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A NONCONFORMING WEAK RESIDUAL ERROR ESTIMATOR FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. A nonconforming weak residual error estimator is presented and analyzed for finite element solutions of linear elliptic partial differential equations. The treatment of the flux jumps across element edges is of special interest. The estimator is obtained by solving local residual problems which do not explicitly involve the jumps and do not require boundary conditions. The estimator handles both interior and edge residuals on each element by a suitable construction of the basis functions for the local problems. Together with the previous conforming estimator, the weak residual error estimation, without the flux jumps, can thus be applied to both odd- and even-order finite element approximations.

1. Introduction

Weak residual error estimators are shown to be very effective for a large class of variational problems [1, 2, 8, 9, 11, 13-19, 21]. The formula used to derive these estimators can be generically expressed as follows: In each finite element τ_i, determine $\tilde{e}_i \in S_i^c$ such that

$$B(\tilde{e}_i, v_i) = F(v_i) - B(u_h, v_i) \quad \forall v_i \in S_i^c, \tag{1.1}$$

where $B(\cdot,\cdot)$ and $F(\cdot)$ are the bilinear and linear forms associated with a given variational problem, $u_h \in H$ is its finite element solution in some Sobolev space H, and S_i^c is a finite element space defined over that element.

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An error estimator is said to be *conforming* if $S_i^c \subset H$ and *nonconforming* if $S_i^c \not\subset H$. The estimators of [1, 2, 8, 9, 11, 14, 19] belong to the conforming type. Nonconforming estimators can be found in [15, 18].

Two important properties of the estimators based on formula (1.1) should be emphasized. First, the (weak) residual term of (1.1) does not explicitly include any flux jumps (the normal derivatives of u_h) across element edges. The treatment of the jumps is one of the major concerns of other estimators in which the jump terms are explicitly involved and/or specific local boundary conditions are imposed for the local problems (see, e.g., [3, 4, 5-8, 10, 20-21, 23, 24]). Second, the performance of the weak residual error estimators is strongly influenced by the construction of the subspace S_i^c which in turn depends on the finite element order of the approximate solution u_h. It is well-known [7] that for second-order elliptic problems the discretization error of odd-order finite element solutions is mainly due to the jumps, while that of even-order approximations occurs principally in the interior of the elements. Therefore, the conforming estimators will fail for odd-order approximations if the support of the basis functions of S_i^c does not cross over element edges [15]. This excludes many important classes of approximations, for example, the linear approximations. On the other hand, if the support does cross over element edges, (1.1) will result in a global system which obviously is not efficient for adaptive computations.

The purpose of this article is to give a more specific treatment of the jumps than that of [15] and [18]. The treatment is based on a particular construction of S_i^c which is essentially an extension of the midpoint shape functions of [10] to the weak residual formulation (1.1). Under the treatment, a new formulation of (1.1) is given and the corresponding error estimator is proved to be equivalent to the exact error in the H^1 semi-norm for linear self-adjoint elliptic partial differential equations.

The main idea of the new formulation is to localize the left-hand side of (1.1) while conforming test functions are used for the right-hand side in order to estimate both the edge and the interior errors. More specifically, formula (1.1) is transformed to the determination of $\tilde{e}_i \in S_i^c$ such that

$$B_h(\tilde{e}_i, v_i) = \frac{1}{2}(F(\hat{v}_i) - B(u_h, \hat{v}_i)) \quad \forall v_i \in S_i^c, \tag{1.2}$$

where $S_i^c \not\subset H$ but $\hat{v}_i \in H$. The conforming shape functions $\hat{v}_i$ are defined by extending the nonconforming functions v_i from the element τ_i to its neighbors. The "nonconforming" bilinear form B_h is a restriction of the conforming bilinear form B to each element. The formulation is first presented for Poisson's equation in Section 2. In this case, the bilinear form $B(\cdot, \cdot)$ corresponds to the

Laplace operator. In Section 3, formula (1.2) is changed to

$$B_h(\tilde{e}_i, v_i) = \frac{1}{2}(F(\hat{v}_i) - A(u_h, \hat{v}_i)) \quad \forall v_i \in S_i^c, \tag{1.3}$$

where the bilinear form $A(\cdot,\cdot)$ corresponds to the more general elliptic operator. A numerical example is given in Section 4.

