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AN EXTENSION OF DHH-ERDÖS CONJECTURE ON CYCLE-PLUS-TRIANGLE GRAPHS

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Abstract. Consider n disjoint triangles and a cycle on the $3n$ vertices of the n triangles. In 1986, Du, Hsu, and Hwang conjectured that the union of the n triangles and the cycle has independent number n. Soon later, Paul Erdös improved it to a stronger version that every cycle-plus-triangle graph is 3-colorable. This conjecture was proved by H. Fleischner and M. Stiebitz. In this note, we want to give an extension of the above conjecture with an application in switching networks.

1. INTRODUCTION

Consider n disjoint triangles and a cycle on the $3n$ vertices of the n triangles. The union of the n triangles and the cycle is called a cycle-plus-triangle graph. In 1986, Du, Hsu, and Hwang [3]† conjectured that every cycle-plus-triangle graph has independent number n, i.e., the maximum independent set contains n vertices. Soon later, Paul Erdös got interested in this conjecture and improved it to a stronger version that every cycle-plus-triangle graph is 3-colorable. Due to Erdös' promotion in his frequent traveling, this conjecture becomes quite well-known during the past ten years. There were several efforts [5, 1] to attack the conjecture and it was finally proved by H. Fleischner and M. Stiebitz [6].

In this note, we want to give an extension of the above conjecture with an application in switching networks.

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2. Extension

Let us first consider a little general graphs. Instead of a cycle, let us consider a union of disjoint cycles on the $3n$ vertices of the n triangles. That is, we consider a graph G constructed by taking the union of n disjoint triangles and a disjount union of cycles on the $3n$ vertices of the n triangle.

Is G still 3-colorable? The answer is MAY BE NOT. In fact, the graph in Figure 1 is not 3-colorable since it contains a clique of size four. But it can be obtained by taking union of four disjoint triangles and a union of three cycles of size four.

This example also shows that a similar conjecture made in [4] is false. The conjecture is as follows: Consider a line graph G of a d-regular graph. Partition the vertex set of G into disjoint subsets of size exactly n with $d \leq n \leq 2d-1$ and for each subset, construct a clique on it. Then the resulting graph G^* is $2d-1$ vertex-colorable. In the above counterexample, we have $d = 2$ and $n = 3$. G consists of three cycles of size 4 and its vertices are divided into four subsets of size exactly n. But, G^* is not $(2d-1)$-colorable.

The graph in Figure 1 is 4-colorable. In general, is G 4-colorable? The answer is YES. In fact, every vertex in G has degree four. It is well-known that a connected 4-regular graph is 4-colorable unless it is a complete graph of order five [2]. Clearly, G cannot have a connected component of size five. Therefore, G is 4-colorable.

The above observation suggests the following conjecture.

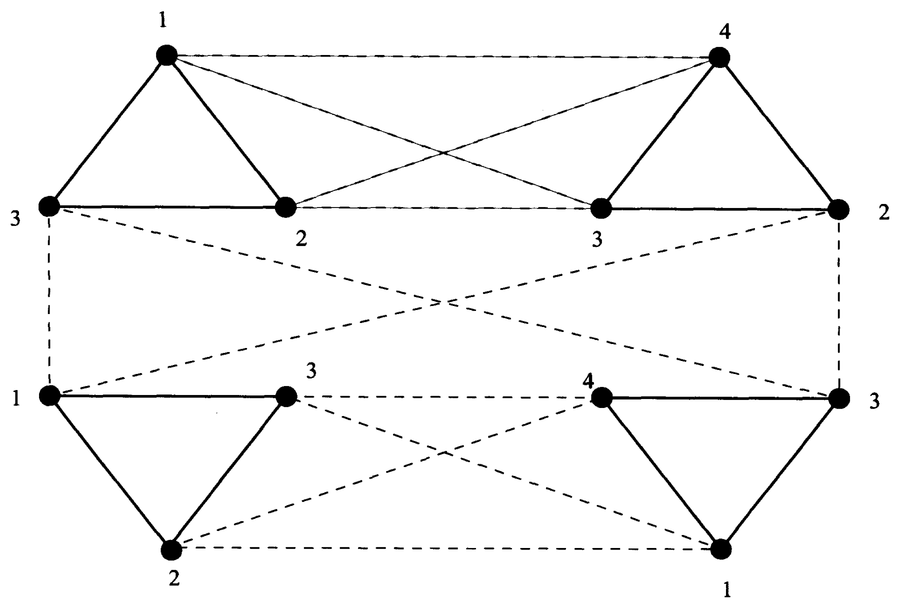

FIG. 1. Not 3-colorable but 4-colorable.

