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ANALYTIC SEMIGROUPS ON $L^p_w(0,1)$ AND ON $L^p(0,1)$ GENERATED BY SOME CLASSES OF SECOND ORDER DIFFERENTIAL OPERATORS

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Abstract. Of concern are singular ordinary differential operators defined on an interval for which the coefficient of ellipticity degenerates at the boundary of the interval. Thus, for instance, of concern is the operator A defined by $Au := \alpha u'' + \beta u'$ on the interval $(0,1)$ where $\alpha \in C[0,1]$, $\alpha > 0$ in $(0,1)$ and $\alpha(0) = 0 = \alpha(1)$. We show that, in many cases, A (with suitable boundary conditions) generates a C_0-semigroup on various (weighted or not) L^p spaces on $(0,1)$ which is analytic. It is the analyticity that is the focal point.

Introduction

Given $\alpha \in C[0,1]$ with $\alpha > 0$ in $(0,1)$, $\alpha(0) = 0 = \alpha(1)$, and β a real-valued function in $L^\infty(0,1)$, we consider the degenerate differential operator

$$Au := \alpha u'' + \beta u'$$

and show that under additional regularity (and compatibility) assumptions on the coefficients α and β the operator A, when restricted to suitable domains, generates an analytic semigroup on the space with weight $L^p_w(0,1)$ or on $L^p(0,1)$, where $1 < p < \infty$.

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Here the weight function w is assumed to be in $C[0,1]$ and $w(x) > 0$ on (0,1), while

$$L^p_w(0,1) := \{u : (0,1) \to \mathbb{C} \,|\, u \,\text{measurable on}\, (0,1) \text{ and } \int_{(0,1)} |u(x)|^p w(x)\, dx < +\infty\}$$

and functions coinciding a.e. are identified. After the classical paper by Feller [13], a wide literature appeared on operators like A where α and β are polynomials, according to many concrete applications. We quote the article [24] by Vespri for several remarkable results and a considerable list of references on the subject, embracing the case of domains in $\mathbb{R}^n$, $n \geq 1$, too.

For our purposes, the most interesting result concerning the operator A, whose domain includes elements u with the so-called Wentzell boundary conditions (i.e. $\lim_{x\to 0^+, x\to 1^-} Au(x) = 0$), is due to Clément and Timmermans in [6]. In that paper the authors gave necessary and sufficient conditions on α and β in order that $(A, D(A))$ with

$$D(A) := \{u \in C[0,1] \cap C^2(0,1) |\ \lim_{x\to 0^+, x\to 1^-} Au(x) = 0\}$$

does generate a C_o-semigroup on $C[0,1]$. Although under general conditions on α and β, the analyticity of the semigroup generated by $(A, D(A))$ is still an open problem, in recent years this property has been established for suitable α and β both in $C[0,1]$ (or particular closed subspaces) and in L^p spaces (with or without weight), provided that the domain of A is correspondingly defined.

For instance, Favini and Romanelli in [11] have recently proved that this holds in $C[0,1]$ if $\sqrt{\alpha} \in C^1[0,1]$ and hence $\sqrt{\alpha}$ is $\mathbb{R}$- admissible (i.e.

$$\int_0^{\frac{1}{2}} (\alpha(x))^{-\frac{1}{2}}\, dx = +\infty = \int_{\frac{1}{2}}^1 (\alpha(x))^{-\frac{1}{2}}\, dx),$$

and

$$\frac{\beta}{\sqrt{\alpha}} \in C[0,1],\ \beta(0) = 0 = \beta(1).$$

This last condition is basic in order to transform the resolvent equation $(\lambda - A)u = f$ into an equivalent problem in

$$C_0(\mathbb{R}) := \{v \in C(\mathbb{R}) |\ \lim_{|t|\to\infty} v(t) = 0\},$$

where some results presented by Lunardi in the monograph [17] can be applied.

Notice that the functions

$$\alpha(x) := x^m(1-x)^m, \qquad m \in \mathbb{N},$$

are $\mathbb{R}$-admissible if and only if $m \geq 2$, so that the most important case given by $m = 1$ cannot be treated in this way, but very recently this case was studied in [4].

Neverthless, in [10] we showed that $A^1 u := x(1-x)u''$ with domain $D(A^1) := \{u \in H^1(0,1) | \alpha u'' \in H_o(0,1)\}$, does generate a holomorphic semigroup on $H^1(0,1)$.

The result is obtained by a technique previously applied in [2], exploiting the well-known Lions method of sesquilinear forms on Hilbert spaces. In [2], Barbu, Favini and Romanelli studied more precisely generation properties for the operator $\alpha\Delta$ in the setting of the spaces $L^p_{\alpha^{-1}}(\Omega)$, where Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary, α is continuous on the closure of Ω, strictly positive on Ω and vanishes on its boundary. The analyticity results are gotten via a direct approach to the resolvent operator and, when $n \geq 2$, they hold for p depending on n.

On the other hand, Campiti, Metafune and Pallara in [5] studied regularity of the semigroup generated by A, when $\alpha \in C^1[0,1]$ and $\beta := \alpha'$, so that Au is written as $\frac{d}{dx}(\alpha \frac{du}{dx})$. In such a case, they prove that the Neumann boundary conditions $\alpha u'(0) = 0 = \alpha u'(1)$ are the natural ones for A.

In what follows both the methods in [2] and in [11] will be adapted to deal with the L^p case, with or without weight.