We briefly remark on the merit of using estimators based on formula (1.3). Although the model problems considered here are of elliptic PDEs, the same principle of the error estimation can be applied to more general problems and more general numerical methods if one refers to the formula in terms of the general bilinear $B(\cdot,\cdot)$ and linear $F(\cdot)$ forms. For example, this approach has been used for finite volume [15, 17], least squares finite element [13, 17], and boundary element [16] computations and for various boundary value problems such as parametrized nonlinear problems [15, 19], PDEs of mixed type [14, 15], semiconductor device simulation model [17], obstacle and free boundary value problems [15, 17], advection-diffusion problems [17], and the Navier-Stokes equations [13, 16, 17] etc.. The most important features of the present error estimation are unification, generality, and simplification. In other words, the principal formula used to derive the estimators for either different numerical methods or various boundary value problems is the same. Moreover, it allows spectral orders to vary in each element since the local space S_i^c can be constructed by using the next higher-order hierachical shape functions [22] to that of u_h in that element. Consequently, a combination of the conforming estimator of [9, 11] and the present noncoforming estimator is suitable for general p- or hp-version finite element methods. The implementation of the estimators is straightforward since formula (1.3) is almost identical to that of approximation, i.e., $A(u_h, v_h) = F(v_h)$. However, we must stress that the approach is not necessarily more efficient than other estimators for typical problems or for certain approximations since (1.3) always results in a solution of local problems whereas some estimators may only require calculation of residual terms or postprocess of higher-order terms (cf. [3, 5, 7, 20, 23, 24]).

2. An Error Estimator for Poisson's Equation

Let Ω be a bounded region in the plane with a Lipschitz boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, where $\partial\Omega_D$ is a nonempty subset of $\partial\Omega$. For any open subset τ of Ω, we denote by $H^1(\tau)$ the usual Sobolev space equipped with the semi-norm $|\cdot|_{1,\tau}$. For simplicity, we write $|\cdot|_1 = |\cdot|_{1,\Omega}$. Consider the boundary value problem

$$\begin{aligned} -\Delta u(x) &= f(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega_D, \\ \frac{\partial u}{\partial n} &= g && \text{on } \partial\Omega_N. \end{aligned} \tag{2.1}$$

The associated variational problem is to find $u \in H(\Omega)$ such that

$$B(u, v) = F(v) \quad \forall v \in H(\Omega), \tag{2.2}$$

where

$$\begin{aligned} H(\Omega) &:= \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega_D\}, \\ B(u, v) &:= \int_\Omega \nabla u \cdot \nabla v \, dx, \\ F(v) &:= \int_\Omega f v \, dx \; + \int_{\partial\Omega_N} g v \, ds. \end{aligned}$$

For $v \in H(\Omega)$, define the (energy) B-norm

$$\|v\|_B := \sqrt{B(v, v)} = |v|_1$$

and its restriction to any subset τ of Ω

$$\|v\|_{B,\tau} = |v|_{1,\tau} := \left(\int_\tau \nabla v \cdot \nabla v \, dx \right)^{1/2}.$$

To discretize (2.2), we introduce a finite-dimensional subspace $S \subset H(\Omega)$, which is associated with a mesh $T_h = \{\tau_i | i = 1, 2, \cdots, M\}$ on $\bar{\Omega}$. The mesh is characterized by a mesh size h. For any two distinct elements (triangles or rectangles or both) τ_i and τ_j in T_h, $\tau_i \cap \tau_j$ is either empty, a single vertex, or a common edge. Two elements are said to be adjacent if they have a common edge. For a given rectangular element, let $h_{\max}$ and $h_{\min}$ denote the largest and smallest edge lengths, respectively. Then the element edge ratio is defined by $h_{\min}/h_{\max}$. We always assume that the mesh T_h belongs to a *regular* family of meshes on $\bar{\Omega}$. Recall that, see, e.g., [4, 8], the family is regular if all angles of its triangular elements and all edge ratios of rectangular elements are bounded below by some constant $\sigma > 0$. Shape regularity does not require a mesh to be globally quasi-uniform, but it does imply local quasi-uniformity of the mesh; an important property that will be exploited below.

The Lax-Milgram theorem asserts that there is a unique solution $u_h \in S$ satisfying the equation

$$B(u_h, v_h) = F(v_h) \quad \forall v_h \in S. \tag{2.3}$$

Let $e = u - u_h$ denote the exact error of the approximate solution u_h. Substituting $u = u_h + e$ into problem (2.2), we have

$$B(e, v) = F(v) - B(u_h, v) \quad \forall v \in H(\Omega). \tag{2.4}$$

Weak residual error estimators are derived by some sort of approximation of the error equation (2.4). The approximation is based on an enlarged subspace $\bar{S} \subset H(\Omega)$ defined by

$$\bar{S} = S \oplus S^c, \qquad S \cap S^c = \{0\}, \quad S \neq \{0\}, \quad S^c \neq \{0\}. \tag{2.5}$$

Here S^c is a complementary space of S in $\bar{S}$ and preserves the conformity. Hence there exists a unique function $u_{\bar{h}} \in \bar{S}$ satisfying

$$B(u_{\bar{h}}, v) = F(v) \quad \forall v \in \bar{S}. \tag{2.6}$$

In general, $u_{\bar{h}}$ is a better approximation of u than u_h. In error analysis, this observation is frequently expressed as the following saturation assumption.

