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# LINE DIGRAPH ITERATIONS AND DIAMETER VULNERABILITY*

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**Abstract.** Many interconnection networks can be constructed with line digraph iterations. In this paper, we will establish a general theorem on diameter vulnerability based on the line digraph iteration which improves and generalizes several existing results in the literature.

## 1. Introduction

Many interconnection networks can be constructed with line digraph iterations, such as de Bruijn digraphs [2], Kautz digraphs [12], generalized de Bruijn digraphs [5, 13], Imase-Itoh digraphs [10], large bipartite digraphs [15], and large generalized cycles [7]. One may study the properties of those networks by taking advantage of line digraph iterations. However, this should be done with caution. In fact, some argument that works for line graph iteration may not work for line digraph iteration. For example, the line graph of a graph which has $d$ edge-disjoint paths of length at most $\ell$ between any two vertices must have $d$ vertex-disjoint paths of length at most $\ell + 1$ between any two vertices. The proof is quite simple. For any two different vertices $u$ and $v$ in the line graph $L(G)$, consider the corresponding edges $(x_u, y_u)$ and $(x_v, y_v)$ in the original graph $G$. We must have $x_u \neq x_v$ or $y_u \neq y_v$. Without loss of generality, assume $x_u \neq x_v$. From $d$ edge-disjoint paths of length at most $\ell$ between $x_u$ and $x_v$, it is easy to construct $d$ vertex-disjoint paths of length at most $\ell + 1$ between $u$ and $v$ in $L(G)$. However, this argument doesn't work so simply for line digraph iteration. For two different vertices $u$ and $v$ in the

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line digraph $L(G)$ of a digraph $G$, consider the corresponding edges $(x_u, y_u)$ and $(x_v, y_v)$ in $G$. In order to construct $d$ vertex-disjoint paths from $u$ to $v$, we need to find $d$ edge-disjoint paths from $y_u$ to $x_v$. When $y_u = x_v$, those edge-disjoint paths are actually cycles whose existences do not follow easily from the assumption on $G$.

Many researchers [1, 14, 8, 6] have noticed this trouble in dealing with line digraph iterations and tried to add additional conditions on the seed digraph to overcome the trouble [6, 14, 15]. In this paper, we want to make one more contribution in this direction. We will propose a set of conditions on seed digraphs and show that with such conditions, every digraph obtained from the seed digraph through line digraph iterations has certain diameter vulnerability. This will improve and generalize several existing results.

## 2. Seed Digraphs

An *internal* vertex of a path in a digraph is a vertex on the path other than endpoints. Note that a vertex can be both an endpoint and an internal vertex in a path. However, this would not occur in a simple path. A *simple* path has no repeated vertex. An *edge-simple* path has no repeated edge. Two paths are *edge-disjoint* if they do not have any edge in common. There are three concepts about vertex-disjointness.

Two paths are *weakly vertex-disjoint* if they do not have an internal vertex in common. Two paths are *vertex-disjoint* if they are edge-disjoint and weakly vertex-disjoint. Two paths are *strongly vertex-disjoint* if no internal vertex on one path is on the other path. $(u, v)$ and $(u, v, w, v)$ are weakly vertex-disjoint, but not vertex-disjoint. $(u, w', v)$ and $(u, v, w, v)$ for $w \neq w'$ are vertex-disjoint but not strongly vertex-disjoint. Thus, these three concepts are different. However, for two simple paths between the same endpoints, these three concepts are equivalent. Note that vertex-disjoint simple paths can be obtained from vertex-disjoint paths by deleting some cycles.

Let $1 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_c$. An $(\ell_1, \ell_2, \cdots, \ell_c)$-*seed* is a digraph satisfying the following conditions:

(a) For any two vertices $u$ and $v$, there exist $c$ vertex-disjoint simple paths from $u$ to $v$, of lengths at most $\ell_1, \ell_2, \cdots, \ell_c$, respectively.

(b) For any two edges $(u, u')$ and $(v', v)$, there are $c - 1$ vertex-disjoint paths from $u$ to $v$, of lengths at most $\ell_2, \ell_3, \cdots, \ell_c$, respectively, satisfying one of the following conditions:

(b1) These $c-1$ paths are simple paths of length at least one, none involving edges $(u, u')$ and $(v', v)$.