The contents of the paper are organized as follows.

In Section 1, in analogy with the technique applied in [11] for the continuous case, we establish a preliminary lemma, which allows us to introduce a suitable lattice isomorphism between $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ and $L^p(\mathbb{R})$ $(1 < p < \infty)$, provided that $\sqrt{\alpha}$ belongs to $C^1[0,1]$. Therefore, in Section 2, Theorem 2.1, we show that if in addition $\frac{\beta}{\sqrt{\alpha}} \in L^\infty(0,1)$, then $(A, D(A_p))$ generates a holomorphic semigroup on $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$, $1 < p < \infty$. (Here $u \in D(A_p)$ if and only if $u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ and

$$\int_{(0,1)} \alpha^{\frac{p-1}{2}}(x)|u'(x)|^p\,dx < \infty, \int_{(0,1)} \alpha^{\frac{2p-1}{2}}(x)|u''(x)|^p\,dx < \infty,$$

where the derivatives of u are viewed in the sense of distributions.)

Moreover then

$$D(A_p) = \{u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1) |\, \alpha u'' + \beta u' \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)\},$$

according to Theorem 2.3.

In order to replace $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ by $L^p(0,1)$, we need some more regularity on α. In fact, in Theorem 2.4 we see that if, in addition, $\alpha \in W^{2,\infty}(0,1)$ and $\alpha^{\frac{1}{2p}} \in C^1[0,1]$, then A generates a holomorphic semigroup on $L^p(0,1)$, $1 <$

$p < \infty$, provided that its domain $D'(A_p)$ consists of all $u \in L^p(0,1)$ such that both the integrals

$$\int_{(0,1)} \alpha^{\frac{p}{2}}(x)|u'(x)|^p\,dx,\ \int_{(0,1)} \alpha^p(x)|u''(x)|^p\,dx$$

are finite.

In particular, if $1 < p < \infty$ and

$$\alpha \in C^2[0,1],\ \beta \in C[0,1],\ \frac{\beta}{\sqrt{\alpha}} \in L^\infty(0,1),$$

then $D'(A_p)$ coincides with

$$\{u \in L^p(0,1)|\, \alpha u'' + \beta u' \in L^p(0,1)\},$$

as shown in Theorem 2.7. Notice that any $u \in D'(A_p)$ satisfies

$$\lim_{x\to 0^+, x\to 1^-} (\alpha(x))^{\frac{1}{2p}} u(x) = 0 = \lim_{x\to 0^+, x\to 1^-} (\alpha(x))^{\frac{p+1}{2p}} u'(x)$$

and hence $\alpha(x)u'(x) \to 0$ as $x \to 0^+$, $x \to 1^-$.

Since $\beta = \alpha'$ is allowed, we see that under last assumptions our operator $(A, D'(A_p))$ coincides with that one considered by Campiti, Metafune and Pallara in [5, Theorem 2.9]. By uniqueness of generators, our slightly more restrictive regularity assumptions guarantee better infomation on the behaviour of $u \in D'(A_p)$ at the boundary.

Section 3 contains extensions of [2] to $\beta \neq 0$. Indeed, in Theorem 3.2 we show that if α, $\beta \in C[0,1]$ with

$$\alpha(0) = 0 = \alpha(1),\ \alpha(x) > 0 \text{ in } (0,1),\ \frac{\beta}{\alpha} \in L^1(0,1)$$

and the domain $D(A)$ of A is the completion of $C_o^\infty(0,1)$ with respect to the norm

$$\|u\|^2_{D(A)} := \|u\|^2_{L^2_w} + \|u'\|^2_{L^2} + \|\alpha u'' + \beta u'\|^2_{L^2_w},$$

where $w(x) := \frac{1}{\alpha(x)} e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt}$, then A is self-adjoint and nonpositive on $L^2_w(0,1)$. Let us observe that under these assumptions $L^2_w(0,1)$ coincides algebraically with $L^2_{\alpha^{-1}}(0,1)$. Applications of all these results are given to the case

$$\alpha(x) := x^j(1-x)^j, \quad \beta(x) := x^k(1-x)^k,$$

with $j \geq 2$ and suitable $k \geq 1$.

In Section 4 we show how the technique developed in Section 3 allows us to handle also degenerate second order differential operators whose domains consist of functions defined on the unbounded interval $[0,+\infty)$.

For the sake of brevity, we confine our treatment to two prominent examples. The first one concerns the operator associated to the linear Kompaneets equation, already studied by Goldstein in [15] and by Wang in [25]. We improve some results of Wang, who obtained differentiability properties under slightly more restrictive regularity assumptions on the coefficients. In particular, we give a precise justification to some basic a-priori estimates and integrations by parts in the resolvent equation.

The second example treats the operator $Au := xu'' + au'$, where $a \in \mathbb{R}$, $x \in (0,+\infty)$ and the considered space is $L^2_{x^{a-1}}(0,+\infty)$.

It was introduced by Feller in [13] and investigated in the very interesting paper [3] by Brezis, Rosenkrantz and Singer, too. We also indicate how the behaviour of an element $u \in D(A)$ near 0 and ∞ depends in an essential way on the constant a.

1. Preliminaries

Let us prove some preliminary results which will allow us in Section 2 to check the analyticity of the semigroup generated by the operator $Au := \alpha u'' + \beta u'$, defined in suitable subspaces of $(L^p_{\alpha^{-\frac{1}{2}}}(0,1), \|.\|_{\alpha^{-\frac{1}{2}}})$, where $1 \le p < \infty$, by applying techniques analogous to those used in [11], in the setting of the space $C[0,1]$.