Assumption 1. *The finite element solutions u_h and $u_{\bar{h}}$ of* (2.3) *and* (2.6) *satisfy the inequality*

$$\|u - u_{\bar{h}}\|_B \leq \rho \|u - u_h\|_B, \tag{2.7}$$

where $\rho \in [0, 1)$ is a constant independent of the mesh size h.

Let $\gamma = \sup\{B(w, v)\| \ w \in S_h, \ \|w\|_B = 1, \ v \in S^c_{\bar{h}}, \ \|v\|_B = 1\}$. Then, obviously, we have $\gamma < 1$ by the definition of the space S^c in (2.5). However, it is not clear that the constant γ is uniformly independent of the mesh parameter h for all adaptive meshes. It is shown, for example, in [8] and [12] that this is indeed the case for the inner product induced by the bilinear form that corresponds to the general elliptic PDEs.

Lemma 1. *There exists a constant $\gamma \in [0, 1)$ independent of h such that*

$$|B(w, v)| \leq \gamma \|w\|_B \ \|v\|_B \ \forall w \in S, \ \forall v \in S^c. \tag{2.8}$$

Conforming error estimators are obtained by solving the *conforming* error problem: Determine $e^c \in S^c$ such that

$$B(e^c, v) = F(v) - B(u_h, v) \quad \forall v \in S^c. \tag{2.9}$$

The following result is well-known (see [8, 9, 11, 14, 18]).

Theorem 1. *Let $u \in H(\Omega)$ and $u_h \in S$ be the solutions of* (2.2) *and* (2.3), *respectively, and $e = u - u_h$. If Assumption* 1 *holds for S and $\bar{S}$, then* (2.9) *has a unique solution $e^c \in S^c$ and*

$$(1 - \rho)\sqrt{1 - \gamma^2} \|e\|_B \leq \|e^c\|_B \leq \|e\|_B, \tag{2.10}$$

where $\gamma \in [0,1)$ and $\rho \in [0,1)$ are independent of h.

In view of the bilinear form $B(\cdot,\cdot)$, the equation (2.9) can be considered as a general formula that leads to various error estimators for a large class of variational problems in which the problem (2.1) is only a special case (see [1, 2, 9, 11, 13-19]). A general framework of this type of estimators cast in an abstract variational setting is presented in [15]. However, the conforming complementary subspace S^c in (2.9) would result in a global system of linear equations if its individual basis functions have supports on more than one element. On the other hand, if the basis functions have supports only on their individual elements, the resulting error estimator will not be effective for odd-order finite element solutions u_h since, for such a case, the major errors occur on the edges of elements [7].

Many estimators have been proposed to handle these errors (flux jumps) across elements (see e.g. [3, 5-8, 10, 20, 21, 23, 24]). All those estimators involve the jumps. We propose here another way to treat the jumps. The objective of the treatment is to retain the weak residual term of (2.9) without explicitly involving the jumps. The resulting error estimator handles the jumps in a more specific way than that of [15]. Moreover, assumptions required for the present estimator are weaker than that of [15]. The jumps are handled indirectly by a proper construction of the basis functions of the complementary space S^c. We require the conforming space S^c to have a *locally affine* basis [11] and satisfy (2.5).

For simplicity, we assume that the approximation is linear, i.e., S consists of piecewise linears. The following results hold for more general approximation with some technical modifications. We define

$$S^c = \text{span}\{\phi_{ij}\} \subset H(\Omega), \tag{2.11}$$

where each ϕ_{ij} is a basis function with the nodal point at the center of the common edge of the adjacent pair τ_i and τ_j and with the support on $\tau_i \cup \tau_j$, for all $\tau_i \in T_h$. Note that $\phi_{ji} = \phi_{ij}$ for all the adjacent pairs. For each element $\tau_i \in T_h$ the restriction of ϕ_{ij} on τ_i can be identified with a shape function on a fixed reference triangle or rectangle $\hat{\tau}$ via the affine transformation which maps the reference element $\hat{\tau}$ one-to-one onto τ_i. In particular, we may consider that ϕ_{ij} are side-mode shape functions of degree two (cf. [22]). The resulting enlarged space $\bar{S}$ will then be hierarchical.

We again assume that the enlarged space so constructed satisfies Assumption 1. Hence, the estimate (2.10) still holds for this particular conforming subspace. Note that, under definition (2.11), the conforming problem (2.9) results in a global system of equations.

