TAIWANESE JOURNAL OF MATHEMATICS
Vol. 3, No. 3, pp. 323-338, September 1999

CONVERGENCE RESULTS FOR A FAST ITERATIVE METHOD IN LINEAR SPACES

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Abstract. We provide convergence theorems for a fast iterative method to solve nonlinear operator equations in a Banach space. The same method under stronger conditions was found to be of order four, under standard Newton-Kantorovich type assumptions. The monotone convergence of this method in a partially ordered topological space is also examined here.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0. \tag{1}$$

In the first section, F is a nonlinear operator defined on some convex subset D of a Banach space E_1 with values in a Banach space E_2. In the second section, E_1 and E_2 are assumed to be partially ordered topological spaces [4, 6, 10, 11].

We recently introduced the method given by

$$y_n = x_n - F'(x_n)^{-1}F(x_n), \tag{2}$$

$$H(x_n, y_n) = F'(x_n)^{-1}\left(F'\left(x_n + \frac{2}{3}(y_n - x_n)\right) - F'(x_n)\right), \tag{3}$$

Received November 22, 1997.
Communicated by F.-B. Yeh.
1991 *Mathematics Subject Classification*: 47H17, 65H10, 65J15.
Key words and phrases: Newton's method, Banach space, Fréchet-derivative, majorant method, partially ordered topological space.

$$x_{n+1} = y_n - \frac{3}{4}H(x_n, y_n)\left(I - \frac{3}{2}H(x_n y_n)\right)(y_n - x_n) \tag{4}$$

for all $n \geq 0$, and for some $x_0 \in D$. Here $F'(x_n)$ denotes a linear operator which is the Fréchet-derivative of the operator F evaluated at $x = x_n$. We showed that under standard Newton-Kantorovich hypotheses, the order of convergence of the iteration $\{x_n\}$ $(n \geq 0)$ to a locally unique solution x^* of equation (1) is four [5, 6]. We used Lipschitz-type hypotheses on the second Fréchet-derivative of F as well as a hypothesis on an upper bound of the same derivative. Despite the fact that these results can apply to solve multilinear operator equations [1], and in other special cases, in general, it is difficult to verify these conditions. That is why, here we relax these conditions in the first section using only Lipschitz-hypotheses on the first Fréchet-derivative only.

These results can easily be extended under weaker Hölder continuity assumptions or to include nondifferentiable operators (see for example [2] and [3] respectively for Newton's method).

In the second section we examine the monotone convergence of the same method in a partially ordered topological space setting [4, 6, 10, 11].

For a background on two step iterative methods, we refer the reader to [5, 6], and the references there. Note that all previous methods mentioned above are slower than our method.

2. Convergence Analysis

We will need to introduce the constants

$$t_0 = 0, \quad s_0 \geq \|y_0 - x_0\|, \quad \beta \geq \|F'(x_0)^{-1}\| \text{ for some } x_0 \in D, \tag{1.1}$$

$$a = 1 - \beta M R_1, \tag{1.2}$$

$$\begin{aligned} a_0 =& 1 - \beta M \left(\frac{R_1 + R}{2}\right) \text{ for fixed } R_1 \text{ and } R \text{ with } 0 \leq R_1 \leq R, \\ & \text{and some } M > 0, \end{aligned} \tag{1.3}$$

the sequences

$$\bar{a}_n = 1 - \beta M \|x_n - x_0\|, \tag{1.4}$$

$$a_n = 1 - \beta M t_n, \tag{1.5}$$

$$\bar{h}_{n+1} = \frac{M}{2}\left[\|x_{n+1} - y_n\|^2 + 2\|x_n - y_n\|^2 \left(1 + \frac{2\beta M \|y_n - x_n\|}{3(1 - \beta M \|x_n - x_0\|}\right)\right], \tag{1.6}$$

$$(1.7)\qquad h_{n+1}=\frac{M}{2}\left[(t_{n+1}-s_n)^2+2(s_n-t_n)\left(1+\frac{2\beta M(s_n-t_n)}{3(1-\beta M t_n)}\right)\right],$$

$$(1.8)\qquad \bar{b}_n=\frac{\beta M\|y_n-x_n\|}{2(1-\beta M\|x_n-x_0\|)}\left(1+\frac{\beta M\|y_n-x_n\|}{1-\beta M\|x_n-x_0\|}\right)\|y_n-x_n\|,$$

$$(1.9)\qquad b_n=\frac{\beta M(s_n-t_n)}{2(1-\beta M t_n)}\left(1+\frac{\beta M(s_n-t_n)}{1-\beta M t_n}\right)(s_n-t_n),$$

$$(1.10)\qquad s_{n+1}=t_{n+1}+\frac{\beta h_{n+1}}{a_{n+1}},$$

$$(1.11)\qquad t_{n+1}=s_n+b_n,$$

$$(1.12)\qquad e_{n+1}=\beta\left[1-\frac{\beta M}{2}(\|x^*-x_0\|+\|x_{n+1}-x_0\|)\right]^{-1}$$

and the function

$$(1.13)\qquad T(r)=s_0+\frac{Mr}{2(1-\beta Mr)}\left[r+2+\frac{4\beta Mr}{3(1-\beta Mr)}+\frac{\beta Mr^2}{1-\beta Mr}\right]$$

on $[0,R]$.

