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NORMAL STRUCTURE AND THE ARC LENGTH IN BANACH SPACES

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Abstract. Let X be a Banach space, $X_2 \subseteq X$ be a two dimensional subspace of X, and $S(X) = \{x \in X, \|x\| = 1\}$ be the unit sphere of X. The relationship between the normal structure and the arc length in X is studied. Let $R(X) = \inf\{l(S(X_2)) - r(X_2) : X_2 \subseteq X\}$, where $l(S(X_2))$ is the circumference of $S(X_2)$ and $r(X_2) = \sup\{2(\|x+y\| + \|x-y\|) : x, y \in S(X_2)\}$ is the least upper bound of the perimeters of the inscribed parallelogram of $S(X_2)$. The main result is that $R(X) > 0$ implies X has the uniform normal structure.

1. Introduction

In a series of papers, Schäffer made use of the concept of geodesic to study the unit sphere of a Banach space X (see [13] for the complete references). He introduced the following two notations: $m(X) = \inf\{\delta(x, -x) : x \in S(X)\}$, and $M(X) = \sup\{\delta(x, -x) : x \in S(X)\}$ where $S(X)$ is the unit sphere of X and $\delta(x, -x)$ the shortest length of arcs joining antipodal points on $S(X)$. He called 2m(X) the girth, and 2M(X) the perimeter of X. These parameters were used to study reflexivity and isomorphism of Banach spaces among other things. But besides L_1 spaces, $C(K)$ spaces and Hilbert spaces, the values of these parameters are difficult to obtain.

We introduced a geometric parameter $J(X) = \sup\{\|x+y\| \bigwedge \|x-y\|\} : x, y \in S(X)\}$, a simplification of Schäffer's girth and perimeter, into a Banach space X (see [7] for the complete references). We proved that $J(X) < 3/2$ implies the uniform normal structure, which, in turn, implies the fixed point property. It is a well-known result that $\delta(1) > 0$ implies normal structure, where $\delta(\epsilon)$ is the modulus

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of convexity. We gave an example of a Banach space X with $J(X) < 3/2$ and $\delta(1) = 0$ to show the significance of the parameter $J(X)$. We also computed the values of $J(X)$ for some classical Banach spaces (see Appendix, §6), and posted a related question as to whether uniformly nonsquare Banach spaces have the fixed point property.

In this paper, we introduce another geometric parameter, $R(X)$, into a Banach space X, and prove that for a Banach space X, $R(X) > 0$ implies the uniform normal structure. We then give in §4 an example of a Banach space X with $R(X) > 0$ and $J(X) > 3/2$. Significantly, this means that the parameter $R(X)$ is really distinct from $J(X)$. However, whether uniformly nonsquare Banach spaces have the fixed point property is still an open question.

2. Preliminaries

Let X be a normed linear space, and let $S(X) = \{x \in X : \|x\| = 1\}$ be the unit sphere of X.

2-1. Curves in Banach Spaces

A continuous mapping $x(t)$ from a closed interval $[a, b]$ to a Banach space X is called a curve in $X : C = x(t), a \le t \le b$. A curve is called simple if it does not have multiple points. A curve is called closed if $x(a) = x(b)$. A closed curve is called symmetric about the origin if $x \in C$, then also $-x \in C$.

The concept of the length of a curve in Banach spaces resembles the same concept in Euclidean spaces. For curve $C = x(t)$, let P stand for a partition $a = t_0 < t_1 < t_2 < ... < t_i < ... < t_n = b$ of interval $[a, b]$ and $l(C, P) = \sum_{i=1}^{n} \|x(t_i) - x(t_{i-1})\|$, where $x_i(t)$, $i = 0, 1, 2, ..., n$ are called partition points on C. Then the length $l(C)$ of curve $C = x(t), a \le t \le b$, is defined as the least upper bound of $l(C, P)$ for all possible partitions P of $[a, b]$:

$$l(C) = \sup_P\{l(C, P)\}.$$

If $l(C)$ is finite, the curve is called rectifiable.

Let $\|P\| = \max_{1\le i\le n}\{|t_i - t_{i-1}|\}$ for a partition P of $[a, b]$.

Theorem 1 [2, 13]. *If curve C is rectifiable, then for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|P\| < \delta$ implies $l(C) - l(C, P) < \epsilon$. Furthermore, if $\{P_k\}$ is a sequence of partitions of $[a, b]$ with $\|P_k\| \to 0$, then $\lim_{k\to\infty} l(C, P_k) = l(C)$.*

Let $l_a^t(C)$ denote the length of curve $C = x(t)$ from a to t. For a rectifiable curve $C = x(t), a \le t \le b$, the arc length $l_a^t(C)$ is a continuous function of t.

Definition 1 [2, 13]. Let $y(s)$ represent the point $x(t)$ on the curve C for which $l_a^t(C) = s$. Then $C = y(s)$, $0 \leq s \leq l(C)$, is called the standard form of the rectifiable curve C.

For a normed linear space X , we use X_2 to denote a two-dimensional subspace of X. Then $S(X_2)$ is a simple closed curve which is symmetric about the origin and unique up to orientation.