The next step is to localize (2.9) while keeping the (weak) residual form of (2.9) and not requiring local boundary conditions or the jumps. For each basis function $\phi_{ij} \in S^c$, we define

$$\tilde{\phi}^i_j = \begin{cases} \phi_{ij}, & \text{on } \tau_i, \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

Similarly,

$$\tilde{\phi}^j_i = \begin{cases} \phi_{ij}, & \text{on } \tau_j, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the halved-functions (2.12) were first introduced in [8] for solving local residual problems with jumps. For each element τ_i, let $K(i)$ denote an index set of $j \neq i$ such that τ_j is an adjacent element of τ_i. Let $|K(i)|$ denote the number of halved-functions (2.12) constructed on τ_i. We assume that $|K(i)| \geq 1$, i.e., the mesh T_h contains at least two elements. Note that $|K(i)| \leq 3$ if τ_i is triangular and $|K(i)| \leq 4$ if τ_i is rectangular. Define

$$S^c_i = \operatorname{span}\{\tilde{\phi}^i_j \,|\, j \in K(i)\}. \tag{2.13}$$

It is obvious that S^c_i are not contained in $H(\Omega)$ nor is their direct sum

$$S^n = S^c_1 \oplus S^c_2 \oplus \cdots \oplus S^c_M. \tag{2.14}$$

Since all functions in (2.13) vanish at the vertices of the mesh T_h and are continuous in τ_i, $|\cdot|_{1,\tau_i}$ is a norm on S^c_i and consequently the mapping

$$v \mapsto \|v\|_B := \left(\sum_{\tau_i \in T_h} \|v\|^2_{B,\tau_i} \right)^{\frac{1}{2}}$$

is a norm for all $v \in S^n$.

The following result is an extended version of Lemma 3.3 of [11] to the nonconforming case.

Lemma 2. *There exist two positive constants C_1 and C_2 independent of the mesh parameter h such that*

$$C_1 \sum_{k \in K(i)} \|d^i_k \, \tilde{\phi}^i_k\|^2_B \leq \|\tilde{v}_i\|^2_B \leq C_2 \sum_{k \in K(i)} \|d^i_k \, \tilde{\phi}^i_k\|^2_B \tag{2.15}$$

for each element τ_i, $i = 1, 2, \cdots, M$, and for any $\tilde{v}_i = \sum_{k \in K(i)} d^i_k \, \tilde{\phi}^i_k \in S^c_i$.

Proof. We use the affine mapping $\theta_{\tau_i} : \hat{\tau} \mapsto \tau_i$ for each element $\tau_i \in T_h$ to define

$$\hat{v}_i = \tilde{v}_i \circ \theta_{\tau_i}$$

on the fixed reference element $\hat{\tau}$. Then

$$\hat{v}_i = \sum_{k \in K(i)} d_k^i \, \tilde{\phi}_k^i \circ \theta_{\tau_i} = \sum_{k \in K(i)} d_k^i \, \hat{\phi}_k^i,$$

where $\hat{\phi}_k^i = \tilde{\phi}_k^i \circ \theta_{\tau_i}$, $k \in K(i)$. Each function $\hat{\phi}_k^i$ can be identified with a basis function in a finite-dimensional space $\hat{S}^c$ that is defined on the reference element $\hat{\tau}$. More specifically, let

$$\hat{S}^c = \operatorname{span}\{\hat{\phi}_1, \cdots, \hat{\phi}_l\}, \quad l = \max\{K(i) \mid i = 1, \cdots, M\}.$$

Then $\hat{\phi}_k^i \in \{\hat{\phi}_1, \cdots, \hat{\phi}_l\}$. Since all basis functions of $\hat{S}^c$ vanish at the vertices of the reference element $\hat{\tau}$, the mapping

$$\hat{v} \mapsto |\hat{v}|_{1,\hat{\tau}} = \left(\int_{\hat{\tau}} \nabla \hat{v} \cdot \nabla \hat{v} \, d\hat{x} \right)^{\frac{1}{2}}, \qquad \hat{v} \in \hat{S}^c,$$

defines a norm in $\hat{S}^c$. Hence, the norms $\left(\sum_{k \in K(i)} |d_k^i \, \hat{\phi}_k^i|_{1,\hat{\tau}}^2 \right)^{1/2}$ and $|\hat{v}_i|_{1,\hat{\tau}}$ are equivalent for the space $\hat{S}^c$ is finite-dimensional. Moreover, since the mapping θ_{τ_i} is affine and meshes are regular, we also have another equivalence relation between the norms $|\tilde{v}_i|_{1,\tau_i}$ and $|\hat{v}_i|_{1,\hat{\tau}}$. All constants appear in each equivalence are independent of the mesh parameter h. We thus have the inequalities (2.15). ∎