We can now state and prove the result:

Theorem 1.1. *Let $F:D\subseteq E_1\to E_2$ be a nonlinear operator whose Fréchet-derivative satisfies the Lipschitz condition*

$$(1.14)\qquad \|F'(x)-F'(y)\|\le M\|x-y\| \text{ for all } x,y\in D \text{ and some } M>0.$$

Moreover, assume:

(i) *there exists a minimum nonnegative number R_1 such that*

$$(1.15)\qquad T(R_1)\le R_1;$$

(ii) *the numbers R, R_1, with $R_1\le R$, are such that the constants, a and a_0, given by* (1.2) *and* (1.3) *respectively, are positive and R is such that*

$$(1.16)\qquad U(x_0,R)=\{x\in E_1 \mid \|x-x_0\|\le R\}\subseteq D.$$

Then

(a) *the scalar sequences $\{t_n\}$ $(n\ge 0)$ generated by* (1.10) *and* (1.11) *is monotonically increasing and bounded above by its limit, which is number R_1;*

(b) *the sequence* $\{x_n\}$ $(n \geq 0)$ *generated by* (2)–(4) *is well-defined, remains in* $U(x_0, R_1)$ *for all* $n \geq 0$*, and converges to a solution* x^* *of the equation* $F(x) = 0$*, which is unique in* $U(x_0, R)$*.*

Moreover, the following estimates are true for all $n \geq 0$*,*

$$\|y_n - x_n\| \leq s_n - t_n, \tag{1.17}$$

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n, \tag{1.18}$$

$$\|x^* - x_n\| \leq R_1 - t_n, \tag{1.19}$$

$$\|x^* - x_n\| \leq R_1 - s_n, \tag{1.20}$$

$$\|F(x_{n+1})\| \leq \bar{h}_{n+1} \leq h_{n+1}, \tag{1.21}$$

$$\|x^* - x_{n+1}\| \leq e_{n+1}\bar{h}_{n+1} \leq R_1 - t_{n+1} \tag{1.22}$$

and

$$\|y_n - x_n\| \leq \|x^* - x_n\| + \frac{\beta M}{2\bar{a}_n}\|x_n - x^*\|^2. \tag{1.23}$$

Proof. (a) By (1.1), (1.10) and (1.11), we deduce that the sequence $\{t_n\}$ $(n \geq 0)$ is monotonically increasing and nonnegative. By the same relations, we can easily get $t_0 \leq s_0 \leq t_1 \leq s_1 \leq R_1$. Let us assume that $t_k \leq s_k \leq t_{k+1} \leq s_{k+1} \leq R_1$ for $k = 0, 1, 2, \ldots, n$. Then by relations (1.10) and (1.11), we can have in turn

$$\begin{aligned}
t_{k+2} = {} & t_{k+1} + \frac{M\beta}{2(1 - \beta M t_{k+1})}\left[(t_{k+1} - s_n)^2 + 2(s_k - t_k)\left(1 + \frac{2\beta M(s_k - t_k)}{3(1 - \beta M t_k)}\right)\right] \\
& + \frac{\beta M(s_{k+1} - t_{k+1})}{2(1 - \beta M t_{k+1})}\left(1 + \frac{\beta M(s_{k+1} - t_{k+1})}{1 - \beta M t_{k+1}}\right)(s_{k+1} - t_{k+1}) \\
& \leq t_{k+1} + \frac{M\beta}{2(1 - \beta M R_1)}\left[(t_{k+1} - s_k)^2 + 2(s_k - t_k) + \frac{4\beta M(s_k - t_k)^2}{3(1 - \beta M R_1)}\right. \\
& \qquad \left. + (s_{k+1} - t_{k+1})^2 + \frac{\beta M(s_{k+1} - t_{k+1})3}{1 - \beta M R_1}\right] \\
& \leq \cdots \leq s_0 + \frac{M\beta}{2(1 - \beta M R_1)}\left[R_1^2 + 2R_1 + \frac{4\beta M R_1^2}{3(1 - \beta M R_1)} + \frac{\beta M R_1^3}{1 - \beta M R_1}\right] \\
& = T(R_1) \leq R_1,
\end{aligned}$$

by (1.15) (we have used the fact that $(t_{k+1} - s_k)^2 + (s_{k+1} - t_{k+1})^2 \leq r(s_{s+1} - s_k))$.

Hence, the scalar sequences $\{x_n\}$ $(n \geq 0)$ is bounded above by R_1.

By hypothesis (1.15), R_1 is the minimum positive zero of the equation $T(r) - r = 0$ in $[0, R_1]$ and from the above $R_1 = \lim_{n\to\infty} t_n$.

