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ON ULTRAREGULAR INDUCTIVE LIMITS

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Abstract. An inductive limit $(E,t) = \mathrm{ind}\,(E_n,t_n)$ is said to have property (P) if every closed absolutely convex neighborhood in (E_n,t_n) is closed in (E_{n+1},t_{n+1}). This property was introduced and investigated by J. Kucera. In this paper we give some equivalent descriptions of property (P) and prove that property (P) implies ultraregularity. Particularly, if all (E_n,t_n) are metrizable locally convex spaces, we have: (E,t) is ultraregular if and only if (E,t) is a strict inductive limit and for each $n \in \mathbb{N}$, there is $m = m(n) \in \mathbb{N}$ such that $\overline{E}_n^E \subset E_m$; (E,t) has property (P) if and only if (E,t) is a strict inductive limit and each E_n is closed in (E_{n+1},t_{n+1}).

1. Introduction

We keep the notations of [1]. Let $(E_1,t_1) \subset (E_2,t_2) \subset \cdots$ be a sequence of locally convex spaces and the inclusions $i_{n,n+1} : (E_n,t_n) \to (E_{n+1},t_{n+1})$ be continuous for all $n \in \mathbb{N}$. Then $(E_n,t_n)_{n\in\mathbb{N}}$ is said to be an inductive sequence of locally convex spaces. If $E = \bigcup_{n=1}^{\infty} E_n$ is endowed with the finest locally convex topology t (in fact, also the finest linear topology; see [1, p.45]) such that the injections $i_n : (E_n,t_n) \to E$ are continuous for all $n \in \mathbb{N}$, then (E,t) is called the inductive limit of the inductive sequence $(E_n,t_n)_{n\in\mathbb{N}}$ and denoted by $(E,t) := \mathrm{ind}\,(E_n,t_n)$. If every (E_n,t_n) is a metrizable locally convex space (resp. a Fréchet space), then $(E,t) = \mathrm{ind}\,(E_n,t_n)$ is called an (LM)-space (resp. an (LF)-space). If for each $n \in \mathbb{N}$, t_{n+1} induces the topology t_n on E_n, then $(E,t) = \mathrm{ind}\,(E_n,t_n)$ is called a strict inductive limit. Certainly, any bounded set in (E_n,t_n) is also bounded in (E,t), but a bounded set in

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(E,t) need not be contained and bounded in some (E_n,t_n). The Dieudonné-Schwartz Theorem ([3, §4, Prop.4] or [16, p.59]) states that a set $B \subset E$ is t-bounded if and only if it is contained and bounded in some (E_n,t_n), provided that $(E,t) = \mathrm{ind}\,(E_n,t_n)$ is a strict inductive limit and each E_n is closed in (E_{n+1},t_{n+1}). The various extensions of Dieudonné-Schwartz Theorem have been considered, for example, in [2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19] etc. As in [1], we call an inductive limit $(E,t) = \mathrm{ind}\,(E_n,t_n)$ to be

(a) α-regular if for each bounded set B in (E,t), there exists $n = n(B) \in \mathbb{N}$ such that B is contained in E_n;

(b) regular if for each bounded set B in (E,t), there exists $n = n(B) \in \mathbb{N}$ such that B is contained and bounded in (E_n,t_n).

By Dieudonné-Schwartz Theorem, we know that a strict inductive limit $(E,t) = \mathrm{ind}\,(E_n,t_n)$ is regular if each E_n is closed in (E_{n+1},t_{n+1}). In [7], Kucera introduced the notion of ultraregular inductive limits as follows.

(c) An inductive limit $(E,t) = \mathrm{ind}\,(E_n,t_n)$ is called to be ultraregular if (E,t) is α-regular and each set $B \subset E_n$, which is bounded in (E,t), is also bounded in (E_n,t_n).