(b2) These $c - 1$ paths are edge-simple paths of length at least two, none starting with edge $(u, u')$ and ending at edge $(v', v)$.

The following are examples.

**Example 1.** Kautz digraph $K(d,1)$ is the complete digraph on $d+1$ vertices without loop [12]. We claim that $K(d,1)$ is a $(1,\underbrace{2,\cdots,2}_{d-2},3)$-seed. For any two vertices $u$ and $v$, there are $d$ paths from $u$ to $v$, edge $(u,v)$ and paths $(u,w,v)$ for $w \neq u,v$, meeting condition (a). To verify condition (b), consider two edges $(u,u')$ and $(v',v)$. If $u \neq v'$ and $u' \neq v$, then we first consider paths $(u,w,v)$ for $w \neq u,v',u',v$. When $u'=v'$ and $u=v$, they are $d-1$ simple paths meeting condition (b1). When ($u \neq v$ and $u'=v'$) or ($u=v$ and $u' \neq v'$), they are $d-2$ simple paths together with $(u,v)$ or $(u,v',u',v)$ meeting condition (b1). When $u \neq v$ and $u' \neq v'$, they are $d-3$ simple paths together with $(u,v)$ and $(u,v',u',v)$ meeting condition (b1). If $u=v'$ and $u' \neq v$, then consider $d-2$ paths $(u,w,v)$ for $w \neq u,v,u'$ together with path $(u,v',u',v)$. Note that $u=v$ or $u'=v'$ implies $u \neq v'$ and $u' \neq v$ since no loop exists. Thus, we have $u \neq v$ and $u' \neq v'$. Hence, these $d-1$ paths meet condition (b2). Similarly, we can deal with the case that $u \neq v'$ and $u'=v$. Finally, if $u=v'$ and $u'=v$, then $d-1$ paths $(u,w,v)$ for $w \neq u,v',u',v$ meet condition (b1).

**Example 2.** De Bruijn digraph $B(d,1)$ is the complete digraph on $d$ vertices with all loops. From Example 1, it is easy to see that $B(d,1)$ is a $(1,\underbrace{2,\cdots,2}_{d-3},3)$-seed.

**Example 3**. The complete bipartite digraph on a pair of vertex sets each of cardinality $d$ ($d \geq 3$) is a $(2,\underbrace{3,\cdots,3}_{d-2},4)$-seed. For any two vertices $u$ and $v$ in the same part, there are $d$ vertex-disjoint paths $(u,w,v)$ for $w$ in the other part. For any two vertices $u$ and $v$ in different parts, consider a perfect matching $(u,v)=(u_1,v_1)$, $(u_2,v_2)$, $\cdots$, $(u_d,v_d)$ such that all $u_i$'s are in one part and all $v_i$'s are in the other part. There are $d$ vertex-disjoint simple paths $(u,v)$, $(u,v_i,u_i,v)$ for $i=2,\cdots,d$, meeting condition (a). For two different edges $(u,u')$ and $(v',v)$ with $u$ and $v$ in the same part, consider vertex-disjoint paths $(u,w,v)$ for $w \neq u',v'$. If $u'=v'$, then they are $d-1$ simple paths meeting condition (b1). If $u' \neq v'$, then these $d-2$ simple paths together with a path $(u,v',v,u',v)$ meet condition (b2). For two different edges $(u,u')$ and $(v',v)$ with $u$ and $v$ in the same part. If $u \neq v'$ and $u' \neq v$, consider a perfect matching $(u,u')$, $(w_1,w_1')$, $\cdots$, $(w_{d-2},w'_{d-2})$, $(v,v)$ in $G$. $(u,w_i',w_i,v)$ for $i=1,2,\cdots,d-2$ and $(u,v)$ form $d-1$ paths meeting condition (b1). If $u=v'$ and $u' \neq v$, then consider a perfect matching $(u,u'),(u_1,u_1'),\cdots,(u_{d-2},u'_{d-2}),(w,v)$. Then $(u,u_i',u_i,v)$ for $i=1,2,\cdots,d-2$ and $(u,v',w,v)$ are $d-1$ vertex-disjoint paths meeting condition (b2). If $u \neq v'$ and $u' = v$, then we can verify condition (b2) similarly. If $u = v'$ and $u' = v$, then consider a perfect matching $(u, u')$, $(w_1, w'_1)$, $\cdots$, $(w_{d-1}, w'_{d-1})$ in $G$. $(u, w'_i, w_i, v)$ for $i = 1, 2, \cdots, d-1$ are $d-1$ paths meeting condition (b1).