(b) Using (2), (3), (4) and (1.1), we get $x_1, y_0 \in U(x_0, R_1)$, and that estimates (1.17) and (1.18) are true for $n = 0$. Let us assume that they are true for $k = 0, 1, 2, \ldots, n-1$. In fact, by the induction hypothesis

$$\begin{aligned}\|x_{k+1} - x_0\| &\leq \|x_{k+1} - y_0\| + \|y_0 - x_0\| \leq \|x_{k+1} - y_k\| + \|y_k - y_0\| + \|y_0 - x_0\| \\ &\leq \cdots \leq (t_{k+1} - s_k) + (s_k - s_0) + s_0 \leq t_{k+1} \leq R_1,\end{aligned}$$

and

$$\begin{aligned}\|y_{k+1} - x_0\| &\leq \|y_{k+1} - y_0\| + \|y_0 - x_0\| \\ &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - y_k\| + \|y_k - y_0\| + \|y_0 - x_0\| \\ &\leq \cdots \leq (s_{k+1} - t_{k+1}) + (t_{k+1} - s_k) + (s_k - s_0) + s_0 \leq s_{k+1} \leq R_1.\end{aligned}$$

That is, $x_n, y_n \in U(x_0, R_1)$ for all $n \geq 0$.

Using hypothesis (1.14), we have

$$\|F'(x_0)^{-1}\| \, \|F'(x_k) - F'(x_0)\| \leq \beta M \|x_k - x_0\| \leq \beta M t_k \leq \beta M R_1 < 1,$$

since $a > 0$. It now follows from the Banach lemma on invertible operators that $F'(x_k)$ is invertible, and

$$\|F'(x_n)^{-1}\| \leq \frac{\beta}{\bar{a}_n} \leq \frac{\beta}{a_n}. \tag{1.24}$$

By (2)–(4), we can easily obtain the approximation

$$\begin{aligned}F(x_{n+1}) = &\int_0^1 [F'(y_n + t(x_{n+1} - y_n)) - F'(y_n)](x_{n+1} - x_n)\, dt \\ &+ \int_0^1 [F'(x_n + t(y_n - x_n)) - F'(x_n)](y_n - x_n)\, dt \\ &- \frac{3}{4}\left(F'\left(\frac{x_n + 2y_n}{3}\right) - F'(x_n)\right)(y_n - x_n) \\ &- \frac{1}{2}\Bigg\{(F'(y_n) - F'(x_n)) \\ &- \frac{3}{2}\left(F'\left(\frac{x_n + 2y_n}{3}\right) - F'(x_n)\right)\Bigg\} H(x_n, y_n)(y_n - x_n).\end{aligned} \tag{1.25}$$

By the induction hypotheses, (1.14) and (1.25), we can have in turn

$$\begin{aligned}\|F(x_{n+1})\| \leq & \frac{M}{2}\|x_{n+1}-y_n\|^2+\frac{M}{2}\|x_n-y_n\|^2+\frac{M}{2}\|y_n-x_n\|^2\\ &+\frac{M}{2}\|y_n-x_n\|^2\frac{2\beta M\|y_n-x_n\|}{1-\beta M\|x_n-x_0\|}\\ &+\frac{M}{2}\|y_n-x_n\|^2\frac{2\beta M\|y_n-x_n\|}{1-\beta M\|x_n-x_0\|}\\ &=\bar{h}_{n+1}\leq h_{n+1},\end{aligned}$$

by (1.6) and (1.7).

By relations (2), (1.6), (1.7) and (1.24), we get

$$\|y_{n+1}-x_{n+1}\| \leq \|F'(x_{n+1})^{-1}\|\cdot\|F(x_{n+1})\| \leq \frac{\beta\bar{h}_{n+1}}{\bar{a}_{n+1}} \leq \frac{\beta h_{n+1}}{a_{n+1}} = s_{n+1}-x_{n+1},$$

by (1.10), which shows (1.17) for all $n \geq 0$.

Similarly from (3), (4) and the above

$$\begin{aligned}\|x_{n+1}-y_n\| \leq & \frac{3}{4}\|H(x_n,y_n)\|\left(1+\frac{3}{2}\|H(x_n,y_n)\|\right)\|y_n-x_n\|\\ &\leq \bar{b}_n \leq b_n = t_{n+1}-s_n\end{aligned}$$

which shows (1.18) for all $n \geq 0$.

It now follows from estimates (1.17) and (1.18) that the sequence $\{x_n\}$ $(n \geq 0)$ is Cauchy in a Banach space E_1 and as such, it converges to some $x^* \in U(x_0, R_1)$ with $F(x^*) = 0$ (by (2)).

To show uniqueness, we assume that there exists another solution y^* of equation (1) in $U(x_0, R)$.