In fact, Dieudonné and Schwartz proved that a strict inductive limit $(E,t) = \mathrm{ind}\,(E_n,t_n)$ is ultraregular if each E_n is closed in (E_{n+1},t_{n+1}). Moreover, Kucera [7] introduced the following property (P) and investigated the relationship between property (P) and ultraregularity.

(d) An inductive limit $(E,t) = \mathrm{ind}\,(E_n,t_n)$ is said to have property (P) if every closed absolutely convex neighborhood in (E_n,t_n) is closed in (E_{n+1},t_{n+1}).

In this paper, we shall see that property (P) is indeed a very strong property. We shall obtain some equivalent descriptions of property (P) and prove that property (P) implies ultraregularity. This improves the related result of Kucera [7]. For an (LM)-space $(E,t) = \mathrm{ind}\,(E_n,t_n)$, we shall give respectively the essential characteristics of ultraregularity and property (P) as follows:

(E,t) is ultraregular if and only if (E,t) is a strict inductive limit and for each $n \in \mathbb{N}$, there is $m = m(n) \in \mathbb{N}$ such that $\overline{E}_n^E \subset E_m$, where $\overline{E}_n^E$ denotes the closure of E_n in (E,t);

(E,t) has property (P) if and only if (E,t) is a strict inductive limit and each E_n is closed in (E_{n+1},t_{n+1}).

For an (LF)-space $(E,t) = \mathrm{ind}\,(E_n,t_n)$, we shall show that (E,t) is ultraregular if and only if (E,t) has property (P) and this is the case if and only if (E,t) is a strict inductive limit and each E_n is closed in (E_{n+1},t_{n+1}).

2. Equivalent Descriptions of Property (P)

We begin this section with the following basic observation.

Lemma 1. *Let* $(E,t) = \text{ind}\,(E_n,t_n)$ *be an inductive limit of locally convex spaces. If* $(E_n,t_n)' = (E_n,t_{n+1}|E_n)'$ *for every* $n \in \mathbb{N}$, *then for any* $m \in \mathbb{N}$, *each* $f_m \in (E_m,t_m)'$ *can be extended to a linear functional* $f \in (E,t)'$. *Here for any locally convex space* (X,τ), $(X,\tau)'$ *denotes the topological dual of* (X,τ).

Proof. Without loss of generality, we assume that $m = 1$. Suppose that $f_1 \in (E_1,t_1)'$. Since $(E_1,t_1)' = (E_1,t_2|E_1)'$, by the Hahn-Banach extension theorem [17, p.49], f_1 can be extended to $f_2 \in (E_2,t_2)' = (E_2,t_3|E_2)'$. Again, f_2 can be extended to $f_3 \in (E_3,t_3)' = (E_3,t_4|E_3)', \cdots$. Repeating this process infinitely, we obtain a linear functional f on E such that $f|E_n = f_n$ for every $n \in \mathbb{N}$. Since $f|E_n = f_n \in (E_n,t_n)'$ for every n, we conclude that $f \in (E,t)'$ (see [17, p.54]). Clearly $f|E_1 = f_1$. ∎

Now we are going to prove our first main result, which gives some equivalent descriptions of property (P).

Theorem 1. *Let* $(E,t) = \text{ind}\,(E_n,t_n)$ *be an inductive limit of locally convex spaces, then the following statements are equivalent*:

- (i) (E,t) *has property* (P), *i.e., every closed absolutely convex neighborhood of* 0 *in* (E_n,t_n) *is closed in* (E_{n+1},t_{n+1}).
- (ii) *Every closed convex set in* (E_n,t_n) *is closed in* (E_{n+1},t_{n+1}) (*see* [10, H-4]).
- (iii) *Every closed convex set in* (E_n,t_n) *is closed in* (E,t).

Proof. The implications (iii)$\Longrightarrow$(ii)$\Longrightarrow$(i) are obvious.