**Example 4.** Fiol and Yebra [8] defined a family of bipartite digraphs $BD(d, n)$ as follows: The vertex set is $Z_2 \times Z_n = \{(\alpha, i) \mid \alpha \in Z_2, i \in Z_n\}$. There is an edge from $(\alpha, i)$ to $(1-\alpha, (-1)^\alpha d(i+\alpha)+t)$ for every $t = 0, 1, \cdots, d-1$. This family of digraphs has the property $BD(d, dn) = L(BD(d, n))$. It is easy to see that $BD(d, d)$ is the complete bipartite digraph and hence a $(2, \underbrace{3, \cdots, 3}_{d-2}, 4)$-seed. We next show that $BD(d, d^2+1)$ is a $(3, \underbrace{4, \cdots, 4}_{d-2}, 5)$-seed.

For any vertex $u$, denote by $V_u$ the set of all vertices which receive an edge from $u$. Suppose $V_u = x_1, x_2, \cdots, x_d$. It is an important property that $\{u\} \cup V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_d}$ is a partition of $\{0\} \times Z_n$ or $\{1\} \times Z_n$. A consequence of this property is that every vertex in the part containing $u$ receives an edge from $V_u$ except $u$. Consider another vertex $v$. Suppose $v$ is in the part not containing $u$. If $v \notin V_u$, then, for every $V_{x_i}$, $v$ receives an edge $(y_i, v)$ from it. Thus, $d$ paths $(u, x_i, y_i, v)$ for $i = 1, 2, \cdots, d$ satisfy (a). If $v \in V_u$, say $v = x_1$, then $v$ receives an edge $(y_i, v)$ from each $V_{x_i}$ for $i = 2, 3, \cdots, d$. Thus, $d$ paths $(u, v)$, $(u, x_i, y_i, v)$ for $i = 2, 3, \cdots, d$ satisfy (a). Now, suppose $v$ is in the part containing $u$ (of course $u \neq v$). Clearly, $v$ receives edges from $d$ vertices, say, $z_1, z_2, \cdots, z_d$. Then, exactly one of them belongs to $V_u$, say, $z_1 = x_1 \in V_u$. Note that each $V_{x_i}$ for $i = 2, 3, \cdots, d$ has a vertex $y_i$ such that edge $(y_i, z_i)$ exists. Therefore $(u, x_1, v)$, $(u, x_i, y_i, z_i, v)$ for $i = 2, 3, \cdots, d$ satisfy (b1).

To verify (b), consider two edges $(u, u')$ and $(v', v)$. In the previous $d$ paths from $u$ to $v$, if only one path contains either edge $(u, u')$ or $(v', v)$, then the remaining $d-1$ paths satisfy (b1). Therefore, we may assume that there are two paths which contain either edge $(u, u')$ or $(v', v)$. First, suppose $u$ and $v$ are in different parts. If $u' = x_i$ and $v' = y_j$ with $i \neq j$, then we can add a new path $(u, x_j, y'_j, x_i, y_i, v)$ to the remaining $d-2$ paths, where $y'_j \in V_{x_j}$. If $u' = v$ and $v' = y_j$ with $y_j \neq u$, then the remaining $d-2$ paths together with a new path $(u, x_j, y_j, w, u, v)$ for some $w \in V_{y_j}$ satisfy (b2). Next, suppose $u$ and $v$ are in the same part. Assume $u' = x_i$. If $v' = z_j$ with $i \neq j$ and $j \geq 2$, then the remaining $d-2$ paths together with a new path $(u, x_j, w, x_i, y_i, z_i, v)$ $(i > 1)$ or $(u, x_j, w, x_i, v)$ $(i = 1)$ for some $w \in V_{x_j}$ satisfy (b2). If $v' = x_1$, then the remaining $d-2$ paths together with a new path $(u, x_1, w, x_i, y_i, z_i, v)$ for some $w \in V_{x_1}$ satisfy (b2).