Theorem 2 [2, 13]. *Let X_2 be a two-dimensional Banach space, and K_1, K_2 be closed convex subsets of X_2 with nonvoid interiors. If $K_1 \subseteq K_2$, then $l(\partial(K_1)) \leq l(\partial(K_2))$, where $l(\partial(K_i))$ denotes the length of the circumference of K_i, $i = 1, 2$.*

Theorem 3 [13]. $l(S(X_2)) \leq 8$; $l(S(X_2)) = 8$ *if and only if* $S(X_2)$ *is a parallelogram.*

Theorem 4 [13]. $l(S(X_2)) \geq 6$; $l(S(X_2)) = 6$ *if and only if* $S(X_2)$ *is an affinely regular hexagon.*

2-2. Normal Structure in Banach Spaces

In 1948, Brodskii and Milman [1] introduced the following geometric concepts:

Definition 2. A bounded, convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$ such that $\sup\{\|x_0 - y\|,\ y \in H\} < d(H)$, where $d(H) = \sup\{\|x - y\|, x, y \in H\}$ denotes the diameter of H. A Banach space X is said to have normal structure if every bounded, convex subset of X has normal structure. A Banach space X is said to have weak normal structure if for each weakly compact convex set K in X that contains more than one point has normal structure. X is said to have uniform normal structure if there exists $c, 0 < c < 1$, such that for any subset K as above, there exists $x_0 \in K$ such that $\sup\{\|x_0 - y\|,\ y \in K\} < c \cdot (d(K))$.

For a reflexive Banach space X, the normal structure and weak normal structure coincide.

In 1964, Kirk [10] proved that if a weakly compact subset K of X has normal structure then any nonexpansive mapping on K has a fixed point. Since then much attention has been focused on normal structure. Whether or not a Banach space has normal structure depends on the geometry of the unit sphere. We refer the interested reader to [4, 5, 6, 7, 8, 11, 15, 16].

Lemma 1 [5]. *Let X be a Banach space without weak normal structure. Then for any $\epsilon, 0 < \epsilon < 1$, there exists a sequence $\{z_n\} \subseteq S(X)$ with $z_n \to^w 0$, and*

$$1 - \epsilon < \|z_{n+1} - z\| < 1 + \epsilon$$

for sufficiently large n and any $z \in co\{z_k\}_{k=1}^n$.

Lemma 2 [7]. *Let X be a Banach space without weak normal structure. Then for any $\epsilon,\ 0 < \epsilon < 1$, there exist $x_1,\ x_2,\ x_3$ in $S(X)$ satisfying*

(i) $x_2 - x_3 = ax_1$ *with* $|a - 1| < \epsilon$,

(ii) $|\|x_1 - x_2\| - 1|$, $|\|x_3 - (-x_1)\| - 1| < \epsilon$, *and*

(iii) $\|(x_1 + x_2)/2\|$, $\|(x_3 - x_1)/2\| > 1 - \epsilon$.

The geometric meaning of the lemma can be succinctly described as follows: if X does not have weak normal structure, then there exists an inscribed hexagon in $S(X)$ with length of each side arbitrarily closed to 1 (by (i) and (ii)), and with at least four sides whose distance to $S(X)$ are arbitrarily small (by (iii)).

3. Parameter $R(X)$ and Normal Structure

For a Banach space X, let $B(X) = \{x \in X : \|x\| \leq 1\}$ be the ball of X, $B_0(X) = B(X) \setminus S(X)$ be the interior of $B(X)$. If $K \subseteq X$, let co(K) be the convex hull of subset K of X.

If $x, y \in S(X_2)$, then $2(\|x + y\| + \|x - y\|)$ is the perimeter of inscribed parallelogram with vertices $x, y, -x$, and $-y$ of $S(X_2)$.

Let $r(X_2) = \sup\{2(\|x + y\| + \|x - y\|) : x, y \in S(X_2)\}$. Then $r(X_2) \leq l(S(X_2))$ by Theorem 2.

Definition 3. For a Banach space X, define $R(X) = \inf\{l(S(X_2)) - r(X_2) : X_2 \subseteq X\}$.

For a Hilbert space H, $R(H) = 2\pi - 4\sqrt{2}$.

Theorem 5. *If X is a Banach space with $R(X) > 0$, then X is uniformly nonsquare.*

Proof. Suppose X is not uniformly nonsquare. For any $\epsilon > 0$, there exist $x, y \in S(X)$ such that both $\|x+y\|$ and $\|x-y\| > 2-(\epsilon/4)$ [9]. let X_2 be the two-dimensional space spanned by x and y. Then $r(X_2) \geq 2(\|x+y\|+\|x-y\|) > 8-\epsilon$, and hence $R(X) = \inf\{l(S(X_2)) - r(X_2) : X_2 \subseteq X\} < \epsilon$. Since ϵ can be arbitrarily small, we have $R(X) = 0$. ∎