(i)$\Longrightarrow$(ii). Assume that $(E,t) = \text{ind}(E_n,t_n)$ has property (P). First, E_n, as a special closed absolutely convex 0-neighborhood in (E_n,t_n), is closed in (E_{n+1},t_{n+1}). Next we shall prove that $(E_n,t_n)' = (E_n,t_{n+1}|E_n)'$, where $(E_n,t_n)'$ and $(E_n,t_{n+1}|E_n)'$ denote the topological duals of (E_n,t_n) and $(E_n, t_{n+1}|E_n)$ respectively. Since $t_n \supset t_{n+1}|E_n$, we have $(E_n,t_n)' \supset (E_n,t_{n+1}|E_n)'$. For any $f \in (E_n,t_n)'$ and any $\epsilon > 0$, denote the set $\{x \in E_n : |f(x)| \le \epsilon\}$ by $(|f| \le \epsilon)$. Clearly $(|f| \le \epsilon)$ is a closed absolutely convex 0-neighborhood in (E_n,t_n). By (P), $(|f| \le \epsilon)$ is closed in (E_{n+1},t_{n+1}). Thus we have:

$$\{x \in E_n : f(x) = 0\} = \bigcap_{\epsilon>0}\{x \in E_n : |f(x)| \le \epsilon\} = \bigcap_{\epsilon>0}(|f| \le \epsilon)$$

is closed in (E_{n+1},t_{n+1}). Certainly $\{x \in E_n : f(x) = 0\}$ is closed in $(E_n, t_{n+1}|E_n)$ and $f \in (E_n,t_{n+1}|E_n)'$. Hence $(E_n,t_n)' = (E_n,t_{n+1}|E_n)'$. From this, (E_n,t_n) and $(E_n,t_{n+1}|E_n)$ have the same closed convex sets (see [17, p.132] or

[20, p.224]). Thus each closed convex set in (E_n, t_n) is closed in $(E_n, t_{n+1}|E_n)$. Since E_n is closed in (E_{n+1}, t_{n+1}), each closed convex set in (E_n, t_n) is closed in (E_{n+1}, t_{n+1}). That is, (ii) holds.

(ii)$\Longrightarrow$(iii). By (ii), any closed convex set B in (E_n, t_n) is closed in (E_{n+1}, t_{n+1}). Thus B is also a closed convex set in (E_{n+1}, t_{n+1}). Again by (ii), B is closed in (E_{n+2}, t_{n+2}). Repeating this process, we conclude that each closed convex set in (E_n, t_n) is closed in (E_m, t_m) for all $m \geq n$. Let A be any fixed closed convex set in (E_n, t_n) and $x \in E \backslash A$. There exists $m \geq n$ such that $x \in E_m$. Since A is closed in (E_m, t_m) and $x \in E_m \backslash A$, by the Hahn-Banach separation theorem, there exists $f_m \in (E_m, t_m)'$ such that $\operatorname{Re} f_m(x) > \sup\{\operatorname{Re} f_m(y) : y \in A\}$. By (ii), (E_n, t_n) and $(E_n, t_{n+1}|E_n)$ have the same closed convex sets. Hence $(E_n, t_n)' = (E_n, t_{n+1}|E_n)'$ for every n. By Lemma 1, $f_m \in (E_m, t_m)'$ can be extended to a linear functional $f \in (E, t)'$. Thus $\operatorname{Re} f(x) = \operatorname{Re} f_m(x)$ and $\operatorname{Re} f(y) = \operatorname{Re} f_m(y)$ for every $y \in A$. Hence

$$\operatorname{Re} f(x) > \sup\{\operatorname{Re} f(y) : y \in A\} = \sup\{\operatorname{Re} f(y) : y \in \overline{A}^E\}.$$

From this, $x \notin \overline{A}^E$ and hence $A = \overline{A}^E$. That is to say, each closed convex set in (E_n, t_n) is closed in (E, t). Namely, (iii) holds. ■

In [7], Kucera proved that if (P) holds and each (E_n, t_n) is fast complete, then (E, t) is ultraregular. On fast complete spaces, i.e. Mackey complete spaces, please refer to [1, p.77]. In fact, the condition that each (E_n, t_n) is fast complete is superfluous. By using Theorem 1, we have:

Corollary 1. *If* (P) *holds, then* (E, t) *is ultraregular and each* E_n *is closed in* (E, t).