In the following, we shall always assume that α, β are real-valued functions on $[0,1]$ with $\alpha \in C[0,1]$ such that

$$\alpha(x) > 0, \quad for \quad x \in (0,1), \qquad \alpha(0) = 0 = \alpha(1) \tag{1.1}$$

and $\beta \in L^\infty(0,1)$.

Let us recall some definitions and notations.

Definition 1.1. A function $\alpha : [0,1] \to \mathbb{R}$ is called $\mathbb{R}$-admissible if the following properties hold:

(i) $\alpha \in C[0,1]$;

(ii) α satisfies (1.1) ;

(iii) $\int_0^{\frac{1}{2}} \frac{1}{\alpha(x)}\, dx = +\infty = \int_{\frac{1}{2}}^1 \frac{1}{\alpha(x)}\, dx$.

Now, let us consider the Banach space $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ $(1 \le p < \infty)$ equipped with the weighted norm

$$\|u\|_{\alpha^{-\frac{1}{2}},p} := \left(\int_0^1 \frac{|u(x)|^p}{\sqrt{\alpha(x)}}\, dx \right)^{\frac{1}{p}}.$$

Let us state the following basic result.

Lemma 1.2. *If* $\alpha \in C[0,1]$ *satisfies* (1.1) *and*

$$\sqrt{\alpha} \in C^1[0,1], \tag{1.2}$$

then the mapping $\Phi : (0,1) \to \mathbb{R}$ *given by*

$$\Phi(x) := \int_{\frac{1}{2}}^{x} \frac{ds}{\sqrt{\alpha(s)}}, \qquad x \in (0,1), \tag{1.3}$$

belongs to $C^1(0,1)$ *and is strictly increasing together with its inverse* $\phi : \mathbb{R} \to (0,1)$*, which is differentiable in* $\mathbb{R}$.

In addition, if $1 \le p < \infty$*, the operator* T_ϕ *defined by*

$$T_\phi u := u \circ \phi$$

possesses the following properties:

(1) $T_\phi(L^p_{\alpha^{-\frac{1}{2}}}(0,1)) = L^p(\mathbb{R})$*, with* $\|T_\phi u\|_{L^p(\mathbb{R})} = \|u\|_{\alpha^{-\frac{1}{2}},p}$ $(u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1))$;
(2) $T_\phi(D(B_p)) = W^{1,p}(\mathbb{R})$,

where

$$D(B_p) := \{u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1) \cap W^{1,p}_{loc}(0,1) |\, \sqrt{\alpha}u' \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)\}.$$

Proof. First of all, let us remark that assumption (1.2) implies that $\sqrt{\alpha}$ is $\mathbb{R}$-admissible. Now, let us consider the mapping Φ as given in (1.3) and observe that Φ is differentiable in $(0,1)$ with $\Phi'(x) = 1/\sqrt{\alpha(x)}$ for $x \in (0,1)$. Then, $\Phi \in C^1(0,1)$ and is strictly increasing and invertible. Its inverse $\phi : \mathbb{R} \to (0,1)$ is differentiable on $\mathbb{R}$ and satisfies

$$\lim_{t\to-\infty} \phi(t) = 0, \quad \lim_{t\to+\infty} \phi(t) = 1,\ \phi'(t) = \frac{1}{\Phi'(\phi(t))} = \sqrt{\alpha(\phi(t))},\ t \in \mathbb{R}.$$

In order to show (1), we consider $u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)$. Thus, $u \circ \phi$ is measurable on $\mathbb{R}$ and, moreover,

$$\int_{\mathbb{R}} |u \circ \phi(t)|^p\, dt = \int_0^1 \frac{|u(x)|^p}{\sqrt{\alpha(x)}}\, dx < +\infty.$$

Conversely, from $v \in L^p(\mathbb{R})$ it follows that

$$\int_0^1 \frac{|v(\Phi(x))|^p}{\sqrt{\alpha(x)}}\, dx = \int_{\mathbb{R}} |v(y)|^p\, dy < +\infty.$$

Hence, $T_\phi(L^p_{\alpha^{-\frac{1}{2}}}(0,1)) = L^p(\mathbb{R})$. In addition, if $u \in D(B_p)$, then

$$\begin{aligned}\int_{\mathbb{R}} |(u\circ\phi)'(t)|^p\,dt &= \int_{\mathbb{R}} |u'(\phi(t))\phi'(t)|^p\,dt\\ &= \int_0^1 \frac{|u'(x)|^p(\sqrt{\alpha(x)})^p}{\sqrt{\alpha(x)}}\,dx < +\infty.\end{aligned}$$

On the other hand, for all $v \in W^{1,p}(\mathbb{R})$, the mapping $v\circ\Phi$ is measurable and, moreover,

$$\begin{aligned}\int_0^1 \frac{|\sqrt{\alpha(x)}(v\circ\Phi)'(x)|^p}{\sqrt{\alpha(x)}}\,dx &= \int_0^1 \frac{|\sqrt{\alpha(x)}v'(\Phi(x))\Phi'(x)|^p}{\sqrt{\alpha(x)}}\,dx\\ &= \int_{\mathbb{R}} |v'(t)|^p\,dt < +\infty.\end{aligned}$$

Since $W^{1,p}(\mathbb{R}) = \overset{o}{W}{}^{1,p}(\mathbb{R})$, we observe that any $u \in D(B_p)$ necessarily satisfies $\lim\limits_{x\to 0^+, x\to 1^-} u(x) = 0$.