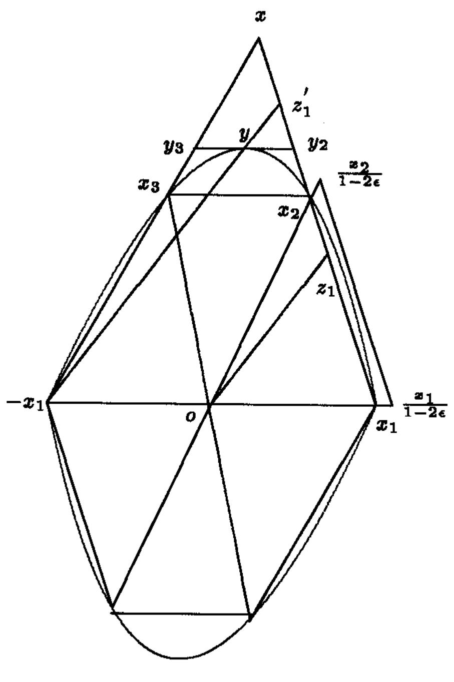
Lemma 3 [7]. *Let $x, y \in B(X)$ and $0 < \epsilon < 1$ such that $\|x+y\|/2 > 1-\epsilon$. Then for any $c, 0 \le c \le 1$, and $z = cx + (1-c)y \in \mathrm{co}(\{x, y\})$, the line segment connecting x and y, we have $\|z\| > 1 - 2\epsilon$.*

Theorem 6. *If X is a Banach space with $R(X) > 0$, then X has normal structure.*

Proof. $R(X) > 0$ implies that X is uniformly nonsquare, and hence reflexive [9]. Normal structure and weak normal structure coincide.

Suppose X dose not have weak normal structure. For $\epsilon > 0$, let x_1, x_2 and x_3 in S(X) satisfy the conditions in Lemma 2. Let X_2 be the two-dimensional space spanned by x_1 and x_2 (see the Figure below).

Then there exists an $x \in X$ and a real number α such that $x - x_2 = \alpha(x_2 - x_1)$ and $x - x_3 = \alpha(x_3 + x_1)$. Let $y_2 \in$ the line segment $co(\{x, x_2\})$ and $y_3 \in$ the line segment $co(\{x, x_3\})$, and β be the real number such that $y_2 - x_2 = \beta(x_2 - x_1)$, $y_3 - x_3 = \beta(x_3 + x_1)$, $co(\{y_2, y_3\}) \bigcap S(X) \neq \emptyset$, and $co(\{y_2, y_3\}) \subseteq X_2 \backslash B_0(X_2)$. Then $y_2 - y_3 = \delta(x_2 - x_3)$, where $0 < \delta < 1$. Furthermore, we can take ϵ small enough such that $\beta < (1+\epsilon)/(1-\epsilon) < 2$. Then



Figure

$$\|y_2 - x_2\| = \beta\|x_2 - x_1\| \le \beta(1+\epsilon) \le \beta + 2\epsilon,$$
$$\|y_3 - x_3\| = \beta\|x_3 + x_1\| \le \beta(1+\epsilon) \le \beta + 2\epsilon,$$
$$\frac{\|x_2 - x_1\|}{\|x - x_2\|} = \frac{2 - \|x_2 - x_3\|}{\|x_2 - x_3\|} \ge \frac{1-\epsilon}{1+\epsilon} \ge 1 - 2\epsilon,$$
$$\delta = \frac{\|y_2 - y_3\|}{\|x_2 - x_3\|} = \frac{\|x - y_2\|}{\|x - x_2\|} = 1 - \frac{\|y_2 - x_2\|}{\|x - x_2\|}$$
$$= 1 - \frac{\|y_2 - x_2\|}{\|x_2 - x_1\|} \cdot \frac{\|x_2 - x_1\|}{\|x - x_2\|} \le 1 - \beta(1 - 2\epsilon) \le 1 - \beta + 4\epsilon,$$

and

$$\|y_2 - y_3\| = \delta\|x_2 - x_3\| \le (1 - \beta + 4\epsilon)(1+\epsilon) \le 1 - \beta + 4\epsilon + 2\epsilon = 1 - \beta + 6\epsilon.$$

Therefore, the length of the curve from x_2 to x_3 on $S(X_2)$ is less than or equal to $2(\beta + 2\epsilon) + 1 - \beta + 6\epsilon = 1 + \beta + 10\epsilon$.

Since $\|x_1 + x_2\|/2 > 1 - \epsilon$, for any $z \in \text{co}(\{x_1, x_2\})$, we have $\|z\| > 1 - 2\epsilon$. So, the line segment $\text{co}(\{x_1/(1-2\epsilon), x_2/(1-2\epsilon)\}) \subseteq X_2 \backslash B_0(X_2)$, and hence the length of the curve from x_1 to x_2 on $S(X_2) \le$ the sum of the lengths of the line segments $\text{co}(\{x_1, x_1/(1-2\epsilon)\})$, and $\text{co}(\{x_1/(1-2\epsilon), x_2/(1-2\epsilon)\})$, and $\text{co}(\{x_2/(1-2\epsilon), x_2\}) \le 2/(1-2\epsilon) - 2 + (1+\epsilon)/(1-2\epsilon) \le 1 + 8\epsilon$.

Similarly, the length of the curve from x_3 to $-x_1$ on $S(X_2)$ is less than or equal to $1 + 8\epsilon$.

