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THE DENSITY OF QUOTIENTS FROM TWO DIFFERENT MÜNTZ SYSTEMS

S. P. Zhou

Abstract. The present paper discusses the density of quotients from two different Müntz systems. An interesting and nontrivial generalization of a result of Somorjai is established.

1. Introduction

From Müntz theorem (cf. [2]), it is well known that the set of combinations of $\{x^{\lambda_n}\}$ for

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

is dense in the space of all continuous functions on $[0, 1]$ (which is denoted by $C_{[0,1]}$) if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

For Müntz rational approximation, the story is different. Somorjai [6] showed that the rational combinations of $\{x^{\lambda_n}\}$ always form a dense set in $C_{[0,1]}$ for any increasing sequence of nonnegative distinct numbers $\{\lambda_n\}$. Bak and Newman [1] further generalized this surprising result to include sequences of nonnegative distinct numbers $\{\lambda_n\}$. Other related materials could be found in [3, 4] and [9].

On the other hand, approximation by quotients from two different Müntz systems is always an interesting but hard topic. Turán in his well-known "problem paper" [8] repeated an open problem which was initially raised by Newman:

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Problem LXXXIII (Turán [8]). *Find conditions on two sequences* $\{\lambda_j\}$ *and* $\{\lambda_j^*\}$ *which assure that every continuous function can be approximated arbitrarily close by rational functions having in their numerators only powers belonging to* $\{\lambda_j\}$ *and in their denominators only powers belonging to* $\{\lambda_j^*\}$[1].

Somorjai [7] constructed an example to show that the set of quotients of two different Müntz systems is not always dense in $C_{[0,1]}$. Explicitly he proved the following result:

Theorem 1. *Assume that for an integer* j_0, $\{\lambda_j\}_{j>j_0}$ *and* $\{\lambda_j^*\}_{j>j_0}$ *are disjoint sets and their union (as a monotone increasing sequence) has Hadamard gaps*[2]. *Then the set of quotients* $R(\Lambda^*/\Lambda)$ *is not dense in* $C_{[0,1]}$.

On the other hand, Somorjai [7] pointed out that the condition $|\lambda_j - \lambda_j^*| = O(1)$ for $j = 1, 2, \cdots$ is sufficient (under the condition that $\{\lambda_j\} \cap \{\lambda_j^*\} \neq \emptyset$) for the density of $R(\Lambda^*/\Lambda)$ in $C_{[0,1]}$.

There has been no further progress on this topic since then.

The intention of the present paper is to give a nontrivial generalization of the result of Somorjai by employing some new ideas. As particular examples, we include the following applications:

Corollary 1. *Let* $\Lambda = \{n^\gamma\}_{n=0}^\infty$, $\gamma \geq 2$, $\Lambda^* = \{n^\gamma \pm n^\rho\}_{n=0}^\infty$, $0 \leq \rho < 1$. *Then* $R(\Lambda^*/\Lambda)$ *is dense in* $C_{[0,1]}$.

Corollary 2. *Let* $\Lambda = \{q^n\}_{n=0}^\infty$, $q > 1$, $\Lambda^* = \{q^n \pm n^\rho\}_{n=0}^\infty$, $0 \leq \rho < 1$. *Then* $R(\Lambda^*/\Lambda)$ *is dense in* $C_{[0,1]}$.

We give the proof of Corollary 1 here. Since $(n+1)^\gamma - n^\gamma \geq \gamma n^{\gamma-1} \geq \gamma n$, and $d_n^\gamma = \pm n^\rho$, $0 \leq \rho < 1$, condition (2) of Theorem 2 is satisfied. Then $R(\Lambda^*/\Lambda)$ is dense in $C_{[0,1]}$ by Theorem 2.

2. Main Result

We are given two sequences of nonnegative distinct increasing numbers $\{\lambda_j\}_{j=1}^\infty$ and $\{\lambda_j^*\}_{j=1}^\infty$. If $\Lambda \cap \Lambda^* \neq \emptyset$, select $\lambda_{N_1^\alpha}$ to be the smallest common element λ_{N_0} of these two sets and let $d_1^\alpha = 0$. Otherwise choose $\lambda_{N_1^\alpha} = \lambda_1$, and

$$|d_1^\alpha| := \min_{j \geq 1}\{|\lambda_j^* - \lambda_1|\}.$$

[01] We denote such a class of rational functions by $R(\Lambda^*/\Lambda)$.