Thus, assertion (2) holds. ∎

Remark 1.3. Let us recall that the family of operators $(T_p(t))_{t\in\mathbb{R}}$ given by

$$T_p(t)u(x) := u(x+t), \quad t \in \mathbb{R},\ u \in L^p(\mathbb{R}),\ x \in \mathbb{R},$$

with $1 \le p < \infty$ is a C_0-group on $L^p(\mathbb{R})$, having as generator $(A_p, D(A_p))$, where

$$\begin{aligned}D(A_p) &:= W^{1,p}(\mathbb{R}),\\ A_p u &:= u' (\text{in the sense of distributions}),\ \text{for}\, u \in D(A_p)\end{aligned}$$

(see [19, A-I, p.10]). Hence, if α satisfies the assumptions of Lemma 1.2, then for every $1 \le p < +\infty$, the operator $B_p u := \sqrt{\alpha}u'$ with domain $D(B_p)$ defined as in the above lemma, generates the C_0-group on $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ obtained by $(T_p(t))_{t\in\mathbb{R}}$ via the similarity associated to T_ϕ, according to [19, A-I, 3.0, p.13].

2. Analyticity in $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ and in $L^p(0,1)$ $\quad(1 \le p < \infty)$

In this section, we shall show that the restrictions of the operator $Au := \alpha u'' + \beta u'$ to certain domains generate analytic semigroups both in the spaces $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$, introduced at the beginning of Section 1, and in the spaces $L^p(0,1)$ $(1 < p < \infty)$.

In analogy with [11, Theorem 2.4], we prove the following result.

Theorem 2.1. *Let $\alpha \in C[0,1]$ and $\beta \in L^\infty(0,1)$ be such that* (1.1) *and* (1.2) *hold. In addition, assume*

$$\frac{\beta}{\sqrt{\alpha}} \in L^\infty(0,1). \tag{2.1}$$

Then the operator $C_p u := \alpha u'' + \beta u'$ with domain

$$D(C_p) := \{u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1) | \int_0^1 |u'(x)|^p \alpha(x)^{\frac{p-1}{2}}\, dx < +\infty, \\ \int_0^1 |u''(x)|^p \alpha(x)^{\frac{2p-1}{2}}\, dx < +\infty\},$$

generates an analytic semigroup on $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ for $1 < p < \infty$.

Proof. In view of Lemma 1.2 and Remark 1.3, the operator $(B_p, D(B_p))$ defined as follows

$$D(B_p) := \{u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1) | \sqrt{\alpha} u' \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)\}, \\ B_p u := \sqrt{\alpha} u', \qquad u \in D(B_p),$$

generates a C_0-group of isometries on $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$. From assumption (1.2), it follows that the operator $(B_p^2, D(B_p^2))$, where

$$D(B_p^2) = \{u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1) | \sqrt{\alpha} u' \in L^p_{\alpha^{-\frac{1}{2}}}(0,1),\ \sqrt{\alpha}(\sqrt{\alpha} u')' \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)\}, \\ B_p^2 u = \sqrt{\alpha}(\sqrt{\alpha})' u' + \alpha u'' = (\sqrt{\alpha})' B_p u + \alpha u'', \qquad u \in D(B_p^2),$$

generates an analytic semigroup on $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$, according to [19, A-II, Corollary 1.13].

Then the operator C_p with domain $D(B_p^2)$ coincides with

$$C_p u = B_p^2 u + (\frac{2\beta - \alpha'}{2\sqrt{\alpha}}) B_p u, \quad u \in D(B_p^2),$$

and, in view of assumptions (1.2) and (2.1), it is easily seen that

$$D(C_p) = D(B_p^2) = \{u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1) | \int_0^1 |\sqrt{\alpha(x)} u'(x)|^p \frac{dx}{\sqrt{\alpha(x)}} < +\infty, \\ \int_0^1 |\alpha(x) u''(x)|^p \frac{dx}{\sqrt{\alpha(x)}} < +\infty\},$$

where $\frac{2\beta-\alpha'}{2\sqrt{\alpha}} \in L^\infty(0,1)$ by virtue of assumptions (1.2) and (2.1) too.

Now, by Kallman-Rota inequality (see e.g. [14, Theorem 9.8, p.65]), we have that

$$\|B_p u\|^2_{\alpha^{-\frac{1}{2}},p} \le 4\|B_p^2 u\|_{\alpha^{-\frac{1}{2}},p}\|u\|_{\alpha^{-\frac{1}{2}},p}$$

for all $u \in D(B_p^2)$. Hence, taking also into account the inequality

$$2ab \le \epsilon a^2 + \frac{b^2}{\epsilon} \qquad (\epsilon > 0,\ a, b \in \mathbb{R}),$$

we obtain that the operator $\frac{2\beta-\alpha'}{2\sqrt{\alpha}}B_p$ is B_p^2-bounded with B_p^2-bound equal to 0 in the sense of [16, p.190].