Therefore, $l(S(X_2)) = 2$ (length of the curve from x_1 to x_2 on $S(X_2)$ + length of the curve from x_2 to x_3 on $S(X_2)$ + length of the curve from x_3 to $-x_1$ on $S(X_2) \le 2(2(1+8\epsilon) + 1 + \beta + 10\epsilon) = 2(3 + \beta + 26\epsilon) = 6 + 2\beta + 52\epsilon$. We have

$$l(S(X_2)) \le 6 + 2\beta + 52\epsilon. \tag{3.1}$$

On the other hand, let $y \in \text{co}(\{y_2, y_3\}) \cap S(X_2)$. There must exist an $z_1' \in \text{co}(\{x_2, x\})$ and $z_1 \in \text{co}(\{x_1, x\})$ such that $y \in \text{co}(\{-x_1, z_1'\})$, and $\|z_1' + x_1\| = 2\|z_1\|$. If $z_1 \in \text{co}(\{x_1, x_2\})$, then $\|z_1\| \ge 1 - 2\epsilon$, and hence $\|z_1' + x_1\| = 2\|z_1\| \ge 2(2 - 2\epsilon) = 2 - 4\epsilon$. Since $\|z_1' - y\|/\|z_1\| = \|y - y_2\|/\|x_1\|$, $\|z_1' - y\| = \|z_1\|(\|y - y_2\|) \le \|y - y_2\|$, and $\|y + x_1\| = \|z_1' + x_1\| - \|z_1' - y\| \ge 2 - 4\epsilon - \|y - y_2\|$. If $z_1 \in \text{co}(\{x_2, x\})$, then $\|z_1' + x_1\| = 2\|z_1\| \ge 2$. We need the following fact.

Fact: Suppose $u = x_2 + t(x_2 - x_1)$, $u_1 = x_2 + t_1(x_2 - x_1)$, $t_1 \ge 0$, and $\|u_1 + x_1\| \ge 2$. Then $\|u + x_1\|$ is an increasing function of t on $[t_1, \infty)$.

Proof of the fact: Let $U(x, a) = \{y \in X : \|y - x\| \le a\}$ and $S(x, a) = \{y \in X : \|y - x\| = a\}$ be the unit ball and the unit sphere of X with center at x and radius a, respectively. Since $\|x_2-(-x_1)\| \leq 2$, there exists $v_1 \in \mathrm{co}(\{x_2, u_1\})$ such that $v_1 \in S(-x_1, 2)$.

If $\|u+x_1\|$ is not an increasing function, let $t_1 \leq t_2 \leq t_3$ such that $\|u_2+x_1\| = b > \|u_3+x_1\|$, where $u_2 = x_2 + t_2(x_2 - x_1)$, and $u_3 = x_2 + t_3(x_2 - x_1)$. Since $v_1 \in \mathrm{co}(\{x_2, u_1\}) \subseteq \mathrm{co}(\{x_2, u_2\})$, $b \geq 2$ by the convexity of $U(-x_1, 2)$.

Consider $v_2 = 2(u_2 + x_1)/b - x_1$, and $v_3 = 2(u_3 + x_1)/b - x_1$. Then $v_2 \in S(-x_1, 2)$, and $v_3 \in U(-x_1, 2) \setminus S(-x_1, 2)$. Since $u_2 = cv_1 + (1-c)u_3$, where $0 \leq c \leq 1$, we have $u_2 + x_1 = c(v_1 + x_1) + (1-c)(u_3 + x_1)$, $\|u_2 + x_1\| \leq c\|v_1 + x_1\| + (1-c)(\|u_3 + x_1\|)$, and $b\|v_2 + x_1\|/2 \leq 2c + b(1-c)\|v_3 + x_1\|/2$. Therefore $\|v_2+x_1\| < 2(2c+(1-c)b)/b \leq 2$. This contradicts with $v_2 \in S(-x_1, 2)$.

From the previous fact we have $\|z_1^{'}+x_1\| \leq \|x+x_1\| \leq 2(1+\epsilon)/(1-\epsilon) \leq 2+6\epsilon$. Hence $\|z_1\| \leq 1 + 3\epsilon$, and $\|z_1^{'} - y\| = \|z_1\|(\|y - y_2\|) \leq (1 + 3\epsilon)(\|y - y_2\|)$. So, $\|y + x_1\| = \|z_1^{'} + x_1\| - \|z_1^{'} - y\| \geq 2 - (1 + 3\epsilon)\|y - y_2\| \geq 2 - 4\epsilon - \|y - y_2\|$. Finally, we proved $\|y + x_1\| \geq 2 - 4\epsilon - \|y - y_2\|$.

Similarly, $\|y - x_1\| \geq 2 - 4\epsilon - \|y - y_3\|$.