Then from hypothesis (1.14), we get

$$\begin{aligned}\|F'(x_0)^{-1}\| & \int_0^1 \|F'(x^*+t(y^*-x^*))-F'(x_0)\|\,dt\\ &\leq \beta M\int_0^1 \|x^*+t(y^*-x^*)-x_0\|\,dt\\ &\leq \beta M\int_0^1 [(1-t)\|x^*-x_0\|+t\|y^*-x_0\|]\,dt\\ &\leq \beta M\left(\frac{R_1+R_2}{2}\right)<1,\ \text{since } a_0>0.\end{aligned}$$

It now follows that the linear operator $\int_0^1 F'(x^*+t(y^*-x^*))\,dt$ is invertible, and from the approximation

$$F(y^*)-F(x^*)=\int_0^1 F'(x^*+t(y^*-x^*))\,dt(y^*-x^*),$$

it follows that $x^* = y^*$.

Estimates (1.19) and (1.20) follow easily from estimates (1.17) and (1.18).

Finally using the triangle inequality, and the approximations

$$\begin{aligned} x_{n+1} - x^* &= B_{n+1}^{-1} F(x_{n+1}), \\ B_{n+1} &= \int_0^1 F'(x^* + t(x_{n+1} - x^*))\, dt, \\ y_n - x_n &= x^* - x_n + F'(x_n)^{-1} \left\{ \int_0^1 [F'(x_n + t(x^* - x_n)) - F'(x_n)] \cdot (x^* - x_n) \right\} dt \end{aligned}$$

and the estimate

$$\begin{aligned} \|F'(x_0)^{-1}\| \int_0^1 \|F'(x^* + t(x_{n+1} - x^*)) - F'(x_0)\|\, dt \\ \le \beta M \int_0^1 \|x^* + t(x_{n+1} - x^*) - x_0\|\, dt \\ \le \beta M \textstyle\int_0^1 [(1-t)\|x^* - x_0\| + t\|x_{n+1} - x_0\|]\, dt \\ \le \beta M R_1 < 1 \text{ since } a > 0, \end{aligned}$$

and

$$\|B_{n+1}^{-1}\| \le e_{n+1},$$

where e_{n+1} is given by (1.12), we can immediately obtain estimates (1.22) and (1.23).

That completes the proof of the theorem. ■

Note that estimates (1.22) and (1.23) can be solved for $\|x_n - x^*\|$ for all $n \ge 0$.

We can show that under the hypotheses $a > 0$, $a_0 > 0$ in the above theorem the sequences $\{s_n\}$, $\{t_n\}$ $(n \ge 0)$ and the function T can be replaced by

$$\|y_n - x_n\| \le v_n - w_n \tag{1.26}$$

and

$$\|x_{n+1} - y_n\| \le w_{n+1} - v_n, \tag{1.27}$$

where

$$w_{n+1} = v_n + \frac{15\beta M}{8(1 - \beta M w_n)} (v_n - w_n)^2, \quad w_0 = t_0 = 0, \tag{1.28}$$

$$v_{n+1} = w_{n+1} + \frac{M\beta}{2(1 - \beta M w_{n+1})}((w_{n+1} - v_n)^2 + 4(v_n - w_n)^2), \tag{1.29}$$

and

$$T_1(r) = s_0 + \frac{4M\beta r^2}{2(1-\beta Mr)} + \frac{15M\beta r^2}{8(1-\beta Mr)}\,. \tag{1.30}$$

It can then easily be seen that under the hypotheses of the theorem

$$\|y_n - x_n\| \le s_n - t_n \le v_n - w_n,$$
$$\|x_{n+1} - y_n\| \le t_{n+1} - s_n \le w_{n+1} - v_n,$$

and

$$\|x_n - x^*\| \le R_1 - t_n \le R^* - v_n, \text{ for all } n \ge 0 \text{ (provided that } R^* \le R),$$

where R^* is the minimum nonnegative zero of the equation $T_1(r) - r = 0$ on $[0, R^*]$.

Let us now introduce the scalar function

$$g(t) = \frac{k}{2}t^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}$$

for some fixed numbers k, β, η, with k, $\beta > 0$ and $\eta \ge 0$, the constants

$$r_1 = \frac{1-\sqrt{1-2h}}{h}\eta, \quad r_2 = \frac{1+\sqrt{1-2h}}{h}\eta, \quad \eta = \frac{r_1}{r_2},$$
$$k_1 = \left(M^2 + \frac{N}{6\beta}\right)^{1/2}, \quad h_1 = .46568\ldots,$$

and the iterations for all $n \ge 0$,

$$p_n = q_n - \frac{g(g_n)}{g'(q_n)}, \quad q_0 = 0,$$
$$q_{n+1} = p_n - \frac{3}{4}H_n\left(1 - \frac{3}{2}H_n\right)(p_n - q_n),$$
$$H_n = g'(q_n)^{-1}\left(g'\left(p_n + \frac{2}{3}(p_n - q_n)\right) - g'(p_n)\right),$$

and

$$\alpha_n = \frac{(1-\theta^2)\eta}{1 - \frac{1}{\sqrt[3]{5}}(\sqrt[3]{5}\theta)^{4n}}(\sqrt[3]{5}\theta)^{4^n-1}.$$

