.8724279835390948	-0.4017777777777778

3. Numerical Results

Finally, by computer simulations, various numerical results concerning with the forms (2.5) and (2.8) are listed as follows:

(a) For Corollary 2,

take m = 10, 30, 50, 100, 300, 500 and 1000 in

$$\sum_{k=3}^{m} \frac{1}{k}\left(\frac{-c}{z-c}\right)^k \left[\frac{(z-c)^2}{(k-1)(k-2)} - \frac{1}{2}z^2\right]$$

and $X = \frac{c^4}{4(z-c)^2}$, our numerical result is given in Table 1.

(b) For Corollary 3,

take m = 10, 20, 30, 50, 100, 300, and 500 in

$$\sum_{k=4}^{m} \frac{1}{k}\left(\frac{-c}{z-c}\right)^k \left[\frac{(z-c)^3}{(k-1)(k-2)(k-3)} + \frac{1}{6}z^3\right]$$

and $Y = \frac{1}{36}\frac{c^4}{(z-c)^3}\,(3z^2 - 3cz + 2c^2)$, our numerical result is given in Table 2.

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$$\sum_{n=1}^{\infty} \frac{(n-1)!\, 2^{n-1}}{\prod_{k=0}^{n}(2k+3)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(n-1)!\,(n+1)\, 2^{n-1}}{\prod_{k=0}^{n+1}(2k+3)}$$

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$$R_{m,\beta} = (-1)^m \sum_{k=1}^{\infty} \frac{(-1)^k \cdot (m+k-1)!}{(m-1)!\, k} \cdot \frac{\Gamma(-m-k-\beta)}{\Gamma(-\beta)}$$

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