Conjecture 1. *Consider a graph H with maximum degree m. Let $L(H)$ be the line graph of H. Divide all vertices of $L(H)$ into disjount groups of size at most n. Connect all vertices in each group into a clique. If $m \leq n$, then the resulting graph is $(m+n)$-colorable.*

The reader may have question on the coloring number $m+n$. Why do we use $m+n$ instead of $m+n-1$? In fact, for the above example, we have $m=2$ and $n=3$ and the resulting graph is $(m+n-1)$-colorable. The following example may provide an explanation.

Let H be a complete graph of order four. Let a, b, c, d be vertices of H. The line graph $L(H)$ of H contains six vertices ab, cd, ac, bd, ad, bc. Now, we divide them into three groups $\{ab, cd\}$, $\{ac, bd\}$, $\{ad, bc\}$. Connect every two vertices in the same group with an edge. The resulting graph G is a complete graph of order six. Thus, it cannot be 5-colorable. However, we can have $m=3$, $n=3$ and $m+n-1=5$ (note: $m \leq n$). Actually, in this example, each group has size 2 (less than 3). Therefore, this is also an example to explain why we need condition $m \leq n$. In fact, if we remove the condition $m \leq n$, then the example fits the condition $m=3$ and $n=2$. In this case, $m+n=5$. However, G is not 5-colorable.

Theorem 1. *Conjecture 1 holds for $m=2$ and 3.*

Proof. It is a well-known fact that every graph with maximum degree $\Delta \geq 3$ must be $(\Delta+1)$-colorable and, furthermore, it is Δ-colorable unless the graph contains a subgraph isomorphic to the complete graph of order $\Delta+1$, i.e., a clique of size $\Delta+1$ [2]. Note that the resulting graph in Conjecture 1 has maximum degree $2(m-1)+n-1$. For $m=2$, $(2(m-1)+n-1)+1 = m+n$ and for $m=3$, $2(m-1)+n-1 = m+n$. Thus, it suffices to show that for $m=3$, the resulting graph does not contain a clique of size $n+4$. For contradiction, suppose that the resulting graph contains a clique Q of size $n+4$. Since it has maximum degree $n+3$, the clique Q must be a connected component of it. Thus, we may assume, without loss of generality, that the resulting graph itself is the clique Q. Now, we want to prove that Q cannot be obtained in the way described in Conjecture 1. To do so, we consider the problem of removing disjoint cliques of size at most n to obtain a graph with maximum degree at most $2(m-1)=4$. Since every vertex in Q has degree $m+n=n+3$, each removed clique has to have size n in order to have degree $n-1$ at each vertex. It follows that $n|(n+4)$. Since $n \geq m=3$, we must have $n=4$. Thus, Q is a clique of size 8. Removing two disjoint cliques of size 4 from Q results in a graph P as shown in Figure 2. This graph P cannot be the line graph $L(H)$ of a graph H with maximum degree at most three. In fact, P is 4-regular. If $P = L(H)$, then H must be 3-regular. So, each vertex of P must

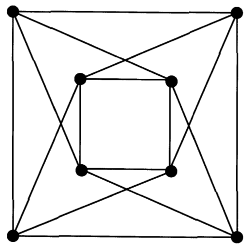

FIG. 2. Graph P.

be adjacent to four vertices which can be divided into two pairs such that vertices in the same pair are adjacent. However, this is not true to P, a contradiction. ■

Now, we propose a direct generalization of DHH-Erdös conjecture as follows.

Conjecture 2. *Consider a m-regular m-connected graph H. Let $L(H)$ be the line graph of H. Suppose all vertices of $L(H)$ can be divided into disjount groups of size exactly n. Add a clique of size n on vertices in each group. The resulting graph is $(m + n - 2)$-colorable.*

The results in [6, 5] show that this conjecture holds for $m = 2$.

3. An Application to Switching Networks

Conjecture 1 has an application in switching networks. To see it, let us first introduce some concepts in switching networks.