In [5] and [6], we showed that if

$$\|F''(x)\| \le M, \quad \|F''(x) - F''(y)\| \le N\|x-y\|,$$
$$\|F'(\bar{x}_0)^{-1}\| \le \beta, \quad \|\bar{y}_0 - \bar{x}_0\| \le \eta$$

and

$$h \geq h_1, \quad k \geq k_1,$$

then

$$\|\bar{x}_n - \bar{x}^*\| \leq r_1 - g_n \leq \alpha_n, \quad F(\bar{x}^*) = 0,$$
$$\|\bar{x}_{n+1} - \bar{y}_n\| \leq q_{n+1} - p_n$$

and

$$\|\bar{y}_n - \bar{x}_n\| \leq p_n - q_n,$$

where

$$\bar{y}_n = \bar{x}_n - F'(\bar{x}_n)^{-1}F(\bar{x}_n),$$
$$\bar{x}_{n+1} = \bar{y}_n - \tfrac{3}{4}\bar{H}_n\left(I - \tfrac{3}{2}\bar{H}_n\right)(\bar{y}_n - \bar{x}_n)$$

and

$$\bar{H}_n = F'(\bar{x}_n)^{-1}\left[F'\left(\bar{x}_n + \frac{2}{3}(\bar{y}_n - \bar{x}_n)\right) - F'(\bar{x}_n)\right] \text{ for all } n \geq 0.$$

Hence the order of convergence of iteration (2)–(4) under the hypotheses of Theorem 1.1 is almost four.

3. Monotone Convergence

In this section we will assume that the reader is familiar with the meaning of a divided difference of order one and the notion of a partially ordered topological space, POTL-space [4, 6, 10, 11]. From now on we assume that E_1 and E_2 are POTL-spaces.

We introduce the iterations

$$F(v_n) + [x_n, x_n](w_n - v_n) = 0, \tag{2.1}$$

$$F(x_n) + [x_n, x_n](y_n - x_n) = 0, \tag{2.2}$$

$$-L_n(w_n - v_n) + [x_n, x_n](v_{n+1} - w_n) = 0, \tag{2.3}$$

and

$$-L_n(y_n - x_n) + [x_n, x_n](x_{n+1} - y_n) = 0, \tag{2.4}$$

where

$$\begin{aligned} L_n = \frac{3}{8}&\left[\left[x_n + \frac{2}{3}(y_n - x_n), x_n + \frac{2}{3}(y_n - x_n)\right] - [x_n, x_n]\right] B_n \\ &\cdot\left[3\left[x_n + \frac{2}{3}(y_n - x_n), x_n + \frac{2}{3}(y_n - x_n)\right] - 5[x_n, x_n]\right] \text{ for all } n \geq 0. \end{aligned} \tag{2.5}$$

Here $[x, y]$ denotes a divided difference of order one, and B_n denotes continuous, nonnegative left subinverses of the linear operator $A_n = [x_n, x_n]$ for all $n \geq 0$. Note that the operator L_n can also be written as

$$\begin{aligned} L_n = &\frac{1}{2}\,[[x_n, y_n] + [y_n - x_n] + 2[y_n, y_n]]\; B_n \\ &\cdot [[x_n, y_n] + [y_n, x_n] + 2[y_n, y_n] - 2[x_n, x_n]] \quad \text{for all } n \geq 0. \end{aligned} \tag{2.6}$$

We can now prove the main result:

Theorem 2.1. *Let F be a nonlinear operator defined on a convex subset D of a regular POTL-space E_1 with values in another POTL-space E_2. Let v_0 and x_0 be two points of D such that*

$$v_0 \leq x_0 \quad \text{and} \quad F(v_0) \leq 0 \leq F(x_0). \tag{2.7}$$

Suppose that F has a divided difference of order one on $D_0 = \langle v_0, x_0 \rangle = \{x \in E_1 \mid v_0 \leq x \leq x_0\} \subseteq D$ satisfying

(2.8) $A_0 = [x_0, x_0]$ *has a continuous nonnegative left subinverse* B_0,

$$[x_0, y] \geq 0 \;\; \text{for all } v_0 \leq y \leq x_0, \tag{2.9}$$

$$[x, v] \leq [x, y] \;\; \text{if } v \leq y, \tag{2.10}$$

$$[x, y] + [y, x] + 2[y, y] - 2[x, x] \geq 0 \;\; \text{if } \; y \leq x, \tag{2.11}$$

there exists a positive number c such that

$$\begin{aligned} &[x, y] + [y, x] + 2[y, y] - (c + 2)[x, x] \leq 0, \\ &\frac{c}{2}[[x, y] + [y, x] + 2[y, y]] + [z, x] \leq [p, q] \end{aligned} \tag{2.12}$$

for all $v \leq y \leq p \leq q \leq x$.