**Example 5.** Ferrero and Padró [7] studied two families of digraphs: $BGC(p, d, n) = C_p \otimes B(d, n)$ and $KGC(p, d, n) = C_p \otimes K(d, n)$, where $C_p$

is a directed cycle of length $p$ and operation $\otimes$ is defined as follows: Let $G = (V, E)$ and $G' = (V', E')$. Then $G \otimes G'$ has vertex set $V \times V'$ and edge set $\{((u, u'), (v, v')) \mid (u, v) \in E, (u', v') \in E'\}$. By a similar argument as above, we can verify that $BGC(p, d, d)$ for $p \geq 2$ is a $(p, \underbrace{p+1, \cdots, p+1}_{d-2}, p+2)$-seed and $KGC(p, d, d^p + 1)$ is a $(2p-1, \underbrace{2p, \cdots, 2p}_{d-2}, 2p+1)$-seed.

## 3. Main Results

Our main theorem is as follows.

**Theorem 1.** *Suppose $G$ is an $(\ell_1, \ell_2, \cdots, \ell_c)$-seed. Then for any two vertices $u$ and $v$ in $L^k(G)$, there are $c$ vertex-disjoint simple paths from $u$ to $v$, of lengths at most $k + \ell_1$, $k + \ell_2$, $\cdots$, $k + \ell_c$, respectively.*

*Proof.* We prove it by induction on $k$. For $k = 0$, it is true due to condition (a) in the definition of $(\ell_1, \ell_2, \cdots, \ell_c)$-seed. Next, consider $k \geq 1$. Suppose $u$ and $v$ are two different vertices in $L^k(G)$ and $(x_u, y_u)$ and $(x_v, y_v)$ are corresponding edges in $L^{k-1}(G)$. If $y_u \neq x_v$, then by the induction hypothesis, there exist $c$ vertex-disjoint paths from $y_u$ to $x_v$, of length at most $k - 1 + \ell_1$, $k - 1 + \ell_2$, $\cdots$, $k - 1 + \ell_c$, respectively. From those $c$ paths, it is easy to construct $c$ vertex-disjoint paths from $u$ to $v$, of lengths at most $k + \ell_1$, $k + \ell_2$, $\cdots$, $k + \ell_c$, respectively.

Next, we assume $y_u = x_v$, i.e., the edge $(u, v)$ exists in $L^k(G)$. For each vertex $w$ in $L^k(G)$, find a corresponding edge $(x, y)$ in $L^{k-1}(G)$ and then find path $(a, b, c)$ in $L^{k-1}(G)$ corresponding to vertices $x$ and $y$, .... In this way, we can find a path $(z_1, z_2, \cdots, z_{k+1})$ in $G$ corresponding to vertex $w$ in $L^k(G)$. Conversely, for each path $(z_1, z_2, \cdots, z_{k+1})$ in $G$, we can also find a corresponding vertex $w$ in $L^k(G)$. In fact, there exists a bijective mapping between vertices in $L^k(G)$ and paths of length $k + 1$ in $G$. Thus, we may denote each vertex in $L^k(G)$ by a path of length $k + 1$ in $G$. Consequently, a path of length $\ell + k$ in $G$, $(x_1, x_2, \cdots, x_{\ell+k})$, represents a path of length $\ell$ in $L^k(G)$, $((x_1, \cdots, x_{k+1}), (x_2, \cdots, x_{k+2}), \cdots, (x_\ell, \cdots, x_{\ell+k}))$. Since $y_u = x_v$, $u$ and $v$ can be represented by $(\alpha, x_1, \cdots, x_k)$ and $(x_1, \cdots, x_k, \beta)$.

Consider two edges $(x_k, \beta)$ and $(\alpha, x_1)$. By condition (b) in the definition of $(\ell_1, \ell_2, \cdots, \ell_c)$-seed, there exist $c - 1$ vertex-disjoint edge-simple paths $p_1, p_2, \cdots, p_{c-1}$ from $x_k$ to $x_1$, of lengths at most $\ell_2$, $\cdots$, $\ell_c$, respectively. They satisfy either condition (b1) or (b2). Note that $(\alpha, x_1, \cdots, x_{k-1}, p_i, x_2, \cdots, x_k, \beta)$ for $i = 1, 2, \cdots, c-1$ represent $c - 1$ paths in $L^k(G)$, of lengths at most $k + \ell_2$, $\cdots$, $k + \ell_c$. The condition (b1) or (b2) guarantees that none of these $c - 1$

paths contains edge $(u,v)$. The vertex-disjointness of these $c-1$ paths follows from the following two lemmas.