So, $\|y+x_1\|+\|y-x_1\| \geq 4-8\epsilon-(\|y-y_2\|+\|y-y_3\|) = 4-8\epsilon-\|y_2-y_3\| \geq 4 - 8\epsilon - (1 - \beta + 6\epsilon) = 3 + \beta - 14\epsilon$, and hence

(3.2) $$r(X_2) = \sup\{2(\|x + y\| + \|x - y\|) : x, y \in S(X_2)\} \geq 6 + 2\beta - 28\epsilon.$$

From (3.1) and (3.2), we have $R(X) = \inf\{l(S(X_2)) - r(X_2) : X_2 \subseteq X\} < 80\epsilon$. Since ϵ can be arbitrarily small, we have $R(X) = 0$. ■

4. R(X) and Other Parameters

Let $\delta(\epsilon) = \inf\{1 - (\|x + y\|/2) : \|x - y\| \geq \epsilon, x, y \in S(X)\}, 0 \leq \epsilon \leq 2$, be the modulus of convexity of X. Since $\inf\{1 - (\|x + y\|)/2 : \|x - y\| \geq \epsilon, x, y \in S(X)\} = \inf\{1 - (\|x + y\|)/2 : \|x - y\| = \epsilon, x, y \in S(X)\}$, $0 \leq \epsilon \leq 2$, we have $\delta(\|x - y\|) \leq 1 - (\|x + y\|)/2$, for any $x, y \in S(X)$.

Let $l(X) = \inf\{l(S(X_2)) : X_2 \subseteq X\}$. Then $6 \leq l(X) \leq 8$.

Lemma 4. *For a Banach space X, $\delta(2^-) > 0$, where $\delta(2^-) = \lim_{\epsilon \to 2} \delta(\epsilon)$, implies that X is uniformly nonsquare.*

Proof. If X is not uniformly nonsquare, let $x, y \in S(X)$ be as in Theorem 5. Then $\delta(2 - (\epsilon/4) \leq 1 - (2 - (\epsilon/4))/2 = \epsilon/8$. Letting $\epsilon \to 0$, we have $\delta(2^-) = 0$. ■

Theorem 7. *For a Banach space* X, $\delta(l(X)/4) > 2 - (l(X)/4$ *implies* X *has normal structure.*

Proof. $\delta(l(X)/4) > 2 - (l(X)/4)$ implies $\delta(2^-) > 0$, so X is uniformly nonsquare. Hence X is reflexive, therefore weak normal structure and normal structure coincide.

If X fails to have normal structure, for any $\epsilon > 0$, let x_1, x_2, x_3 and y be in Theorem 6. Then $1-\epsilon \leq \|y-x_1\| \leq 2$, $1-\epsilon \leq \|y+x_1\| \leq 2$, and $l(X)/2-80\epsilon \leq \|y-x_1\| + \|y+x_1\|$, from Theorem 6. So, $l(X)/2 - 2 - 80\epsilon \leq \min\{\|y-x_1\|, \|y+x_1\|\}$, and $\max\{\|y-x_1\|, \|y+x_1\|\} \geq l(X)/4 - 40\epsilon$. Recall that $\delta(\epsilon)$ is an increasing function on $[0,2]$. Thus $\delta(l(X)/4 - 40\epsilon) \leq \delta(\max\{\|y-x_1\|, \|y+x_1\|\}) \leq 1 - \min\{\|y-x_1\|, \|y+x_1\|\}/2 \leq 2 - l(X)/4 + 40\epsilon$. By letting $\epsilon \to 0$, we have $\delta(l(X)/4) \leq 2 - l(X)/4$.

Therefore, $\delta(l(X)/4) > 2 - l(X)/4$ implies normal structure. ■

Corollary 1. *For a Banach space* X *with* $l(X) \geq 7$, *the condition* $\delta(7/4) > 1/4$ *implies* X *has normal structure.*

Proof. The conditions $l(X)/4 \geq 7/4$, $2 - l(X)/4 \leq 1/4$, $\delta(7/4) > 1/4$, and $\delta(\epsilon)$ is an increasing function on [0, 2] imply that $\delta(l(X)/4) > 2 - l(X)/4$. So, X has normal structure from Theorem 7. ■

Since $\delta(3/2) > 1/4$ implies $\delta(7/4) > 1/4$, Corollary 1 improved the result of Corollary 5.6 for the space X with $l(X) \geq 7$ [7].

Corollary 2. *For a Banach space* X, *if there exists an* ϵ, *such that* $0 \leq \epsilon \leq l(X)/4$ *and* $\delta(\epsilon) > ((8 - l(X))/l(X))\epsilon$, *then* X *has normal structure.*

Proof. $\delta(\epsilon)/\epsilon$ is an increasing function on [0, 2] by [12]. $\delta(\epsilon)/\epsilon > (8 - l(X))/l(X)(0 \leq \epsilon \leq l(X)/4)$ implies $\delta(l(X)/4)/(l(X)/4) \geq \delta(\epsilon)/\epsilon > (8 - l(X))/l(X)$, that is, $\delta(l(X)/4) > 2 - l(X)/4$. ■

Theorem 8. *For a Banach space* X, $\delta(2^-) > 1/2$ *implies* X *has normal structure.*