Then there exist two sequences $\{v_n\}$, $\{x_n\}$ $(n \geq 0)$ satisfying the approximations (2.1)–(2.4),

$$v_0 \leq w_0 \leq v_1 \leq \cdots \leq w_n \leq v_{n+1} \leq x_{n+1} \leq y_n \leq \cdots \leq x_1 \leq y_0 \leq x_0,$$

and

$$\lim_{n\to\infty} v_n = v^* \leq x^* = \lim_{n\to\infty} x_n \;\; \text{with} \;\; x^*, v^* \in D_0.$$

Moreover, if the operator A_n is inverse nonnegative, then any solution u of the equation $F(x) = 0$ in D_0 belongs to $\langle v^*, x^* \rangle$.

Proof. Let us define the operator

$$P_1 : \langle 0, x_0 - v_0\rangle \to E_1, \quad P_1(x) = x - B_0(F(v_0) + A_0(x)).$$

This operator is isotone and continuous. We can have in turn

$$\begin{aligned} P_1(0) &= -B_0F(v_0) \geq 0, \quad \text{by (2.7)}, \\ P_1(x_0 - v_0) &= x_0 - v_0 - B_0F(x_0) + B_0(F(x_0) - F(v_0) - A_0(x_0 - v_0)) \\ &\leq x_0 - v_0 + B_0([x_0, v_0] - [x_0, x_0])(x_0 - v_0) \quad \text{by (2.7)} \\ &\leq x_0 - v_0, \end{aligned}$$

since $[x_0, v_0] \leq [x_0, x_0]$ by (2.10).

By Kantorovich's theorem [6, 10], the operator P_1 has a fixed point $z_1 \in \langle 0, x_0 - v_0\rangle$: $P_1(z_1) = z_1$. Set $w_0 = v_0 + z_1$, and we have the estimates

$$\begin{aligned} F(v_0) + A_0(w_0 - v_0) = 0, \\ F(w_0) = F(w_0) - F(v_0) - A_0(w_0 - v_0) \leq 0 \end{aligned}$$

and

$$v_0 \leq w_0 \leq x_0.$$

We define the operator

$$P_2 : \langle 0, x_0 - w_0\rangle \to E_1, \quad P_2(x) = x + B_0(F(x_0) - A_0(x)).$$

This operator is isotone and continuous. We can have in turn

$$\begin{aligned} P_2(0) &= B_0F(x_0) \geq 0, \quad \text{by (2.7)}, \\ P_2(x_0 - w_0) &= x_0 - w_0 + B_0F(w_0) + B_0(F(x_0) - F(w_0) - A_0(x_0 - w_0)) \\ &\leq x_0 - w_0 + B_0([x_0, w_0] - [x_0, x_0])(x_0 - w_0) \quad \text{by (2.7)} \\ &\leq x_0 - w_0, \end{aligned}$$

since $[x_0, w_0] \leq [x_0, x_0]$ by (2.10).

By Kantorovich's theorem, there exists $z_2 \in \langle 0, x_0 - w_0\rangle$ such that $P_2(z_2) = z_2$. Set $y_0 = x_0 - z_1$, and we have the estimates

$$\begin{aligned} F(x_0) + A_0(y_0 - x_0) = 0, \\ F(y_0) = F(y_0) - F(x_0) - A_0(y_0 - x_0) \geq 0 \end{aligned}$$

and

$$v_0 \leq w_0 \leq y_0 \leq x_0.$$

We now define the operator

$$P_3 : \langle 0, x_0 - v_0\rangle \to E_1, \quad P_3(x) = x - B_0(L_0B_0F(v_0) + A_0(x)).$$

where $L_0 = [x_0, x_0] - [x_0, y_0]$.

This operator is isotone and continuous. We have in turn

$$\begin{aligned} P_3(0) &= -B_0 L_0 B_0 F(v_0) \geq 0 \quad \text{by (2.7)}, \\ P_3(x_0 - v_0) &= x_0 - v_0 - B_0 L_0 B_0 F(x_0) + B_0(L_0 B_0(F(x_0) - F(v_0)) \\ &\quad -[x_0, x_0](x_0 - v_0)). \end{aligned}$$

But, by (2.5), (2.6), and (2.10), we can have

$$\begin{aligned} &L_0 B_0(F(x_0) - F(v_0)) - [x_0, x_0](x_0 - v_0) \\ &= (L_0 B_0 [x_0, v_0] - [x_0, x_0])(x_0 - v_0) \leq (L_0 - [x_0, x_0])(x_0 - v_0) \leq 0. \end{aligned}$$