A three-stage Clos network $C(n_1, n_3, r_1, r_2, r_3)$ consists of r_1 many $n_1 \times r_2$ crossbars in the first stage, r_2 many $r_1 \times r_3$ crossbars in the second stage, and r_2 many $r_2 \times n_3$ crossbars in the third stage. Every crossbar in the first stage has an outlet connected to an inlet of every crossbar in the second stage and every crossbar in the second stage has an outlet connected to an inlet of every crossbar in the third stage (Figure 3). There are totally $r_1 n_1$ inlets in the first stage and totally $r_3 n_3$ outlets in the third stage. Denote by I the set of all $r_1 n_1$ inlets in the first stage and by O the set of all $r_3 n_3$ outlets in the third stage. Then a *connection* in a three-stage Clos network is a pair (x, y), where $x \in I$ and $y \in O$. A *route* is a path in the network joining an input crossbar (i.e., a crossbar in the first stage) to an output crossbar (i.e., a crossbar in the third stage) and a route r *realizes* a connection (x, y) if x and y belong to the input crossbar and the output crossbar joined by r, respectively.

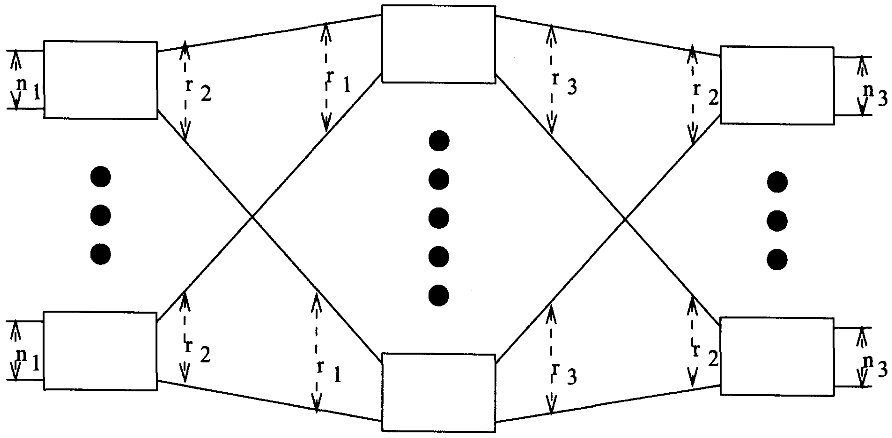

FIG. 3. Three-stage Clos network.

A set of connections is *compatible* if for every $x \in I$, there are at most n_1 connections involving x and for every $y \in O$, there are at most n_3 connections involving y. A *configuration* is a set of routes and it is *compatible* if every edge in the network is used only once. A set of connections is said to be *realizable* if there exists a compatible configuration which contains routes realizing all connections in the set. A network is said to be *rearrangeable* if every compatible set of connections is realizable. It is well-known that a three-stage Clos network $C(n_1, n_3, r_1, r_2, r_3)$ is rearrangeable if and only if $\min(n_1, n_3) \leq r_2$.

A connection c is said to be *compatible* with a compatible set C of connections if $C \cup \{c\}$ is still compatible. A route r is said to be *compatible* with a compatible configuration R if $R \cup \{r\}$ is still compatible. A network is said to be *strictly nonblocking* if for every compatible configuration R realizing a connection set C and every connection c compatible with C, there exists a route r such that r realizes c and is compatible with R. A three-stage Clos network $C(n_1, n_3, r_1, r_2, r_3)$ is strictly nonblocking if and only if $m \geq n_1 + n_3 - 1$.

Above concepts can be easily extended to one-to-many connections. A 1-to-k connection is a $(k+1)$-tuplc $(x; y_1, y_2 \cdots, y_k)$, where $x \in I$ and $y_1, y_2, \cdots, y_k \in O$.

Note that for those y_j's lying in the same output crossbar, the path from x can branch at that output crossbar to reach these y_j's. But for y_j's lying in different output crossbars, the branching has to take place either in the input crossbar or at a center crossbar. It is well-known [8] that if branching at imput crossbars is allowed, then $C(n, m, r) = C(n, n, r, m, r)$ is rearrangeable for 1-to-k connections if $m \geq kn$. However, the case that imput switches do not have the branching capability remains an open problem. Hwang and Lin [7] conjectured that $C(n, m, r)$

is rearrangeable for $1 - to - 2$ connections and meanwhile strictly nonblocking for 1-to-1 connection if $m \geq 2n$. This conjecture can be extended to asymmetric three-stage Clos networks. In fact, this extension has connection to Conjecture 1.