**Lemma 1.** *If $p_i$ and $p_j$ are two vertex-disjoint paths of length at least two from $x_k$ to $x_1$, then $(\alpha, x_1, \cdots, x_{k-1}, p_i, x_2, \cdots, x_k, \beta)$ and $(\alpha,\ x_1,\ \cdots,\ x_{k-1},\ p_j,\ x_2,\ \cdots,\ x_k,\ \beta)$ represent two vertex-disjoint paths in $L^k(G)$.*

*Proof.* Let $p_i = (x_k,\ y_1,\ \cdots,\ y_s,\ x_1)$ and $p_j = (x_k, z_1, \cdots, z_t, x_1)$, $s \geq 1$ and $t \geq 1$. Assume the contrary that the two paths $(\alpha,\ x_1,\ \cdots,\ x_k,\ y_1,\ \cdots,\ y_s,\ x_1,\ \cdots,\ x_k,\ \beta)$ and $(\alpha, x_1, \cdots, x_k, z_1, \cdots,\ z_t, x_1, \cdots, x_k, \beta)$ are not vertex-disjoint in $L^k(G)$. Then an internal vertex in the first path will be identical to an internal vertex of the second path. Note that each internal vertex in the first path has three possible forms $(x_{i'}, \cdots, x_k, y_1, \cdots, y_{i'})$, $(x_{i'}, \cdots, x_k, y_1, \cdots, y_s, x_1, \cdots, x_{i'-s})$, and $(y_{i'}, \cdots, y_s, x_1, \cdots, x_{k+1-s})$, and so does each vertex in the secod path. Thus, there are nine possible cases. However, a contradiction can be found by essentially the same argument. Without loss of generality, let us consider only one such case: $(x_{i'}, \cdots, x_k, y_1, \cdots, y_{i'}) = (z_{t-j'}, \cdots, z_t, x_1, \cdots, x_{k-j'})$, where $i' \geq 1$ and $j' \leq t$. Note that $p_i$ and $p_j$ are vertex-disjoint paths. Thus, the $y$'s part cannot overlap with the $z$'s part. It follows that

$$y_1 = x_{k-j'-i'+1}, \cdots, y_{i'} = x_{k-j'}$$

and

$$z_{t-j'} = x_{i'}, \cdots, z_t = x_{i'+j'}.$$

Therefore,

$$\{\{x_{i'}, \cdots, x_k, x_{k-j'-i'+1}, \cdots, x_{k-j'}\}\} = \{\{(x_{i'}, \cdots, x_{i'+j'}, x_1, \cdots, x_{k-j'}\}\} \tag{1}$$

and

$$\{x_{k-j'-i'+1}, \cdots, x_{k-j'}\} \cap \{x_{i'}, \cdots, x_{i'+j'}\} = \emptyset, \tag{2}$$

where $\{\{\cdots\}\}$ denotes a multiset.

Now, we first consider the subscripts of elements in the multiset on the left-hand side. Since all subscripts are between 1 and $k$, among $k-j'-i'+1, \cdots, k-j'$, at least one appears twice in the sequence $\{i', \cdots, k, k-j'-i'+1, \cdots, k-j'\}$. Note that each subscript appears at most twice in the sequence. Denote by $N_a$ the subset of subscripts appearing $a$-times in the sequence. Then $|N_2| = |N_0| + 1$.

Similarly, denote by $N'_a$ the subset of subscripts appearing $a$-times in the sequence $\{i', \cdots, i' + j', 1, \cdots, k - j'\}$, subscripts of elements in the multiset on the right-hand side of (1). Note that $N_2 \subseteq \{k - j' - i' + 1, \cdots, k - j'\}$ and $N'_2 \subseteq \{i', \cdots, i' + j'\}$. It follows from (2) that

$$\{x_q \mid q \in N_2\} \cap \{x_r \mid r \in N'_2\} = \emptyset.$$

Deleting all $x_q$ for $q \in N_1 \cap N_1'$ from both sides of (1), we obtain

$$\{\{x_q \mid q \in N_2 \cup (N_1 \cap N_0')\}\} = \{\{x_r \mid r \in N_2' \cup (N_1' \cap N_0)\}\}.$$