Observe that Eq. (2.9) can now be localized in each element τ_i if the test and trial functions on the left-hand side are taken to be the halved-functions of (2.12). On the other side, the residual term of (2.9) will involve the residual errors in each pair of adjacent elements τ_i and τ_j as well as the errors on the common edge if the conforming shape function ϕ_{ij} of (2.11) is kept as a test function for the residual term. This motivates us to modify the conforming problem (2.9) and consider the following problem: On each element τ_i, find $\tilde{e}_i \in S_i^c$ such that

$$B_h(\tilde{e}_i, \tilde{\phi}_j^i) = \frac{1}{2}(F(\phi_{ij}) - B(u_h, \phi_{ij})) \quad \forall \tilde{\phi}_j^i \in S_i^c, \tag{2.16}$$

where the bilinear form B_h is "nonconforming" and represents a restriction of B to each element, i.e.,

$$B_h(w_i, v_i) = \int_{\tau_i} \nabla w_i \cdot \nabla v_i \, dx$$

for all w_i and v_i in S_i^c. The uniqueness and existence of $\tilde{e}_i$ is guaranteed since the bilinear form induces a norm in the space S_i^c and the space itself is finite-dimensional. Let e^n denote the sum of the local solutions, i.e.,

$$(2.17) \qquad e^n = \sum_{i=1}^{M} \tilde{e}_i \in S^n.$$

The norm of e^n

$$(2.18) \qquad \|e^n\|_{B_h} := \left(\sum_{i=1}^{M} \|\tilde{e}_i\|_{B_h}^2 \right)^{1/2}$$

can then be used as a (nonconforming) error estimator for the approximate solution u_h of (2.3). We now state the main result of the nonconforming error estimation.

Theorem 2. *With the same assumptions as in Theorem* 1*, let* $\tilde{e}_i$ *be the solution of* (2.16) *and* e^n *be defined by* (2.17)*. Then*

$$(2.19) \qquad (1-\rho)\sqrt{1-\gamma^2}\|e\|_B \le \|e^n\|_{B_h} \le C_4 \, \|e\|_B,$$

where C_4 *is a positive constant,* ρ *and* γ *are given in Theorem* 1*, and all the constants are independent of the mesh size* h*.*

Proof. Let $e^c \in S^c$ be the solution of (2.9). By the definition of S^c in (2.11), the solution can be written as

$$e^c = \frac{1}{2} \sum_i \sum_{k \in K(i)} c_{ik} \, \phi_{ik},$$

where c_{ik} are midpoint nodal values of e^c associated with the adjacent pair τ_i and τ_k. Note that $c_{ik} = c_{ki}$ and $\phi_{ik} = \phi_{ki}$ for all $k \in K(i)$ and the summations will visit each element twice. From (2.9) and (2.16), we have

$$\begin{aligned} B_h(\tilde{e}_i, \tilde{\phi}_k^i) &= \frac{1}{2}(F(\phi_{ik}) - B(u_h, \phi_{ik})) \\ &= \frac{1}{2} B(e^c, \phi_{ik}) \end{aligned}$$

for any fixed $i = 1, 2, \cdots, M$ and for all $k \in K(i)$. Multiplying both sides by c_{ik} and then summing over the index set $K(i)$ yield

$$B_h\Big(\tilde{e}_i, \sum_{k \in K(i)} c_{ik} \tilde{\phi}_k^i\Big) = B_h(\tilde{e}_i, e^c|_{\tau_i}) = \frac{1}{2} B\Big(e^c, \sum_{k \in K(i)} c_{ik} \phi_{ik}\Big).$$

It follows that

$$\begin{aligned}\|e^c\|_B^2 &= B(e^c, e^c)\\ &= B\left(e^c, \frac{1}{2}\sum_i \sum_{k\in K(i)} c_{ik}\,\phi_{ik}\right)\\ &= \sum_i B_h(\tilde{e}_i, e^c|_{\tau_i})\\ &\le \sum_i \|\tilde{e}_i\|_{B_h}\|e^c\|_{B,\tau_i}\\ &\le \frac{1}{2}\sum_i \left(\|\tilde{e}_i\|_{B_h}^2 + \|e^c\|_{B,\tau_i}^2\right)\\ &= \frac{1}{2}\left(\|e^n\|_{B_h}^2 + \|e^c\|_B^2\right).\end{aligned}$$