Thus, according to [14, Corollary 9, p.42], we can conclude that also $(C_p, D(C_p))$ generates an analytic semigroup on $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ and the proof is complete. ∎

Remark 2.2. Assumptions (1.2) and (2.1) of the previous theorem assure that

$$D(C_p) = \{u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1) |\, \sqrt{\alpha}u' \in L^p_{\alpha^{-\frac{1}{2}}}(0,1),\, \alpha u'' + \beta u' \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)\}.$$

Moreover, a function α satisfying (1.2) necessarily has the property

$$|\alpha'(x)| \le c\sqrt{\alpha(x)}, \qquad x \in (0,1),$$

and, therefore, by integration, $\alpha(x) \le cx^2$ near 0, and $\alpha(x) \le c(1-x)^2$ near 1. In addition, for every $u \in D(C_p)$, $u(0) = u(1) = 0$ and

$$\lim_{x\to 0^+, x\to 1^-} \sqrt{\alpha(x)}u'(x) = 0.$$

Concerning the domain $D(C_p)$ we obtain this further result.

Theorem 2.3. *Under the hypotheses of Theorem* 2.1 *we have that the domain $D(C_p)$ coincides with*

$$D'(C_p) := \{u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1) |\, \alpha u'' + \beta u' \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)\}$$

for $1 < p < \infty$.

Proof. Let $1 < p < \infty$ and $(A_p, D(A_p))$, $(B_p, D(B_p))$ be the operators introduced in Remark 1.3 and Theorem 2.1, respectively.

According to Lemma 1.2, let us define

$$v(t) := u(\phi(t))$$

and observe that $v \in L^p(\mathbb{R})$ ($v \in W^{1,p}(\mathbb{R})$, resp.) if and only if $u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ ($u \in D(B_p)$, resp.). Hence, from

$$D(A_p^2) = W^{2,p}(\mathbb{R}) = \{v \in L^p(\mathbb{R}) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}) |\, v'' \in L^p(\mathbb{R})\},$$

it follows that

$$D(B_p^2) = \{u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1) \cap W^{2,p}_{\mathrm{loc}}(0,1) |\, B_p^2 u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)\},$$

i.e., the condition $B_p u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ is automatically satisfied if $B_p^2 u \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)$.

Now, arguing as in Theorem 2.1, we have that $D(C_p) = D(B_p^2)$ and

$$C_p u = B_p^2 u + \frac{2\beta - \alpha'}{2\sqrt{\alpha}} B_p$$

for all $u \in D(B_p^2)$. Since $\frac{2\beta-\alpha'}{2\sqrt{\alpha}} \in L^\infty(0,1)$, $D(B_p^2) = D'(C_p)$ and the assertion holds. ■

In what follows we shall show that analogous results to Theorem 2.1 and Theorem 2.3 hold in $L^p(0,1)$ too, provided that more regularity is supposed for α. In order to accomplish this, we observe that the resolvent estimates are reduced to ones in $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ and hence Theorem 2.1 and Theorem 2.3 play a key role.

Theorem 2.4. *Let $1 < p < \infty$, and let $\alpha \in W^{2,\infty}(0,1), \beta \in L^\infty(0,1)$ satisfy the assumptions* (1.1), (2.1) *and, in addition,*

$$\alpha^{\frac{1}{2p}} \in C^1[0,1]. \tag{2.2}$$

Let $D(A_p)$ consist of all $u \in L^p(0,1)$ such that

$$\int_0^1 |\sqrt{\alpha(x)} u'(x)|^p \, dx < +\infty, \quad \int_0^1 |\alpha(x) u''(x)|^p \, dx < \infty,$$

and let

$$A_p u := \alpha u'' + \beta u', \quad u \in D(A_p).$$

Then $(A_p, D(A_p))$ generates a holomorphic semigroup on $L^p(0,1)$.

Remark 2.5. As it will be noted in the proof of the theorem, any $u \in D(A_p)$ satisfies the boundary conditions

$$\lim_{x \to 0^+, x \to 1^-} \alpha(x)^{\frac{1}{2p}} u(x) = \lim_{x \to 0^+, x \to 1^-} \alpha(x)^{\frac{p+1}{2p}} u'(x) = 0. \tag{2.3}$$

Moreover, Theorem 2.4 refines Theorem 4.6 in Vespri's paper [24, p.365], giving both a unified treatment for all $1 < p < \infty$ and an explicit description of $D(A_p)$.

Proof of Theorem 2.4. Let Re $\lambda > 0$ be sufficiently large and consider

$$\lambda u - \alpha u'' - \beta u' = f \in L^p(0,1), \quad u \in D(A_p). \tag{2.4}$$

Multiplying (2.4) by $\alpha(x)^{\frac{1}{2p}}$ and denoting $\alpha(x)^{\frac{1}{2p}} u(x)$ by $v(x)$ and $\alpha(x)^{\frac{1}{2p}} f(x)$ by $g(x)$, we observe that $g \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ and (2.4) reads equivalently

$$\lambda v(x) - \alpha(x)v''(x) - (\beta(x) - \frac{\alpha'(x)}{p})v'(x) + \gamma(x)v(x) = g(x), \tag{2.5}$$

where $v \in D(C_p)$, $0 < x < 1$, and

$$\gamma(x) := \frac{\alpha'(x)}{(2p)\alpha(x)}\beta(x) + \frac{1}{2p}\alpha''(x) - \frac{(1+2p)(\alpha'(x))^2}{4p^2\alpha(x)}.$$