[02] We say a nonnegative increasing sequence $\{a_n\}$ has Hadamard gaps if there is a number $q > 1$ such that $a_{n+1}/a_n \geq q$ for all $n = 1, 2, \cdots$.

Assume the above minimum is achieved at some $j = j_1$, and write $d_1^\alpha = \lambda_{j_1}^* - \lambda_1$. After $\lambda_{N_k^\alpha}$ and d_k^α, $k = 1, 2, \cdots$, are selected, choose

$$\lambda_{N_{k+1}^\alpha} = \min\{\lambda_n : \lambda_n - \lambda_{N_k^\alpha} \geq \alpha k, n > N_k^\alpha\} \tag{1}$$

for some $\alpha > 0$, and

$$|d_{k+1}^\alpha| := \min_{j \geq 1}\{|\lambda_j^* - \lambda_{N_{k+1}^\alpha}|\}.$$

Assume the above minimum is achieved at some $j = j_{k+1}$, and write $d_{k+1}^\alpha = \lambda_{j_{k+1}}^* - \lambda_{N_{k+1}^\alpha}$. By this way we have defined inductively the sequences $\{\lambda_{N_k^\alpha}\}$ and $\{d_k^\alpha\}$.

Theorem 2. *Let $\Lambda = \{\lambda_j\}_{j=1}^\infty$, $\Lambda^* = \{\lambda_j^*\}_{j=1}^\infty$ be two sequences of nonnegative distinct increasing numbers. Assume for some $\alpha > 0$,*

$$\lim_{n\to\infty} \frac{d_n^\alpha}{n} = 0. \tag{2}$$

Then $R(\Lambda^*/\Lambda)$ forms a dense set in $C_{[0,1]}$ if and only if

$$\Lambda^* \cap \Lambda \neq \emptyset.$$

Proof. The necessity part is obvious. For if $\Lambda^* \cap \Lambda = \phi$, then any rational function $r(x) \in R(\Lambda^*/\Lambda)$ which is continuous on $[0,1]$ must have[3]

$$r(0) = 0.$$

This means, $R(\Lambda^*/\Lambda)$ is not dense in $C_{[0,1]}$.

Now suppose $\Lambda^* \cap \Lambda$ is not empty and is finite, otherwise the conclusion is trivial. Then $d_k^\alpha \neq 0$ for sufficiently large k. Without loss of generality, assume $d_k^\alpha \neq 0$ for $k > 1$. Let λ_{N_0} be the smallest one among those common elements of $\Lambda^* \cap \Lambda$, so $\lambda_{N_1^\alpha} = \lambda_{N_0}$. And for some $\alpha > 0$ (2) holds. For convenience, we write $\{N_j\}$ instead of $\{N_j^\alpha\}, \{d_j\}$ instead of $\{d_j^\alpha\}$, and so on. Fix a sufficiently large N. Set

$$x_j := x_j^N = \frac{j}{N},\ j = 1, 2, \cdots, N,$$

$$Q_j(x) = x^{\lambda_{N_{M_1+j}}} x_M^{-(\lambda_{N_{M_1+M}} - \lambda_{N_1})} \prod_{i=M+1}^{j} x_i^{-(\lambda_{N_{M_1+i}} - \lambda_{N_{M_1+i-1}})}, \quad j > M,$$

$$Q_M(x) = Q_M^*(x) = x^{\lambda_{N_1}},$$

[03] Note that if $\Lambda^* \cap \Lambda = \emptyset$, any rational function $r(x) \in R(\Lambda^*/\Lambda)$ satisfies either $r(0) = 0$ or $r(0) = \infty$.

and

$$Q_j^*(x) = \left(\frac{x}{x_j}\right)^{d_{N_{M_1+j}}} Q_j(x), \quad j > M,$$

where

$$M = [\sqrt{N}], \quad M_1 = [M\sqrt{\epsilon_M}^{-1}] + 1,$$

and

$$\epsilon_n = \max_{j \geq n} \left\{ \frac{|d_j|}{j} \right\}.$$

Then clearly we have $Q_j(x) \in \mathrm{span}\{x^{\lambda_j}\}$, and $Q_j^*(x) \in \mathrm{span}\{x^{\lambda_j^*}\}$. For convenience, with the above notations, we divide the proof into some lemmas.