Hence

$$\|e^c\|_B \le \|e^n\|_{B_h},$$

which together with (2.10) proves the left inequality of (2.19). For the right inequality, we first extend $\tilde{e}_i = \sum_{k\in K(i)} d_k^i\,\tilde{\phi}_k^i \in S_i^c$ to a new function e_i^+ defined by

$$e_i^+ = \sum_{k\in K(i)} d_k^i\,\phi_{ik},$$

which obviously has support on the extended subdomain

$$T_h(i) := \tau_i \cup \left(\cup_{k\in K(i)}\tau_k\right)$$

from the element $\tau_i \in T_h$. Note that $e_i^+ \in S^c$ and

$$\begin{aligned}\|\tilde{e}_i\|_{B_h}^2 &= B_h(\tilde{e}_i, \tilde{e}_i)\\ &= \frac{1}{2}(F(e_i^+) - B(u_h, e_i^+))\\ &= \frac{1}{2}B(e^c, e_i^+)\\ &\le \frac{1}{2}\|e^c\|_{B,T_h(i)}\,\|e_i^+\|_B.\end{aligned} \tag{2.20}$$

Since the shape regularity of T_h implies the local quasi-uniformity of the mesh, the areas of the two adjacent elements τ_i and τ_k are comparable, that is, the ratio of the Jacobians of the two affine transformations θ_{τ_i} and θ_{τ_k} is bounded. Moreover, on the reference elements (triangle or rectangle), the ratio of $|\hat{\phi}_k^i|_{1,\hat{\tau}}$ and $|\hat{\phi}_i^k|_{1,\hat{\tau}}$ is either equal to one (both τ_i and τ_k are of the same type of elements) or fixed (one is triangle and the other is rectangle). Therefore, there is a positive constant C_5, which is independent of the elements τ_i and τ_k, such that

$$|\phi_{ik}|^2_{1,\tau_k} \le C_5 |\phi_{ik}|^2_{1,\tau_i} \tag{2.21}$$

or equivalently

$$\|d^i_k \phi_{ik}\|^2_{B,\tau_k} \le C_5 \, \|d^i_k \phi_{ik}\|^2_{B,\tau_i}. \tag{2.22}$$

Hence,

$$\begin{aligned}
\|e_i^+\|^2_B &= \|e_i^+\|^2_{B,\tau_i} + \sum_{k\in K(i)} \|e_i^+\|^2_{B,\tau_k} \\
&= \|\tilde{e}_i\|^2_{B_h} + \sum_{k\in K(i)} \|d^i_k \phi_{ik}\|^2_{B,\tau_k} \\
&\le \|\tilde{e}_i\|^2_{B_h} + C_5 \sum_{k\in K(i)} \|d^i_k \phi_{ik}\|^2_{B,\tau_i} \\
&= \|\tilde{e}_i\|^2_{B_h} + C_5 \sum_{k\in K(i)} \|d^i_k \tilde{\phi}^i_k\|^2_{B_h} \\
&\le \|\tilde{e}_i\|^2_{B_h} + \frac{C_5}{C_1} \, \|\tilde{e}_i\|^2_{B_h} \qquad \text{(by (2.15))} \\
&= C_6 \, \|\tilde{e}_i\|^2_{B_h},
\end{aligned} \tag{2.23}$$

where $C_6 = 1 + \frac{C_5}{C_1}$. From (2.18), (2.20), and (2.23), we have

$$\begin{aligned}
\|e^n\|^2_{B_h} &= \sum_i \|\tilde{e}_i\|^2_{B_h} \\
&\le \frac{C_6}{4} \sum_i \|e^c\|^2_{B,T_h(i)} \\
&\le \frac{C_6}{4} \max_i\{|T_h(i)|\} \|e^c\|^2_B \\
&\le \frac{5C_6}{4} \|e^c\|^2_B,
\end{aligned} \tag{2.24}$$

where $|T_h(i)|$ denotes the number of elements in the subdomain, which is less than or equal to 5 (a rectangle having 4 adjacent elements at most). The right inequality of (2.19) then follows with $C_4 = \frac{\sqrt{5C_6}}{2}$ which is independent of h. ■

Remark. Obviously, the previous argument holds for more general finite element spaces S and S^c as long as the condition (2.5) is satisfied. Furthermore, if both conforming and nonconforming formulas, i.e., (2.9) and (2.16) are used, we obtain a generic method for all h, p, and hp adaptivity. For example, assuming that the approximation order is p, if the next hierarchical shape functions of degree $p+1$ are internal modes [22], we then use (2.9) to compute the error estimator. Otherwise, we use (2.16) for side modes. Note that the estimators of [9] and [11] based on hierarchical bases are only conforming. Evidently, these estimators will fail for the Laplace equation with linear finite element approximation (see Example 4.1 in [15]). Of course for odd-order approximations, there are many other alternatives, for example, the Bank-Weiser estimator [10]. Nevertheless, we feel that formula (2.16) is a more generic extension of (2.9). This makes implementation of adaptive computations easier for more general applications and numerical methods as mentioned in the introduction. We refer particularly to [17] on how this generic feature can be fully exploited in the object-oriented programming.