Proof. If X fails to have normal structure, let x_1, x_2, x_3 be in Lemma 2. Then $2-4\epsilon \leq \|x_3-x_1\| \leq 2$, $1-\epsilon \leq \|x_3+x_1\| \leq 1+\epsilon$, and $\delta(2-4\epsilon) \leq \delta(\|x_3-x_1\|) \leq 1 - (\|x_3+x_1\|/2) \leq (1+\epsilon)/2$. By letting $\epsilon \to 0$, we have $\delta(2^-) \leq 1/2$. ■

Corollary 3. *For a Banach space* X, *the condition* $\delta(\epsilon) > \epsilon/4$ *implies* X *has normal structure.*

Proof. Since $\delta(\epsilon)/\epsilon$ is an increasing function on [0, 2], from $\delta(2^-)/2 \geq \delta(\epsilon)/\epsilon > 1/4$, we have $\delta(2^-) > 1/2$. ■

Let $r(X) = \sup\{r(X_2) : X_2 \subseteq X\} = \sup\{2(\|x+y\| + \|x-y\|) : x, y \in S(X)\}$.

Proposition 1. *If X is a Banach space, either l_p or $L_p[0,1]$, then $r(X) = 2^{2+(1/p)}$, $1 < p \leq 2$; $r(X) = 2^{2+(1/q)}$, $p > 2$, where $(1/p) + (1/q) = 1$.*

Proof. By using Lagrange multipliers in basic calculus, the function $u+v$, under the constraint $u^p+v^p = a$, assumes its maximum $2^{(p-1)/p} \cdot a^{1/p}$ at $u = v = (a/2)^{1/p}$.

If $1 < p \leq 2$, Clarkson inequality [3, 4]: $\|x+y\|^p + \|x-y\|^p \leq 2(\|x\|^p + \|y\|^p)$, for all $x, y \in S(X)$, implies that $\|x+y\| + \|x-y\| \leq 2^{(p-1)/p} \cdot 2^{2/p} = 2^{(p+1)/p}$.

If $p > 2$, Clarkson inequality $\|x+y\|^p + \|x-y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p)$, for all $x, y \in S(X)$, implies $\|x+y\| + \|x-y\| \leq 2^{(p-1)/p} \cdot (2^p)^{1/p} = 2^{(2p-1)/p}$.

For l_p, $1 < p \leq 2$, let $x = (1, 0, 0, ..., 0, ...)$ and $y = (0, 1, 0, ..., 0, ...)$. Then $\|x+y\| + \|x-y\| = 2^{(p+1)/p}$.

For L_p [0, 1], $1 < p \leq 2$, let

$$x(t) = \begin{cases} 2^{\frac{1}{p}}, & 0 \leq t < \frac{1}{2}, \\ 0, & \frac{1}{2} \leq t \leq 1, \end{cases}$$

and

$$y(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ 2^{1/p}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then

$$\|x(t)+y(t)\| + \|x(t)-y(t)\| = \sqrt[p]{\int_0^1 (2^{\frac{1}{p}})^p \, dt} + \sqrt[p]{\int_0^1 (2^{\frac{1}{p}})^p \, dt} = 2 \cdot 2^{\frac{1}{p}} = 2^{\frac{p+1}{p}}.$$

We have $r(X) = \sup\{2(\|x+y\| + \|x-y\|) : x, y \in S(X)\} = 2^{(1/p)+2}$, $1 < p \leq 2$.

For l_p, $p > 2$, let $x = (2^{-1/p}, 2^{-1/p}, 0, ..., 0, ...)$, $y = (2^{-1/p}, -2^{-1/p}, 0, ..., 0, ...)$. Then $\|x+y\| + \|x-y\| = 2(2^{(p-1)/p}) = 2^{(p+1)/p}$.

For L_p [0, 1], $p > 2$, let

$$x(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq t \leq 1, \end{cases}$$

and

$$y(t) = \begin{cases} -1, & 0 \leq t < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then

$$\|x(t)+y(t)\| + \|x(t)-y(t)\| = \sqrt[p]{\int_{\frac{1}{2}}^1 2^p \, dt} + \sqrt[p]{\int_0^{\frac{1}{2}} 2^p \, dt} = 2(2^{\frac{p-1}{p}}) = 2^{\frac{2p-1}{p}}.$$

We have $r(X) = \sup\{2(\|x+y\| + \|x-y\|) : x, y \in S(X)\} = 2^{3-(1/p)} = 2^{2+(1/q)}, p > 2.$ ■

Theorem 9. *For a Banach space X, $r(X) < l(X)$ implies X has normal structure.*

Proof. $R(X) = \inf\{l(S(X_2)) - r(X_2) : X_2 \subseteq X\} \geq \inf\{l(S(X_2)) : X_2 \subseteq X\} - \sup\{r(X_2) : X_2 \subseteq X\} = l(X) - r(X)$. $r(X) < l(X)$ implies $R(X) = l(X) - r(X) > 0$, which hence implies X has normal structure by Theorem 6. ■

Corollary 4. *For a Banach space X, $r(X) < 6$ implies that X has normal structure.*

Finally, at the end of this section we show that the two parameters are distinct by giving an example of a Banach space X with $R(X) > 0$ and $J(X) > 3/2$.