Therefore, we have

$$P_3(x_0 - v_0) \leq x_0 - v_0.$$

By Kantorovich's theorem, there exists $z_3 \in \langle 0, x_0 - v_0 \rangle$ such that $P_3(z_3) = z_3$. Set $v_1 = w_0 + z_3$, and we have the estimates

$$-L_0(w_0 - v_0) + A_0(v_1 - w_0) = 0$$

and

$$L_0(w_0 - v_0) \geq 0.$$

Furthermore, we can define the operator

$$P_4 : \langle 0, x_0 - v_0 \rangle \to E_1, \quad P_4(x) = x + B_0(L_0 B_0 F(x_0) - A_0(x)).$$

This operator is isotone and continuous. We have in turn

$$\begin{aligned} P_4(0) &= B_0 L_0 B_0 F(x_0) \geq 0 \quad \text{by (2.7)}, \\ P_4(x_0 - v_0) &= x_0 - v_0 + B_0 L_0 B_0 F(v_0) \\ &\quad + B_0(L_0 B_0(F(x_0) - F(v_0)) - A_0(x_0 - v_0)) \leq x_0 - v_0 \end{aligned}$$

(by using the same approach as for P_3). By Kantorovich's theorem, there exists $z_4 \in \langle 0, x_0 - v_0 \rangle$ such that $P_4(z_4) = z_4$. Set $x_1 = y_0 - z_4$, and we have the estimates

$$-L_0(y_0 - x_0) + A_0(x_1 - y_0) = 0$$

and

$$L_0(y_0 - x_0) \leq 0.$$

From the approximation (2.3), we now have

$$v_1 - w_0 = w_0 + B_0 L_0(w_0 - v_0) - w_0 = B_0 L_0(w_0 - v_0) \geq 0.$$

Hence, we obtain $w_0 \leq v_1$. Moreover, from the approximation (2.4), we have

$$x_1 - y_0 = y_0 + B_0 L_0(y_0 - x_0) - y_0 = B_0 L_0(y_0 - x_0) \leq 0.$$

That is, we get $x_1 \le y_0$. Furthermore, we can obtain in turn

$$\begin{aligned} v_1 - x_1 &= w_0 + B_0L_0(w_0 - v_0) - (y_0 - B_0L_0(y_0 - x_0)) \\ &= w_0 - y_0 + B_0L_0(w_0 - v_0 + x_0 - y_0) \\ &= v_0 - B_0L_0F(v_0) - (x_0 - B_0F(x_0)) + B_0L_0(v_0 - B_0F(v_0)) \\ &\quad - B_0L_0(v_0) + B_0L_0(x_0) - B_0L_0(x_0 - B_0F(x_0)) \\ &= v_0 - x_0 - B_0(F(v_0) - F(x_0)) - B_0L_0B_0(F(v_0) - F(x_0)) \\ &= (I - B_0[v_0, x_0] - B_0L_0B_0[v_0, x_0])(v_0 - x_0). \end{aligned}$$

But, using hypotheses (2.11) and (2.12), we have

$$\begin{aligned} B_0L_0B_0[v_0, x_0] + B_0[v_0, x_0] &\le B_0L_0B_0A_0 + B_0[v_0, x_0] \\ &\le B_0L_0 + B_0[v_0, x_0] \le B_0(L_0 + [v_0, x_0]) \\ &\le B_0[p, q] \le B_0A_0 \le I. \end{aligned}$$

We now obtain $v_1 \le x_1$. From all the above, we now have that

$$v_0 \le w_0 \le v_1 \le x_1 \le y_0 \le x_0.$$

By hypothesis (2.10), it follows that the operator A_n has a continuous nonnegative left subinverse B_n for all $n \ge 0$. Proceeding by induction, we can show that there exist two sequences $\{v_n\}$, $\{x_n\}$ $(n \ge 0)$ satisfying (2.1)–(2.4) in a regular space E_1 and as such, they converge to some v^*, $x^* \in D_0$. That is, we have

$$\lim_{n\to\infty} v_n = v^* \le x^* = \lim_{n\to\infty} x_n.$$

If $v_0 \le u \le x_0$ and $F(u) = 0$, then we can obtain

$$\begin{aligned} A_0(y_0 - u) &= A_0(x_0 - B_0F(x_0)) - A_0u = A_0(x_0 - u) - A_0B_0(F(x_0) - F(u)) \\ &= A_0(I - B_0[x_0, u])(x_0 - u) \ge 0, \text{ since } B_0[x_0, u] \le B_0A_0 \le I. \end{aligned}$$

Similarly, we show $A_0(w_0 - u) \le 0$.

If the operator A_0 is inverse nonnegative, then it follows from the above that $w_0 \le u \le y_0$. Proceeding by induction, we deduce that $w_n \le u \le y_n$, from which it follows that $w_n \le v_n \le w_{n+1} \le u \le y_{n+1} \le x_n \le y_n$ for all $n \ge 0$. That is, we have $v_n \le u \le x_n$ for all $n \ge 0$. Hence, we get $v^* \le u \le x^*$.

That completes the proof of the theorem. ∎

In what follows, we shall give some natural conditions under which the points v^* and x^* are solutions of the equation $F(x) = 0$.