Lemma 1. (i) *For* $x_k \leq x < x_{k+1}$, $k = M, M+1, \cdots, N-1$ *and* $j \in \{M, M+1, \cdots, N-1\} \setminus \{k, k+1\}$, *we have*

$$0 \leq \frac{Q_j(x)}{Q_k(x)} \leq C_1 e^{-C_2|j-k|}. \tag{3}$$

(ii) *For* $x \in [0, x_M)$, $j > M$, *we have*

$$0 \leq \frac{Q_j(x)}{Q_M(x)} \leq \prod_{i=M+1}^{j} e^{-\alpha(i-M)(M_1+i-1)/i} \leq C_1 \exp\left(-\frac{C_2}{\sqrt{\epsilon_M}}(j-M)\right), \tag{4}$$

where here and in the sequel, we always use C_i, $i = 1, 2, \cdots$, *to indicate absolute positive constants.*

Proof. (i) Suppose $x_k \leq x < x_{k+1}$, $M \leq k \leq N-1$ and $M \leq j < k$. We check that for $M < j < k$,

$$0 \leq \frac{Q_j(x)}{Q_k(x)} = \prod_{i=j+1}^{k} \left(\frac{x_i}{x}\right)^{\lambda_{N_{M_1+i}} - \lambda_{N_{M_1+i-1}}} \leq \prod_{i=j+1}^{k} \left(\frac{x_i}{x_k}\right)^{\lambda_{N_{M_1+i}} - \lambda_{N_{M_1+i-1}}}$$
$$\leq \prod_{i=j+1}^{k} \left(1 - \frac{k-i}{k}\right)^{\lambda_{N_{M_1+i}} - \lambda_{N_{M_1+i-1}}}.$$

By (1) and the estimate

$$1 - x \leq e^{-x} \text{ for } x \geq 0,$$

it follows that for $M \leq i < k$,

$$\left(1 - \frac{k-i}{k}\right)^{\lambda_{N_{M_1+i}} - \lambda_{N_{M_1+i-1}}} \leq e^{-\alpha(k-i)(M_1+i-1)/k} \leq e^{-\alpha(k-i)(i-1)/k} \leq e^{-C_3}.$$

Therefore

$$\frac{Q_j(x)}{Q_k(x)} \le C_4 e^{-C_5(k-j)}.$$

The argument of the above inequality, apart from constants, is similar for $j = M$.

In case $x_k \le x < x_{k+1}$, $M \le k \le N-1$ and $k+1 < j \le N-1$, in a similar way we calculate that

$$\begin{aligned} 0 \le \frac{Q_j(x)}{Q_k(x)} &= \prod_{i=k+1}^{j} \left(\frac{x}{x_i}\right)^{\lambda_{N_{M_1+i}} - \lambda_{N_{M_1+i-1}}} \\ &\le \prod_{i=k+1}^{j} \left(1 - \frac{i-k-1}{i}\right)^{\lambda_{N_{M_1+i}} - \lambda_{N_{M_1+i-1}}} \\ &\le \prod_{i=k+1}^{j} e^{-\alpha(i-k-1)(M_1+i-1)/i} \le C_6 e^{-C_7(j-k)}. \end{aligned}$$

Altogether for $x_k \le x < x_{k+1}$, $k = M, M+1, \cdots, N-1$ and $j \in \{M, M+1, \cdots, N-1\} \setminus \{k, k+1\}$, we have

$$0 \le \frac{Q_j(x)}{Q_k(x)} \le C_1 e^{-C_2|j-k|}.$$

(ii) When $x \in [0, x_M)$, $j > M$, noticing that

$$\frac{M_1}{M} \ge \frac{1}{\sqrt{\epsilon_M}},$$

we apply a similar argument to obtain that

$$0 \le \frac{Q_j(x)}{Q_M(x)} \le \prod_{i=M+1}^{j} e^{-\alpha(i-M)(M_1+i-1)/i} \le C_1 \exp\left(-\frac{C_2}{\sqrt{\epsilon_M}}(j-M)\right).$$

Lemma 1 is proved. ■

Lemma 2. *Assume $x_k \le x < x_{k+1}$, $M \le k \le N-1$, and $M < j \le N-1$. Then*

$$\frac{|Q_j(x) - Q_j^*(x)|}{Q_k(x)} \le \begin{cases} C_8\sqrt{\epsilon_M}, & if\ j = k, k+1, \\ C_8 e^{-C_9(k-j)}(k-j)\sqrt{\epsilon_M}, & otherwise. \end{cases}$$