3. The Error Estimator for the General Problem

We now apply the nonconforming error estimator to the more general self-adjoint problem

$$\begin{aligned} -\nabla\cdot a(x)\nabla u(x) + b(x)u(x) &= f(x) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega_D, \\ a(x)\frac{\partial u}{\partial n} &= g \qquad \text{on } \partial\Omega_N \end{aligned} \tag{3.1}$$

with Ω as in (2.1), $a(x)$ in $C^1(\bar{\Omega})$, $b(x)$ in $C^0(\bar{\Omega})$. We assume there exist constants $\underline{a}$, $\bar{a}$, $\underline{b}$, $\bar{b}$ such that $0 < \underline{a} \le a(x) \le \bar{a}$, and $0 \le \underline{b} \le b(x) \le \bar{b}$ for $x \in \bar{\Omega}$. The associated variational problem is to find $u \in H(\Omega)$ such that

$$A(u,v) = F(v) \quad \forall v \in H(\Omega), \tag{3.2}$$

where

$$\begin{aligned} H(\Omega) &:= \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega_D\}, \\ A(u,v) &:= \int_\Omega (a\nabla u \cdot \nabla v + buv)\,dx, \\ F(v) &:= \int_\Omega fv\,dx + \int_{\partial\Omega_N} gv\,ds. \end{aligned}$$

Corresponding to (2.3), the finite element solution $u_h \in S$ now satisfies

$$A(u_h, v_h) = F(v_h) \quad \forall v_h \in S. \tag{3.3}$$

The conforming error problem is then to determine $e^c \in S^c$ such that

$$A(e^c, v) = F(v) - A(u_h, v) \quad \forall v \in S^c. \tag{3.4}$$

The bilinear form $A(\cdot,\cdot)$ induces the A-norm $\|\cdot\|_A$. Assume that Assumption 1 holds with the B-norm replaced by the A-norm. Evidently, Theorem 1 is still valid with (2.10) replaced by

$$(1-\rho)\sqrt{1-\gamma^2}\|e\|_A \le \|e^c\|_A \le \|e\|_A. \tag{3.5}$$

Using the Friedrichs inequality [4], it is well-known that the A- and B-norms are equivalent, i.e., for any $v \in H(\Omega)$, we have

$$C_7\|v\|_B \le \|v\|_A \le C_8\|v\|_B, \tag{3.6}$$

where $C_7 = \underline{a}$ and C_8 is some positive constant.

The equivalence clearly suggests that it is simpler, in terms of implementation, to use the bilinear form $B(\cdot,\cdot)$ for the left-hand side of the nonconforming equation (2.16) than to use $A(\cdot,\cdot)$. The nonconforming problem for the general case is thus to determine, in each element τ_i, $\tilde{e}_i \in S_i^c$ such that

$$B_h(\tilde{e}_i, \tilde{\phi}_j^i) = \frac{1}{2}(F(\phi_{ij}) - A(u_h, \phi_{ij})) \quad \forall \tilde{\phi}_j^i \in S_i^c. \tag{3.7}$$

The use of a simpler (Laplace) operator for the left-hand side of local residual problems was suggested in [3]. The result of the previous section remains valid for the general problem (3.1) as follows:

Theorem 3. *Let $u \in H(\Omega)$, $u_h \in S$ and $\tilde{e}_i$ be the solutions of* (3.2), (3.3), *and* (3.7), *respectively. Let $e = u - u_h$ and e^n be defined by* (2.17). *If Assumption* 1 *holds then*

$$C_9\|e\|_B \le \|e^n\|_{B_h} \le C_{10}\,\|e\|_B, \tag{3.8}$$

where C_9 and C_{10} are positive constants independent of the mesh size h.

Proof. The proof proceeds as that of Theorem 2. We use the same notation there, e.g., c_{ik}, e_i^+ etc.. From (3.4) and (3.7), we have

$$\begin{aligned} B_h(\tilde{e}_i, \tilde{\phi}_k^i) &= \frac{1}{2}(F(\phi_{ik}) - A(u_h, \phi_{ik})) \\ &= \frac{1}{2}A(e^c, \phi_{ik}) \end{aligned}$$