Therefore,

$$\{\{x_q \mid q \in N_2\}\} \subseteq \{\{x_r \mid r \in N_1' \cap N_0\}\}.$$

Hence $|N_2| \leq |N_0|$, a contradiction. ∎

**Lemma 2.** *Suppose $p_i = (x_k, x_1)$ and $p_j$ is a simple path of length at least two from $x_k$ to $x_1$. Then $(\alpha,\ x_1,\ \cdots,\ x_{k-1},\ p_i,\ x_2,\ \cdots,\ x_k,\ \beta)$ and $(\alpha,\ x_1,\ \cdots,\ x_{k-1},\ p_j,\ x_2,\ \cdots,\ x_k,\ \beta)$ represent two vertex-disjoint paths in $L^k(G)$.*

*Proof.* Let $p_j = (x_k, z_1, \cdots, z_t, x_1)$. Assume the contrary that the two paths have an internal vertex in common. Since an internal vertex in the path $(\alpha, x_1, \cdots, x_{k-1}, p_j, x_2, \cdots, x_k, \beta)$ has three possible forms, there are three cases. But, a contradiction can be found by the same argument. Without loss of generality, we may consider only one case that $(x_{i'}, \cdots, x_k, x_1, \cdots, x_{i'}) = (x_{k-j'}, \cdots, x_k, z_1, \cdots, z_t, x_1, \cdots, x_{k-j'-t})$.

Construct a digraph $H$ with vertex set $\{x_1, \cdots, x_k, z_1, \cdots, z_t\}$ and edge set

$$\{(y_1, y_1'), (y_2, y_2'), \cdots, (y_{k+1}, y_{k+1}')\},$$

where

$$(y_1, y_2, \cdots, y_{k+1}) = (x_{i'}, \cdots, x_k, x_1, \cdots, x_{i'}),$$
$$(y_1', y_2', \cdots, y_{k+1}') = (x_{k-j'}, \cdots, x_k, z_1, \cdots, z_t, x_1, \cdots, x_{k-j'-t}).$$

(Keep in mind that each edge represents an equality sign.) Clearly, in $H$, $x_{i'}$ has outdegree 2, every vertex in $\{x_{k-j'-t+1}, \cdots, x_{k-j'-1}\}$ has indegree 0 and outdegree 1, every vertex in $\{z_1, \cdots, z_t\}$ has indegree 1 and outdegree 0, and each in the remainder has indegree 1 and outdegree 1. Since the total number of indegees and the total number of outdegrees should be equal, $x_{i'}$ has indegree 1. It follows that starting from $x_{i'}$, if we always choose a new edge to go further, then we must end at a vertex with a larger indegree than outdegree, hence in $\{z_1, \cdots, z_t\}$. Thus, $x_{i'}$ equals one of the $z_h$'s. Similarly, there also exist $t-1$ paths respectively starting from $x_{k-j'-t+1}, \cdots, x_{k-j'-1}$ and ending in $\{z_1, \cdots, z_t\}$. These $t-1$ paths must be totally vertex-disjoint. In fact, two paths having a vertex in common would imply the equality of two $z_h$'s, contradicting the fact that the path $p_j$ is simple. For the same reason, these $t-1$ paths would not pass vertices $x_{i'}$, $x_{k-j'}$, $x_{k-j'-t}$, $x_k$, and $x_1$. Note that $x_{k-j'-t+1}, \cdots, x_{k-j'-1}$ is a consecutive run of size $t-1$, i.e., a set of $t-1$ elements appearing consecutively in $x_1, x_2, \cdots, x_k$. Thus, edges from them will reach a consecutive run of size $t-1$ in $\{x_{k-j'}, \cdots, x_{k-1}\}$,

$\{z_1, \cdots, z_t\}$ or $\{x_2, \cdots, x_{k-j'-t}\}$. In general, they run parallelly such that at each time, they reach a consecutive run of size $t-1$ in $\{x_{k-j'+1}, \cdots, x_{k-1}\}$, $\{z_1, \cdots, z_t\}$ or $\{x_2, \cdots, x_{k-j'-t-1}\}$. This means that these $t-1$ paths would reach either $\{z_1, \cdots, z_{t-1}\}$ or $\{z_2, \cdots, z_t\}$ at the same time. Without loss of generality, assume that the former occurs. Then $x_{k-j'-t+1} = z_1, \cdots, x_{k-j'-1} = z_{t-1}$. It follows that $x_{i'} = z_t$. Now, we follow these $t-1$ paths and consider a path $p$ starting from $x_{k-j'-t}$. When the $t-1$ paths reach a consecutive run $\{x_\ell, x_{\ell+1}, \cdots, x_{\ell+t-1}\}$, path $p$ reaches $x_{\ell-1}$. When the $t-1$ paths reach $\{z_1, \cdots, z_{t-1}\}$, path $p$ would reach $x_k$. Therefore, $x_{i'} = x_{k-j'-t} = x_k$. Hence, $x_k = z_t$, contradicting the fact that $p_j$ is a simple path. ∎