Theorem 2. *Suppose Conjecture 1 holds. If* $r_2 \geq n_1 + n_3$ *and* $n_1 \geq n_3$, *then* $C(n_1, n_3, r_1, r_2, r_3)$ *is rearrangeble for* 1*-to-*2 *connections.*

Proof. Suppose $\{(x_i; y_{2i-1}, y_{2i})\}$ is a set of compatible 1-to-2 connections. That is, at most n_1 many x_i's are the same and at most n_3 many y_j's are the same.

First, consider the case that for each 1-to-2 connection $(x_i; y_{2i-1}, y_{2i})$, $y_{2i-1} \neq y_{2i}$. Let H be the graph with vertex set O and edge set $\{(y_{2i-1}, y_{2i})\}$. Then H has maximum degree at most n_3. Let $L(H)$ be the line graph of H. Divide all vertices of $L(H)$ into disjoint groups such that two vertices (y_{2i-1}, y_{2i}) and (y_{2j-1}, y_{2j}) are in the same group if and only if $x_i = x_j$. Thus, each group has size at most n_1. Connect all vertices in each group into a clique. Since Conjecture 1 is assumed to be true, the resulting graph is (n_1+n_3)-*colorable.* Note that if two vertices (y_{2i-1}, y_{2i}) and (y_{2j-1}, y_{2j}) are in the same color, then we must have $x_i \neq x_j$. Note that each vertex (y_{2i-1}, y_{2i}) represents a connection $(x_i; y_{2i-1}, y_{2i})$. Therefore, if we arrange all 1-to-2 connections in the same color to pass through the same middle switch, then each input switch has at most one connection to this middle switch. Moreover, the middle switch has at most one connection to each output switch since two 1-to-2 connections have the same component in output switches must be adjacent in $L(H)$. Therefore, if $r_2 \geq n_1 + n_3$, $C(n_1, n_3, r_1, r_2, r_3)$ is rearrangeable for 1-to-2 connections.

Now, we consider the general case. If $y_{2i-1} = y_{2i}$ for some 1-to-2 connection $(x_i; y_{2i-1}, y_{2i})$, then we may add a new output switch and change y_{2i} to the new output switch. In this way, we can reduce the general case to the first case. ∎

4. Discussion

The connection of Conjecture 1 to the rearrangeability for 1-to-2 connections may suggest a generalization of Conjecture 1, corresponding to the rearrangeability for 1-to-k connections.

Let V_1 and V_2 be two disjoint sets of vertices. A (h, k)-bipartite hypergraph (V_1, V_2, E) is a hypergraph such that each hyper-edge $e \in E$ contains at most h vertices in V_1 and at most k vertices in V_2. The degree of each vertex is the number of hyper-edges containing the vertex. The generalization can be stated as follows: Consider a $(1, k)$-bipartite hypergraph $G(V_1, V_2, E)$. Suppose each vertex in V_1 has degree at most n and each vertex in V_2 has degree at most m. If $m \leq n$, then G is $(n + 1 + (k - 1)(m - 1))$-edge-colorable, i.e., all hyper-edges of G can be in $(n + 1 + (k - 1)(m - 1))$ colors such that any two hyper-edges in the same color are not adjacent.

Unfortunately, this generalization is false. The following is a conterexample.

Choose $V_1 = \{I_1, I_2, I_3\}$ and $V_2 = \{O_1, O_2, \cdots, O_9\}$. Consider the following edge set E:

$$(I_1; O_1, O_2, O_3),\ (I_1; O_4, O_5, O_6),\ (I_1; O_7, O_8, O_9),$$
$$(I_2; O_1, O_4, O_7),\ (I_2; O_2, O_5, O_8),\ (I_2; O_3, O_6, O_9),$$
$$(I_3; O_1, O_5, O_8),\ (I_3; O_2, O_4, O_9),\ (I_3; O_3, O_5, O_7).$$

Then we have $k = m = n = 3$. But, G is not 8-edge-colorable. In fact, the edge graph of G is a complete graph of order 9.

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