By (2.2), $\alpha^{\frac{1}{2p}} \in C^1[0,1]$ implies $\sqrt{\alpha} \in C^1[0,1]$, while from $\alpha \in W^{2,\infty}(0,1)$ and assumption (2.1) it follows that γI is a bounded linear operator on $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$. Hence, by virtue of Theorem 2.1 the operator

$$D_p v := \alpha v'' + (\beta - \frac{\alpha'}{p})v' - \gamma v$$

with domain $D(D_p) := D(C_p)$ generates an analytic semigroup on $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$, so that

$$\|v\|_{\alpha^{-\frac{1}{2}},p} = \|u\|_{L^p} \le C|\lambda|^{-1}\|g\|_{\alpha^{-\frac{1}{2}},p} = C|\lambda|^{-1}\|f\|_{L^p}.$$

Moreover, there exist $c, c_1 \in \mathbb{R}$ such that the solution v to (2.5) satisfies

$$\int_0^1 |v'(x)|^p \alpha(x)^{\frac{p-1}{2}}\, dx \le c\|f\|^p_{L^p}, \tag{2.6}$$

and

$$\int_0^1 |v''(x)|^p \alpha(x)^{\frac{2p-1}{2}}\, dx \le c_1\|f\|^p_{L^p}. \tag{2.7}$$

We only need to translate (2.6), (2.7) into equivalent conditions on u.

Property (2.6) means

$$\int_0^1 |\frac{1}{2p}(\alpha(x))^{\frac{1}{2p}-1}\alpha'(x)u(x) + (\alpha(x))^{\frac{1}{2p}}u'(x)|^p(\alpha(x))^{\frac{p-1}{2}}\, dx < \infty, \tag{2.8}$$

but, for a suitable $c_2 \in \mathbb{R}$ we obtain

$$\int_0^1 \alpha(x)^{\frac{1-2p}{2}} |u(x)|^p \alpha'(x)^p \alpha(x)^{\frac{p-1}{2}}\, dx \le c_2 \int_0^1 |u(x)|^p\, dx,$$

because of Remark 2.2.

Therefore (2.8) reduces to

$$\int_0^1 (\alpha(x))^{\frac{p}{2}} |u'(x)|^p\, dx < \infty.$$

Concerning (2.7), it reads

$$\int_0^1 (\alpha(x))^{p-\frac{1}{2}} \Big| \frac{1-2p}{4p^2}(\alpha(x))^{\frac{1}{2p}-2}(\alpha'(x))^2 u(x) + \frac{1}{2p}(\alpha(x))^{\frac{1}{2p}-1}\alpha''(x)u(x)$$
$$+\frac{1}{p}(\alpha(x))^{\frac{1}{2p}-1}\alpha'(x)u'(x) + (\alpha(x))^{\frac{1}{2p}}u''(x)\Big|^p\, dx < \infty.$$

On the other hand,

$$\int_0^1 (\alpha(x))^{p-\frac{1}{2}}(\alpha(x))^{\frac{1}{2}-2p}(\alpha'(x))^{2p}|u(x)|^p\, dx$$

is estimated by $c_3\|u\|^p_{L^p}$ for a suitable constant c_3.

Moreover, by Remark 2.2 again,

$$\begin{aligned}\int_0^1 (\alpha(x))^{p-\frac{1}{2}}(\alpha(x))^{\frac{1}{2}-p}|\alpha''(x)|^p|u(x)|^p\, dx &\le c\|u\|^p_{L^p},\\ \int_0^1 (\alpha(x))^{p-\frac{1}{2}}(\alpha(x))^{\frac{1}{2}-p}|\alpha'(x)|^p|u'(x)|^p\, dx &\le c\textstyle\int_0^1 (\alpha(x))^{\frac{p}{2}}|u'(x)|^p\, dx.\end{aligned}$$

Hence (2.7) affirms nothing else than

$$\int_0^1 |\alpha(x)u''(x)|^p\, dx < \infty.$$

Application of Remark 2.2 to $v = \alpha^{\frac{1}{2p}}u$ guarantees that

$$\lim_{x\to 0^+, x\to 1^-} (\alpha(x))^{\frac{1}{2p}}u(x) = 0 = \lim_{x\to 0^+, x\to 1^-} \sqrt{\alpha(x)}(\alpha^{\frac{1}{2p}}u)'(x).$$

Now

$$\sqrt{\alpha(x)}(\alpha(x))^{\frac{1}{2p}-1}\alpha'(x)u(x) = \alpha(x)^{\frac{1}{2p}}\frac{\alpha'(x)}{\sqrt{\alpha(x)}}u(x)$$

tends to 0 as $x \to 0^+$ and $x \to 1^-$. Hence, for all $u \in D(A_p)$, we have the boundary conditions (2.3).

On the other hand, it is an easy matter to verify that if $v \in D(C_p)$ satisfies (2.5), with $g = \alpha^{\frac{1}{2p}} f$, $f \in L^p(0,1)$, then $u := \alpha^{-\frac{1}{2p}} v \in D(A_p)$ satisfies (2.4).

This concludes the proof. ■

Remark 2.6. All $u \in D(A_p)$ satisfy in fact the generalized Neumann boundary conditions of the type

$$\lim_{x\to 0^+, x\to 1^-} \alpha(x)u'(x) = 0,$$

for $\alpha(x)u'(x) = (\alpha(x))^{\frac{p+1}{2p}} u'(x)(\alpha(x))^{\frac{p-1}{2p}}$ and $(\alpha(x))^{\frac{p+1}{2p}} u'(x)$ tends to 0 as $x \to 0^+$ and $x \to 1^-$, in view of Remark 2.5. Our result can be hence compared to that one by Campiti, Metafune and Pallara in [5].