The above two lemmas guarantee that the $c-1$ paths constructed before are actually vertex-disjoint. This completes the proof of Theorem 1. ∎

**Corollary 1 (Du, Hsu, and Lyuu [4]).** *In Kautz digraph $K(d,D) = L^{D-1}(K(d,1))$, for any two vertices, there are $d$ vertex-disjoint paths from one to the other, one of length $D$, $d-2$ of length $D+1$, and one of length $D+2$.*

**Corollary 2 (Imase, Soneoka, and Okada [11]).** *In de Bruijn digraph $B(d,D)(= L^{D-1}(B(d,1)))$, for two vertices, there are $d$ vertex-disjoint paths from one to the other, one of length $D$, $d-3$ of length $D+1$, and one of length $D+2$.*

**Corollary 3 (Cao, Du, and Hsu [3]).** *In bipartite digraph $BD(d, d^{D-1} + d^{D-3})(= L^{D-3}(BD(d, d^2+1)))$, for any two vertices, there are $d$ vertex-disjoint paths from one to the other, one of length $D$, $d-2$ of length $D+1$, and one of length $D+2$.*

This is an improvement of a result in [15].

**Corollary 4 (Ferrero and Padró [7]).** *In bipartite digraph $BD(d, d^{D-1})(= L^{D-2}(BD(d, d)))$, for any two vertices, there are $d$ vertex-disjoint paths from one to the other, one of length $D$, $d-2$ of length $D+1$, and one of length $D+2$.*

**Corollary 5 (Ferrero and Padró [7]).** *In $KGC(p, d, d^{p+k} + d^k)(= L^k(KGC(p, d, d^p + 1)))$, for any two vertices, there are $d$ vertex-disjoint paths from one to the other, one of length $D$, $d-2$ of length $D+1$, and one of length $D+2$, where $D = 2p + k - 1$.*

By a similar argument used in the proof of Theorem 1, we can also show an improvement of the result in Du, Lyuu, and Hsu [6] as follows.

**Theorem 2.** *In de Bruijn $B(d,D)$ and Kautz digraph $K(d,D)$, for any vertex $u$ and vertices $v_1$, $v_2$, $\cdots$, $v_h$ $(0 < \mathrm{h} \le \mathrm{d})$ with $h$ positive integers $d_1$, $d_2$, $\cdots$, $d_h$ $(\mathrm{d}_1 + \mathrm{d}_2 + \cdots + \mathrm{d}_h = \mathrm{d})$, there are $d_1$ simple paths from $u$ to $v_1$, $d_2$ simple paths from $u$ to $v_2$, $\cdots$, $d_h$ simple paths from $u$ to $v_h$. These $d$ simple paths are strongly vertex-disjoint, one of length at most $D$, $d-2$ of length at most $D+1$, and one of length at most $D+2$.*

## 4. Discussion

The main contribution of this paper is the proof technique of Lemma 2. In fact, this is the first time that it appears in a publication. In those previous results that we cited as corollaries in Section 3, the proof was either incomplete or misleading. Typically, in a situation which needs Lemma 2, a statement "this can be proved analogously" appeared. Actually, the proof of Lemma 2 is not analogous to the proof of Lemma 1.

Imase and Itoh [10] proposed a family of digraphs $G_I(d,n)$ with vertex set $Z_n$ and edge set $\{(i, -d(i+1)+r) \mid i \in Z_n, r = 0, 1, \cdots, d-1\}$. When $n = d^D + d^{D-s}$ for odd $s < D$, $G_I(d,n)$ has diameter $D$ and connectivity $d$ [9]. Now, we have the following conjecture.

**Conjecture 1.** *If $s$ is an odd natural number less than $D$, then Imase-Itoh digraph $G_I(d, d^s+1)$ is an $(s, \underbrace{s+1, \cdots, s+1}_{d-2}, s+2)$-seed.*

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