Theorem 2.2. *Under the hypotheses of Theorem* 2.1*, suppose that* F *is continuous at* v^* *and* x^**. If one of the following conditions is satisfied*

(a) $x^* = y^*$;

(b) E_1 *is normal and there exists an operator* $Q : E_1 \to E_2$ $(Q(0) = 0)$ *which has an isotone inverse continuous at the origin and such that* $A_n \leq T$ *for sufficiently large* n;

(c) E_2 *is normal and there exists an operator* $R : E_1 \to E_2$ $(R(0) = 0)$ *continuous at the origin and such that* $A_n \leq R$ *for sufficiently large* n;

(d) *the operators* A_n *are equicontinuous for all* $n \geq 0$; *and*

(e) E_2 *is normal and* $[u, v] \leq [x, y]$ *if* $u \leq x$ *and* $v \leq y$.
Then, we have

$$F(v^*) = F(x^*) = 0.$$

Proof.

(a) Using the continuity of F and $F(v_n) \leq 0 \leq F(x_n)$ we get $F(v^*) \leq v^* \leq F(v^*)$. That is, we obtain $F(x^*) = F(v^*) = 0$.

(b) By (2.1) and (2.2), we get

$$\begin{aligned} 0 &\geq F(v_n) = A_n(v_n - w_n) \geq Q(v_n - w_n), \\ 0 &\leq F(x_n) = A_n(x_n - y_n) \leq Q(x_n - y_n). \end{aligned}$$

Hence, we get

$$0 \geq Q^{-1}F(v_n) \geq v_n - w_n, \quad 0 \leq Q^{-1}F(x_n) \leq x_n - y_n.$$

Since E_1 is normal and $\lim_{n\to\infty}(v_n - w_n) = \lim_{n\to\infty}(x_n - y_n) = 0$, we have $\lim_{n\to\infty} Q^{-1}F(v_n) = \lim_{n\to\infty} Q^{-1}F(x_n) = 0$. Hence, by continuity, we get $F(v^*) = F(x^*) = 0$.

(c) As above, we get

$$0 \geq F(v_n) \geq R(v_n - w_n), \quad 0 \leq F(x_n) \leq R(x_n - y_n).$$

Using the normality of E_2 and the continuity of F and R, we get $F(v^*) = F(x^*) = 0$.

(d) From the equicontinuity of the operator A_n, we have $\lim_{n\to\infty} A_n(v_n - w_n) = \lim_{n\to\infty} A_n(x_n - y_n) = 0$. Hence, by (2.1) and (2.2), $F(v^*) = F(x^*) = 0$.

(e) Using hypotheses (2.9)–(2.12), we get in turn

$$\begin{aligned} 0 \leq F(y_n) &= F(y_n) - F(x_n) - A_n(y_n - x_n) \\ &= (A_n - [y_n, x_n])(x_n - y_n) \leq ([x_0, x_0] - [x^*, x^*])(x_n - y_n). \end{aligned}$$

Since E_2 is normal and $\lim_{n\to\infty}(x_n - y_n) = 0$, we get $\lim_{n\to\infty} F(x_n) = 0$. Moreover, from hypothesis (2.10)

$$[x^*, x^*](x_n - x^*) \le [x^*, x_n](x_n - x^*) = F(x_n) - F(x^*) \le [x_0, x_0](x_n - x^*)$$

and by the normality of E_2, $F(x^*) = \lim_{n\to\infty} F(x_n)$. Hence, we get $F(x^*) = 0$. The result $F(v^*) = 0$ can be obtained similarly.

The proof of the theorem is complete. ■

As in Theorems 2.1 and 2.2, we can prove the following result (see also [4, 6, 10]):

Theorem 2.3. *Assume that the hypotheses of Theorem* 2.1 *are true. Then the approximations*

$$\begin{aligned} y_n &= x_n - B_n F(x_n), \\ x_{n+1} &= y_n + B_n L_n(y_n - x_n), \quad L_n = [x_n, x_n] - [x_n, y_n], \\ w_n &= v_n - B_n F(v_n) \end{aligned}$$

and

$$v_{n+1} = w_n + B_n L_n(w_n - v_n),$$

where the operators B_n are nonnegative subinverses of A_n, generate two sequences $\{v_n\}$ and $\{x_n\}$ satisfying approximations (2.1)–(2.4). *Moreover, for any solution $u \in \langle v_0, x_0\rangle$ of the equation $F(x) = 0$ we have*

$$u \in \langle v_n, x_n\rangle, \quad n \ge 0.$$

Furthermore, assume that the following are true:

(a) *E_2 is a POTL-space and E_1 is a normal POTL-space*;

(b) $\lim_{n\to\infty} x_n = x^*$ *and* $\lim_{n\to\infty} v_n = v^*$;

(c) *F is continuous at v^** and *x^**; *and*

(d) *there exists a continuous nonsingular nonnegative operator T such that $B_n \ge T$ for sufficiently large n.*
Then

$$F(v^*) = F(x^*) = 0.$$

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