Consider an n-dimensional space l_p^n, where $1 \leq p \leq \infty$, and n is a positive integer. The norm is defined by

$$\|(x_1, x_2, ..., x_n)\|_p = \begin{cases} (\Sigma_{j=1}^n |x_j|^p)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \max\{|x_1|, |x_2|, ..., |x_n|\}, & \text{if } p = \infty. \end{cases}$$

This is a subspace of general l_p space, where $1 \leq p \leq \infty$. The norm is defined by

$$\|(x_1, x_2, ..., x_n, ...)\|_p = \begin{cases} (\Sigma_{j=1}^\infty (|x_j|^p)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \sup\{|x_1|, |x_2|, ..., |x_n|, ...\}, & \text{if } p = \infty. \end{cases}$$

From [7],

$$J(l_p) = \begin{cases} 2^{1-\frac{1}{p}}, & \text{if } 2 \leq p < \infty \\ 2, & \text{if } p = \infty. \end{cases}$$

It is easy to show $J(l_p^n) = J(l_p)$ for all n.

Let $n = 2$, and $p > (\log_2(4/3))^{-1}$. Then $J(l_p^2) = 2^{1-(1/p)} > 2 \cdot 3/4 = 3/2$.

On the other hand, $S(l_p^2)$ is a compact set in $\mathbb{R}^2$, so there exist x and $y \in S(l_p^2)$ such that the supremum is assummed at x and y in the definition of $J(l_p^2)$. So, $J(l_p^2) = \|x+y\| = \|x-y\|$, and $r(l_p^2) = 2(\|x+y\| + \|x-y\|) = 4\|x+y\|$.

But l_p^2 is a two-dimensional uniform convex space, so $l(S(l_p^2)) > 4\|x+y\|$ by the definition of arc length. We have $R(l_p^2) > 0$.

We may also use an l_p^n space for any n or the l_p space to establish our purpose, but for the l_p space it is more complicated.

5. The Arc Lengths and Uniform Normal Structure

Let F be a filter on an index set I, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to F, denoted by $\lim_F x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in F$. A filter U on I is called an ultrafilter if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if U is an ultrafilter, then (i): for any $A \subseteq I$, either $A \subseteq U$ or $I \setminus A \subseteq U$; (ii): if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_U x_i$ exists and equals x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty$.

Definition 4 [14]. Let U be an ultrafilter on I and let $N_U = \{(x_i) \in l_\infty(I, X_i) : \lim_U ||x_i|| = 0\}$.

The ultraproduct of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_U$ equipped with the quotient norm.

We will use $(x_i)_U$ to denote the element of the ultraproduct. It follows from remark (ii) above and the definition of quotient norm that

$$\|(x_i)_U\| = \lim{}_U ||x_i||. \tag{5.1}$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X, i \in \mathbb{N}$, for some Banach space X. For an ultrafilter U on $\mathbb{N}$, we use X_U to denote the ultraproduct.

Theorem 10. *For any Banach space X, and for any nontrivial ultrafilter U on $\mathbb{N}$, $R(X_U) = R(X)$.*

Proof. For any $\epsilon > 0$, from the definition of $R(X)$, there exists a two-dimensional subspace $X_2 \subseteq X$ and $x, y \in S(X_2)$ such that for all partitions P of the interval $[0, l(S(X_2))]$ and the corresponding $l(S(X_2), P)$,

$$l(S(X_2), P) - 2(\|x + y\| + \|x - y\|) < R(X) + \epsilon.$$

Let $x_i = x$, and $y_i = y$, for all $i \in \mathbb{N}$. Then $(x_i)_U$, $(y_i)_U \in S((X_U)_2)$, where $(X_U)_2$ is a two dimensional subspace, spanned by $(x_i)_U$, and $(y_i)_U$, of X_U. The projection from X_U to X produces a one-to-one correspondence between the partitions P_U of $[0, l(S((X_U)_2))]$ and the partition P of $[0, l(S(X_2))]$, and $l(S((X_U)_2), P_U) = l(S(X_2), P)$.

Hence $l(S((X_U)_2), P_U) - 2(\|(x_i)_U + (y_i)_U\| + \|(x_i)_U - (y_i)_U\|) = l(S(X_2), P) - 2(\|x + y\| + \|x - y\|) < R(X) + \epsilon$.

Since ϵ can be arbitrarily small, we have proved $R(X_U) \le R(X)$.

To prove the reverse inequality, we choose $(X_U)_2 \subseteq X_U$, $(x_i)_U$, $(y_i)_U \in S((X_U)_2)$, and a partition P_U of $[0, l(S((X_U)_2))]$ such that $l(S((X_U)_2), P_U) > l(S((X_U)_2)) - \epsilon$ and $l(S((X_U)_2), P_U) - 2(\|(x_i)_U + (y_i)_U\| + \|(x_i)_U - (y_i)_U\|) < R(X_U) + \epsilon$.

Without loss of generality, we may assume that $\|x_i\|, \|y_i\| = 1$ for all $i \in \mathbb{N}$, and the norm of each component of the partition on $S((X_U)_2)$ has norm 1 too.