Theorem 2.7. *Let $1 < p < \infty$ and $\alpha \in C^2[0,1], \beta \in C[0,1]$ verify assumptions* (1.1)*,* (2.1) *and* (2.2)*. Then, the domain $D(A_p)$ coincides with*

$$X_p := \{u \in L^p(0,1) | \, \alpha u'' + \beta u' \in L^p(0,1)\}.$$

Proof. Clearly X_p contains $D(A_p)$. Now, if $\operatorname{Re} \lambda > 0$ is sufficiently large, for $f \in L^p(0,1)$ let us consider the resolvent equation

$$\lambda u - \alpha u'' - \beta u' = f \tag{2.9}$$

with $u \in X_p$. Multiplying (2.9) by $(\alpha(x))^{\frac{1}{2p}}$, we see that it is equivalent to the equation in $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$ as follows:

$$\lambda v - \alpha v'' - (\beta - \frac{\alpha'}{p})v' + \gamma v = g, \tag{2.10}$$

where $g(x) := (\alpha(x))^{\frac{1}{2p}} f(x)$, in the unknown v given by $v(x) := (\alpha(x))^{\frac{1}{p}} u(x)$, and with

$$\gamma(x) := \frac{\alpha'(x)}{2p\alpha(x)}\beta(x) + \frac{1}{2p}\alpha''(x) - \frac{(1+2p)(\alpha'(x))^2}{4p^2\alpha(x)}.$$

In view of the new assumptions, (notice that necessarily $\sqrt{\alpha} \in C^1[0,1]$ by (2.2)), the function γ is in $C[0,1]$.

Now we apply Theorem 2.3 and deduce that the operator $(D_p, D(D_p))$ with

$$D(D_p) := \{v \in L^p_{\alpha^{-\frac{1}{2}}}(0,1) | \, \alpha v'' + (\beta - \frac{\alpha'}{p})v' - \gamma v \in L^p_{\alpha^{-\frac{1}{2}}}(0,1)\},$$

$$D_p v := \alpha v'' + (\beta - \frac{\alpha'}{p})v' - \gamma v, \qquad v \in D(D_p),$$

generates an analytic semigroup on $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$.

Thus, denoting by ϕ the same function introduced in Lemma 1.2, it suffices to observe that the present assumptions assure that $\gamma\circ\phi$ is uniformly continuous on $\mathbb{R}$, so that we can apply [17, Theorem 3.1.3, p.73], since $p>1$.

Therefore,

$$\|v\|_{\alpha^{-\frac{1}{2}},p} = \|u\|_{L^p} \le c|\lambda|^{-1}\|g\|_{\alpha^{-\frac{1}{2}},p} = c|\lambda|^{-1}\|f\|_{L^p}$$

and

$$\int_0^1 \frac{|\alpha(x)v''(x) + (\beta(x) - \dfrac{\alpha'(x)}{p})v'(x) - \gamma(x)v(x)|^p}{\sqrt{\alpha(x)}}\,dx \le c^p\|f\|^p_{L^p}, \tag{2.11}$$

where v solves (2.10). Taking into account that $v=\alpha^{\frac{1}{2p}}u$, (2.11) reads equivalently

$$\int_0^1 |\alpha(x)u''(x) + \beta(x)u'(x)|^p\,dx \le c\|f\|^p_{L^p},$$

which characterizes X_p. Since

$$L^p(0,1) = (\lambda - A_p)(D(A_p)) \subseteq (\lambda - A_p)(X_p) = L^p(0,1),$$

necessarily $D(A_p)=X_p$. We want to remark that since v is a solution of (2.10) in $L^p_{\alpha^{-\frac{1}{2}}}(0,1)$, we have

$$\lim_{|t|\to+\infty}(v\circ\phi)(t) = 0,\ \ \lim_{|t|\to+\infty}(v\circ\phi)'(t) = 0,$$

so that

$$\lim_{x\to 0^+, x\to 1^-}(\alpha(x))^{\frac{1}{2p}}u(x) = 0 = \lim_{x\to 0^+, x\to 1^-}\alpha(x)u'(x) = 0.$$

Hence, the proof is complete. ∎

Using the ideas of Remark 2.2 and taking into account Remark 2.5, we deduce the following

Corollary 2.8. *Let* $1<p<\infty$ *and* $\alpha\in C^2[0,1], \beta\in C[0,1]$ *verify assumptions* (1.1), (2.1) *and* (2.2). *Then, the operator* $(K_p, D(K_p))$ *given by*

$$D(K_p) := \{u \in L^p(0,1) \,|\, (\alpha u')' + \beta u' \in L^p(0,1)\},$$
$$K_p u := (\alpha u')' + \beta u',\ \ u\in D(K_p)$$

generates an analytic semigroup on $L^p(0,1)$.

Proof. Under our regularity assumptions, it suffices to observe that $K_p u = \alpha u'' + (\beta + \alpha')u'$ and, on the other hand, $\beta + \alpha'$ has precisely the properties that allow the application of Theorem 2.7. ∎

Example. Let $\alpha(x) := x^j(1-x)^j$, $\beta(x) := x^k(1-x)^k$, where $j, k \geq 1$ and $x \in [0,1]$. Then Theorems 2.1 and 2.3 hold, provided that $j \geq 2$ and $k \geq \frac{j}{2}$. If, in addition, $j \geq 2p$, then Theorems 2.4 and 2.7 apply too. Here j, k need not be integers.