Proof. By Theorem 1, we know that (P) is equivalent to the condition that every closed convex set in (E_n, t_n) is closed in (E, t). Hence each E_n is closed in (E, t). Thus each bounded set in (E, t) is contained in some E_n (see [8]), i.e., (E, t) is α-regular. Besides, (E_n, t_n) and $(E_n, t|E_n)$ have the same closed convex sets, and hence $(E_n, t_n)' = (E_n, t|E_n)'$. Thus (E_n, t_n) and $(E_n, t|E_n)$ have the same bounded sets (see [17, p.132] or [6, p.254]). This implies that if a bounded set B in (E, t) is contained in E_n, then B is also bounded in (E_n, t_n). That is, (E, t) is ultraregular. ■

3. On Ultraregular (LM)-spaces

In this section, we shall discuss the relationship among property (P), ultraregularity and strictness in (LM)-spaces.

Theorem 2. *Let* $(E,t) = \operatorname{ind}(E_n, t_n)$ *be an* (*LM*)*-space. Then* (E,t) *is ultraregular if and only if* (E,t) *is a strict inductive limit and for each* $n \in \mathbb{N}$, *there is* $m = m(n) \in \mathbb{N}$ *such that* $\overline{E}_n^E \subset E_m$.

Proof. Assume that (E,t) is ultraregular. Then (E_n, t_n) and $(E_n, t|E_n)$ have the same bounded sets. Certainly (E_n, t_n) and $(E_n, t_{n+1}|E_n)$ have the same bounded sets. Since (E_n, t_n) and $(E_n, t_{n+1}|E_n)$ both are metrizable and hence bornological, we have $(E_n, t_n) = (E_n, t_{n+1}|E_n)$. This means that (E,t) is a strict inductive limit. By the assumption that (E,t) is ultraregular, (E,t) is α-regular. By [11, Theorem 4], for each $n \in \mathbb{N}$ there is $m = m(n) \geq n$ and an absolutely convex 0-neighborhood U_n in (E_n, t_n) such that $\overline{U}_n^E \subset E_m$. Since $(E,t) = \operatorname{ind}(E_n, t_n)$ is a strict inductive limit, we have $t|E_n = t_n$ (see [17, p.58] or [20, p.159]). Thus there exists an open absolutely convex 0-neighborhood U in (E,t) such that $U \cap E_n \subset U_n$. For any $x \in U \cap \overline{E}_n^E$, there is a net $(x_\delta) \subset E_n$ such that $x_\delta \to x$ in (E,t). Note that U is an open neighborhood of x in (E,t). Hence there exists δ_0 such that $x_\delta \in U$ for all $\delta \geq \delta_0$. Thus $x_\delta \in U \cap E_n$ for all $\delta \geq \delta_0$ and $x \in \overline{U \cap E_n}^E$. Now we have:

$$U \cap \overline{E}_n^E \subset \overline{U \cap E_n}^E \subset \overline{U}_n^E \subset E_m.$$

Thus $k(U \cap \overline{E}_n^E) \subset E_m$ for every $k \in \mathbb{N}$. From this,

$$\overline{E}_n^E = E \cap \overline{E}_n^E = \left(\bigcup_{k=1}^{\infty} kU\right) \bigcap \overline{E}_n^E = \bigcup_{k=1}^{\infty} k\left(U \cap \overline{E}_n^E\right) \subset E_m.$$