Proof. We see that for $k > M$,

$$Q_k(x) - Q_k^*(x) = Q_k(x)\left(1 - (x/x_k)^{d_{M_1+k}}\right).$$

When $x \in [x_k, x_{k+1})$, $k = M, M+1, \cdots, N-1$, by the mean-value theorem ($k = M$ is a trivial case since $Q_k(x) = Q_k^*(x)$, so we suppose $M < k \le N-1$),

$$1 - (x/x_k)^{d_{M_1+k}} = d_{M_1+k} \xi_k^{d_{M_1+k}-1} (1 - x/x_k),$$

where $\xi_k \in (1, x/x_k)$. We have the following inequality: for $k \ge M+1$,

$$\frac{|d_{M_1+k}|}{k} \le C_{10}\sqrt{\epsilon_M}. \tag{5}$$

In fact, since $|d_{M_1+k}|/(M_1+k) \le \epsilon_M$, we have

$$\frac{|d_{M_1+k}|}{k} = \frac{|d_{M_1+k}|}{M_1+k}\frac{M_1+k}{k} \le C_{11}\epsilon_M \sqrt{\epsilon_M}^{-1} \le C_{11}\sqrt{\epsilon_M}.$$

Together with

$$1 \le \frac{x}{x_k} \le \frac{x_{k+1}}{x_k} = 1 + \frac{1}{k},$$

we get

$$\xi_k^{d_{M_1+k}-1} \le \left(1 + \frac{1}{k}\right)^{|d_{M_1+k}|} \le e^{|d_{M_1+k}|/k} \le C_{12}.$$

Meanwhile,

$$\left|1 - \frac{x}{x_k}\right| \le \frac{x_{k+1}}{x_k} - 1 = \frac{1}{k}.$$

Altogether, with (5), we have

$$\frac{|Q_k(x) - Q_k^*(x)|}{Q_k(x)} \le C_{12}\frac{|d_{M_1+k}|}{k} \le C_{13}\sqrt{\epsilon_M}. \tag{6}$$

The proof of the following inequality is exactly the same: for $x \in [x_k, x_{k+1})$, $k = M, M+1, \cdots, N-1$,

$$\frac{|Q_{k+1}(x) - Q_{k+1}^*(x)|}{Q_{k+1}(x)} \le \frac{|d_{M_1+k+1}|}{k+1} \le C_{14}\sqrt{\epsilon_M}. \tag{7}$$

Assume $x_k \le x < x_{k+1}$, $k = M, M+1, \cdots, N-1$ and $M < j < k$. By a similar argument,

$$\begin{aligned} |Q_j(x) - Q_j^*(x)| &= Q_j(x)\left|1 - (x/x_j)^{d_{M_1+j}}\right| \\ &\le Q_j(x)|d_{M_1+j}|\xi_j^{d_{M_1+j}-1}(x_{k+1}/x_j - 1), \end{aligned}$$

where $\xi_j \in (1, x/x_j) \subset (1, x_{k+1}/x_j)$, so that

$$\begin{aligned} \xi_j^{d_{M_1+j}-1} \le \left(\frac{k+1}{j}\right)^{|d_{M_1+j}|} &= \left(1 + \tfrac{k-j+1}{j}\right)^{|d_{M_1+j}|} \\ &\le \exp\left(\frac{|d_{M_1+j}|}{j}(k-j+1)\right). \end{aligned}$$

Therefore with (3), (5), we have

$$\frac{|Q_j(x)-Q_j^*(x)|}{Q_k(x)} \le \exp\left(-C_{15}(k-j)+\sqrt{\epsilon_M}(k-j+1)\right)\sqrt{\epsilon_M}(k-j+1).$$

By noting (2) and $j>M$, we finally get for sufficiently large N that

$$\frac{|Q_j(x)-Q_j^*(x)|}{Q_k(x)} \le C_{16}e^{-C_{17}(k-j)}(k-j)\sqrt{\epsilon_M}. \tag{8}$$