and hence, by (3.6),

$$\begin{aligned}
C_7^2\|e^c\|_B^2 &\le \|e^c\|_A^2 \\
&= A(e^c, e^c) \\
&= \sum_i \frac{1}{2} A\Big(e^c, \sum_{k\in K(i)} c_{ik}\,\phi_{ik}\Big) \\
&= \sum_i B_h\Big(\tilde{e}_i, \sum_{k\in K(i)} c_{ik}\,\tilde{\phi}_k^i\Big) \\
&= \sum_i B_h(\tilde{e}_i, e^c|_{\tau_i}) \\
&\le \sum_i \|\tilde{e}_i\|_{B_h}\|e^c\|_{B,\tau_i} \\
&\le \frac{1}{2}\sum_i \left(\frac{1}{\varepsilon}\|\tilde{e}_i\|_{B_h}^2 + \varepsilon\|e^c\|_{B,\tau_i}^2\right) \\
&= \frac{1}{2}\left(\frac{1}{\varepsilon}\|e^n\|_{B_h}^2 + \varepsilon\|e^c\|_B^2\right)
\end{aligned}$$

for any positive number ε. The left inequality of (3.8) thus follows by choosing $\varepsilon = C_7^2$ and by using (3.5) and (3.6) with the constant $C_9 = C_7^3(1-\rho)\sqrt{1-\gamma^2}/C_8$. Corresponding to (2.20), we have

$$\begin{aligned}
\|\tilde{e}_i\|_{B_h}^2 &= \frac{1}{2}(F(e_i^+) - A(u_h, e_i^+)) \\
&= \frac{1}{2}A(e^c, e_i^+) \\
&\le \frac{1}{2}\|e^c\|_{A,T_h(i)}\,\|e_i^+\|_A \\
&\le \frac{C_8}{2}\|e^c\|_{A,T_h(i)}\,\|e_i^+\|_B.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|e^n\|_{B_h}^2 &= \sum_i \|\tilde{e}_i\|_{B_h}^2 \\
&\le \frac{C_6C_8^2}{4}\sum_i \|e^c\|_{A,T_h(i)}^2 \\
&\le \frac{5C_6C_8^2}{4}\|e^c\|_A^2,
\end{aligned}$$

which proves the right inequality of (3.8) with $C_{10} = C_8^2\sqrt{5C_6}/2$. ■

4. Numerical Example

We consider Laplace's equation

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega = (-1,1)\times(0,1), \\ u &= 0 \quad \text{on } \partial\Omega_D, \\ \frac{\partial u}{\partial n} &= g \quad \text{on } \partial\Omega_N, \end{aligned}$$

where $\partial\Omega_D = \{(x,y) : x \in [-1,1],\ y = 0\}$, $\partial\Omega_N = \partial\Omega\backslash\partial\Omega_D$ and g is defined so that, in polar coordinates, the exact solution u becomes $u = c \cdot r^{1/2} \sin\frac{\alpha}{2}$, where c=0.0700754 [7].

For our computations, the domain is partitioned by squares using the 1-irregular refinement scheme [17] and bilinear elements are used to define S. The halved-functions (2.12) are constructed by using the following shape functions

$$\begin{aligned} \hat{\phi}_1(\xi,\eta) &= (1-\xi^2)(1-\eta)/2, \quad \hat{\phi}_2(\xi,\eta) = (1+\xi)(1-\eta^2)/2, \\ \hat{\phi}_3(\xi,\eta) &= (1-\xi^2)(1+\eta)/2, \quad \hat{\phi}_4(\xi,\eta) = (1-\xi)(1-\eta^2)/2, \end{aligned}$$

defined on the reference element $\hat{\tau} = \{(\xi,\eta) : |\xi| \le 1,\ |\eta| \le 1\}$.

An adaptive process using the nonconforming estimator (2.18) begins with the initial mesh shown in Fig. 1 and ends with the final mesh shown in Fig. 2. The performance of the estimator is shown in Table I, where NN denotes the number of nodes.

This type of model problems has been used to test various estimators (see, e.g., [1, 7, 10, 20]). As mentioned in [20], error estimators based on the solution of local residual problems give more accurate global error assessment as well as better local refinement indicators. This indeed has been observed from the numerical results of [1, 7, 10, 20] and this paper. The quality of our nonconforming estimator (the last column in Table I) coincides with that

Fig. 1. Initial mesh.

Fig. 2. Final mesh.

TABLE I. Effectivity of the estimator

NN	$\|e\|_B$	$\frac{\|e^n\|_{B_h}}{\|e\|_B}$
6	.0231	1.091
15	.0179	1.057
37	.0081	1.093
68	.0053	1.019
165	.0030	0.990
412	.0017	0.984
816	.0012	0.984
1563	.0008	0.986
2215	.0007	0.987
3184	.0005	0.988

of [1] in which the conforming estimator was first proposed for even-order finite element approximations. Finally, in comparison with the strong residual nonconforming estimator of [10], which involves the flux jumps, our estimator also improves in terms of effectiveness and implementation.

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