Conversely, suppose that (E,t) is a strict inductive limit and for each $n \in \mathbb{N}$, there is $m = m(n) \geq n$ such that $\overline{E}_n^E \subset E_m$. By the assumption that for each $n \in \mathbb{N}$, there is $m = m(n) \geq n$ such that $\overline{E}_n^E \subset E_m$, we conclude that (E,t) is α-regular (see [10, Theorem 1]). Moreover, since $(E_n, t_n) = (E_n, t|E_n)$, each set $B \subset E_n$, which is bounded in (E,t), is also bounded in (E_n, t_n). Thus (E,t) is ultraregular. ∎

Corollary 2. *Let* $(E,t) = \operatorname{ind}(E_n, t_n)$ *be an* (*LM*)*-space. Then the following statements are equivalent*:

(i) (E,t) *has property* (P).

(ii) (E,t) *is ultraregular and each* E_n *is closed in* (E_{n+1}, t_{n+1}).

(iii) (E,t) *is a strict inductive limit and each* E_n *is closed in* (E_{n+1}, t_{n+1}).

Proof. (i)$\Longrightarrow$(ii). It follows from Corollary 1.

(ii)$\Longrightarrow$(iii). It follows from Theorem 2.

(iii)$\Longrightarrow$(i). It is obvious. ∎

For an (LF)-space $(E,t) = \mathrm{ind}\,(E_n, t_n)$, we even have a stronger result. In [14, Theorem 4], we already proved that (E,t) is regular if and only if $(\overline{E}_n^E, t|\overline{E}_n^E)$ is fast complete for every $n \in \mathbb{N}$. Now we shall see that (E,t) is ultraregular if and only if $(E_n, t|E_n)$ is fast complete for every $n \in \mathbb{N}$. For brevity, we call an inductive limit β-ultraregular if each set $B \subset E_n$, which is bounded in (E,t), is also bounded in (E_n, t_n). For (LF)-spaces, we have the following:

Theorem 3. *Let* $(E,t) = \mathrm{ind}\,(E_n, t_n)$ *be an* (*LF*)*-space. Then the following statements are equivalent*:

(i) (E,t) *is ultraregular.*

(ii) (E,t) *is* β*-ultraregular.*

(iii) $(E_n, t|E_n)$ *is fast complete for every* n*.*

(iv) (E,t) *is a strict inductive limit.*

(v) (E,t) *has property* (P).

Proof. (i)$\Longrightarrow$(ii). It is obvious.

(ii)$\Longrightarrow$(iii). For any bounded set B in $(E_n, t|E_n)$, denote the closed absolutely convex hull of B in (E_n, t_n) by $\overline{\Gamma(B)}^{E_n}$. By (ii), $(E_n, t|E_n)$ and (E_n, t_n) have the same bounded sets. Hence B is also bounded in (E_n, t_n) and $\overline{\Gamma(B)}^{E_n}$ is a closed absolutely convex bounded set in (E_n, t_n). Since (E_n, t_n) is a Fréchet space, $\overline{\Gamma(B)}^{E_n}$ is a Banach disk in (E_n, t_n) and hence it is also a Banach disk in $(E_n, t|E_n)$. Thus each bounded set B in $(E_n, t|E_n)$ is contained in the Banach disk $\overline{\Gamma(B)}^{E_n}$ in $(E_n, t|E_n)$. That is, $(E_n, t|E_n)$ is fast complete.

(iii)$\Longrightarrow$(iv). See [16, Lemma 3].

(iv)$\Longrightarrow$(v). Since (E,t) is a strict inductive limit, $(E_n, t_n) = (E_n, t_{n+1}|E_n)$ for every n. Since (E_n, t_n) is complete, E_n is closed in (E_{n+1}, t_{n+1}). Thus each closed convex set in (E_n, t_n) is closed in (E_{n+1}, t_{n+1}) and hence (E,t) has property (P).

(v)$\Longrightarrow$(i). It follows from Corollary 1. ■

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