When $x_k \le x < x_{k+1}$, $k=M, M+1, \cdots, N-1$ and $k+1<j\le N-1$, the argument is similar. We have

$$|Q_j(x)-Q_j^*(x)| \le Q_j(x)|d_{M_1+j}|\xi_j^{d_{M_1+j}-1}\left(1-x_k/x_j\right),$$

where $\xi_j \in (x_k/x_j, 1)$. The same calculation as the above case leads to

$$\begin{aligned}\frac{|Q_j(x)-Q_j^*(x)|}{Q_k(x)} &\le \exp\left(-C_{18}(j-k)+\frac{|d_{M_1+j}|}{j}\right)\frac{|d_{M_1+j}|}{j}(j-k)\\ &\le C_{19}e^{-C_{20}(j-k)}(j-k)\sqrt{\epsilon_M}.\end{aligned} \tag{9}$$

Lemma 2 is proved. ■

For any given $f(x)\in C_{[0,1]}$, define

$$r_N(f,x)=\frac{\sum_{j=M}^{N-1} f(x_j)Q_j^*(x)}{\sum_{j=M}^{N-1} Q_j(x)}.$$

Write

$$\begin{aligned}f(x)-r_N(f,x) &= \frac{\sum_{j=M}^{N-1}(f(x)-f(x_j))Q_j(x)}{\sum_{j=M}^{N-1} Q_j(x)}\\ &\quad +\frac{\sum_{j=M+1}^{N-1} f(x_j)(Q_j(x)-Q_j^*(x))}{\sum_{j=M}^{N-1} Q_j(x)}\\ &=: \textstyle\sum_1(x)+\sum_2(x).\end{aligned}$$

By (3), for $x\in[x_k, x_{k+1})$, $k=M, M+1, \cdots, N-1$, we have

$$\begin{aligned}\left|\textstyle\sum_1(x)\right| &\le \frac{\sum_{j=k}^{k+1}|f(x)-f(x_j)|Q_j(x)+\sum_{M\le j\le N-1, j\ne k,k+1}|f(x)-f(x_j)|Q_j(x)}{\sum_{j=M}^{N-1} Q_j(x)}\\ &\le 2\omega(f,N^{-1})+\sum_{M\le j\le N-1, j\ne k,k+1}\omega\left(f,\frac{|k-j+1|}{N}\right)C_{21}e^{-C_{22}|k-j|}\\ &\le C_{23}\omega(f,N^{-1})\sum_{j=0}^{\infty} je^{-C_{24}j} \le C_{25}\omega(f,N^{-1}).\end{aligned}$$

Similarly, when $x \in [0, x_M)$, in view of (4), we have

$$|\Sigma_1(x)| \le C_{26}\left(\omega(f, M/N) + \omega(f, N^{-1})\right) \le C_{27}\omega(f, M/N).$$

Altogether for $x \in [0, 1]$, it is deduced that

$$|\Sigma_1(x)| \le C_{28}\omega(f, M/N). \tag{10}$$

At the same time, by applying (6)–(9), for $x \in [x_M, 1]$, we get

$$|\Sigma_2(x)| \le C_{29} \max_{0\le x\le 1} |f(x)|\sqrt{\epsilon_M}\sum_{j=0}^{\infty} je^{-C_{30}j} \le C_{31} \max_{0\le x\le 1} |f(x)|\sqrt{\epsilon_M}.$$

While for $x \in [0, x_M)$, it follows from (4) that

$$|\Sigma_2(x)| \le C_{32} \max_{0\le x\le 1} |f(x)| \sum_{j=0}^{\infty} e^{-C_{33}j\sqrt{\epsilon_M}^{-1}} =: C_{32}\sigma_M \max_{0\le x\le 1} |f(x)|.$$

Combining the above estimates, we have that for $x \in [0, 1]$,

$$|\Sigma_2(x)| \le C_{34} \max_{0\le x\le 1} |f(x)| \left(\sqrt{\epsilon_M} + \sigma_M\right). \tag{11}$$

Because $M = [\sqrt{N}]$, $\lim_{N\to\infty} M/N = 0$. Together with $\lim_{N\to\infty} \epsilon_M = 0$ (note $\lim_{N\to\infty} M = \infty$) as well as $\lim_{N\to\infty} \sigma_M = 0$, by combining (10)–(11), we get for $x \in [0, 1]$,

$$\lim_{N\to\infty} |f(x) - r_N(f, x)| = 0.$$

Thus Theorem 2 is proved. ■

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State Key Laboratory of Oil/Gas Reservoir Geology
and Exploitation
Southwest Institute of Petroleum
Nangchong, Sichuan 637001 China

and

Department of Mathematics
Hangzhou University
Hangzhou, Zhejiang 310028 China