From Theorem 1 and (5.1), $l(S((X_U)_2)) = \sup_{P_U}\{l(S((X_U)_2), P_U)\}$
$= \sup_{P_U}\{\lim_U\{l(S(X_2^i), (P_U)_i)\}\} = \lim_U\{\sup_{P_U}\{l(S(X_2^i), (P_U)_i)\}\}$
$= \lim_U\{l(S(X_2^i))\}$, where X_2^i is a two-dimensional subspace spanned by x_i, and y_i, and $(P_U)_i$, a projection of the partition P_U to X_2^i, is a partition of $S(X_2^i)$ for all $i \in \mathbb{N}$.

From remarks (i) and (ii) of ultrafilter and by (5.1) and the paragragh above, the sets
$J = \{i \in \mathbb{N}, l(S((X_U)_2), P_U) - 2(\|(x_i)_U + (y_i)_U\| + \|(x_i)_U - (y_i)_U\|) < R(X_U) + \epsilon\}$,
$K = \{i \in \mathbb{N}, l(S((X_U)_2), P_U) > l(S((X_U)_2)) - \epsilon\}$, and
$M = \{i \in \mathbb{N}, l(S(X_2^i)) < l(S((X_U)_2)) + \epsilon\}$
are all in U. So the intersection $J \bigcap K \bigcap M$ is in U too, and is hence not empty.

Let $i \in J \bigcap K \bigcap M$. We have $l(S(X_2^i), (P_U)_i) - 2(\|x_i + y_i\| + \|x_i - y_i\|) < R(X_U) + \epsilon$, $l(S(X_2^i), (P_U)_i) > l(S((X_U)_2)) - \epsilon$, and $l(S(X_2^i)) < l(S((X_U)_2)) + \epsilon$.

So, $l(S(X_2^i)) - 2(\|x_i + y_i\| + \|x_i - y_i\|) < R(X_U) + 3\epsilon$. Hence $R(X) < R(X_U) + 3\epsilon$. Since ϵ can be arbitrarily small, $R(X) \leq R(X_U)$. ■

Similarly, we can prove the following two theorems:

Theorem 11. *For any Banach space X, and for any nontrivial ultrufilter U on $\mathbb{N}$, $r(X_U) = r(X)$.*

Theorem 12. *For any Banach space X, and for any nontrivial ultrufilter U on $\mathbb{N}$, $l(X_U) = l(X)$.*

Theorem 13. *If X is a Banach space with $R(X) > 0$, then X has uniform normal structure.*

Proof. The idea of the proof is the same as the proof of Theorem 4.4 in [7]. Suppose that $R(X) > 0$, and that X does not have uniform normal structure. We find a sequence $\{C_n\}$ of bounded closed convex subsets of X such that for each n,

$0 \in C_n, d(C_n) = 1$, and
$\text{rad}(C_n) = \inf\{\sup\{\|x - y\|, y \in C_n\}, x \in C_n\} > 1 - \frac{1}{n}$.

Let U be any nontrivial ultrafilter on $\mathbb{N}$, and let

$$C = \{(x_n)_U : x_n \in C_n, n \in \mathbb{N}\}.$$

Then C is a nonempty bounded closed convex subset of X_U. It follows from the above properties of C_n that $d(C) = \mathrm{rad}(C) = 1$, so X_U does not have normal structure. On the other hand, from Theorem 10, $R(X_U) = R(X) > 0$. This contradicts Theorem 6, and hence X must have uniform normal structure. ■

Similarly, we can prove the following theorem:

Theorem 14. *For a Banach space X, $\delta(l(X)/4) > 2 - (l(X)/4)$ implies that X has uniform normal structure.*

Ttheorem 15. *For a Banach space X, $r(X) < l(X)$ implies that X has uniform normal structure.*

Theorem 16. *For a Banach space X, $r(X) < 6$ implies that X has uniform normal structure.*

6. Appendix

In this section, I summarize some results about the parameters $\delta(\epsilon)$, $r(X)$ and $J(X)$ for some classical Banach spaces.

Theorem 17 [7]. *Let X be either l_p or $L_p\ [0,1]$, where $1 \leq p \leq \infty$. Then $J(X) = 2^{1/p}$, if $1 < p \leq 2$; $J(X) = 2^{1-(1/p)}$, if $2 < p < \infty$; and $J(X) = 2$, if $p = 1$ or ∞.*

Theorem 18 [4, p. 148]. *Let X be either l_p or $L_p\ [0,1]$, where $1 < p < \infty$. Then, $\delta(\epsilon)$ satisfies the equation: $(1-\delta(\epsilon)+(\epsilon/2))^p + (1-\delta(\epsilon)-(\epsilon/2))^p = 2$, if $1 < p \leq 2$; $\delta(\epsilon) = 1-(1-(\epsilon/2)^p)^{1/p}$, if $2 < p < \infty$.*

Theorem 19. *For the spaces $l_1, l_\infty, L_1\ [0,1]$ and $L_\infty\ [0,1]$, we have $\delta(\epsilon) \equiv 0$.*

Proof. From [7], for any Banach space X, $J(X) < \epsilon$ if and only if $\delta(\epsilon) > 1-(\epsilon/2)$. So, it is a direct result of Theorem 17. ■

The values of $r(X)$ are shown in Proposition 1.

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