3. A Generation Theorem for $Au := \alpha u'' + \beta u'$ in $L^2_w(0,1)$

Here we shall consider the space $L^2_w(0,1)$, with a suitable w, in order to obtain analyticity results for the semigroup generated by $Au := \alpha u'' + \beta u'$, where the mappings α and β are more general than in Section 2. The arguments we shall use are similar to those in [2], as already observed in the Introduction.

We begin our discussion by stating a preliminary lemma.

Lemma 3.1. *Let $\alpha, \beta \in C[0,1]$ be such that α satisfies (1.1) and $\frac{\beta}{\alpha} \in L^1(0,1)$. If we introduce*

$$w(x) := \frac{1}{\alpha(x)} e^{\int_{\frac{1}{2}}^{x} \frac{\beta(t)}{\alpha(t)}\, dt}, \qquad x \in (0,1), \tag{3.1}$$

then the operator $(A, D(A))$ with

$$Au := \alpha u'' + \beta u', \qquad u \in D(A),$$

and $D(A)$ given by the completion of $C_0^\infty(0,1)$ with respect to the norm

$$\|u\|^2_{D(A)} := \|u\|^2_{L^2_w} + \|u'\|^2_{L^2} + \|\alpha u'' + \beta u'\|^2_{L^2_w}$$

is self-adjoint and nonpositive in $L^2_w(0,1)$. Moreover, the domain of A is equivalently described by

$$\begin{aligned} D(A) = \{ & u \in L^2_w(0,1) \cap H^1_o(0,1) \,|\, u \in H^2_{loc}(0,1), \\ & \frac{1}{w}\big(e^{\int_{\frac{1}{2}}^{x} \frac{\beta(t)}{\alpha(t)}\, dt} u'\big)' \in L^2_w(0,1)\}. \end{aligned} \tag{3.2}$$

Proof. $\mathbf{1}^{st}$ **step**.

We show that A is symmetric and nonpositive. Indeed, if $u, v \in D(A)$, there exist two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $C_0^\infty(0,1)$ such that

$$\|u_n - u\|_{D(A)} \to 0, \quad \|v_n - v\|_{D(A)} \to 0.$$

Thus

$$\begin{aligned}
\langle Au, v\rangle_{L^2_w} &= \int_0^1 (\alpha(x))^{-1} e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} (\alpha(x)u''(x) + \beta(x)u'(x))\overline{v}(x)\,dx \\
&= \int_0^1 e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} (u''(x) + \frac{\beta(x)}{\alpha(x)} u'(x))\overline{v}(x)\,dx \\
&= \int_0^1 \frac{d}{dx}(e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} u'(x))\overline{v}(x)\,dx \\
&= \lim_{n\to\infty} \int_0^1 \frac{d}{dx}(e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} u_n'(x))\overline{v}_n(x)\,dx \\
&= \lim_{n\to\infty} \{[e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} u_n'(x)\overline{v}_n(x)]_{x=0}^{x=1} - \int_0^1 e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} u_n'(x)\overline{v}_n'(x)\,dx\} \\
&= -\lim_{n\to\infty} \int_0^1 e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} u_n'(x)\overline{v}_n'(x)\,dx \\
&= -\int_0^1 e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} u'(x)\overline{v}'(x)\,dx \\
&= \overline{-\int_0^1 e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt}\overline{u'(x)} v'(x)\,dx} = \langle u, Av\rangle_{L^2_w}.
\end{aligned}$$

Note also that the above calculation (with $u = v$) shows that

$$\langle Au, u\rangle_{L^2_w} = -\int_0^1 e^{\int_{\frac{1}{2}}^x \frac{\beta(x)}{\alpha(x)}\,dx} |u'(x)|^2\,dx < 0 \quad (\text{unless}\, u \equiv \text{constant}),$$

whence Au is nonpositive.

2nd **step.** Let us observe that $D(A) \subseteq L^2_w(0,1) \cap H^1_o(0,1)$ and if we interpret $\alpha u'' + \beta u'$ as $w^{-1}(e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}} u')'$, then by arguing as in [18] we deduce that $D(A)$ satisfies (3.2).

3rd **step.** Since a symmetric operator which is *onto* is self-adjoint (see e.g. [26, Chapter VII 3, Corollary, p.199]), in order to complete the proof it suffices to show that $I - A$ is onto $X := L^2_w(0,1)$. For this goal, we observe that if V is the Hilbert space obtained by the completion of $C_o^\infty(0,1)$ with respect to the norm

$$\|u\|_V := (\|u\|_X^2 + \int_0^1 e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} |u'(x)|^2\,dx)^{\frac{1}{2}},$$

which is equivalent to $(\|u\|_X^2 + \|u'\|_{L^2}^2)^{\frac{1}{2}}$ by the assumption $\frac{\beta}{\alpha} \in L^1(0,1)$, then the equation

$$u - Au = f, \tag{3.3}$$

where $f \in X$, admits a weak formulation : to find $u \in V$ such that

$$\int_0^1 w(x)u(x)\overline{\phi(x)}\,dx + \int_0^1 e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} u'(x)\overline{\phi'(x)}\,dx = \int_0^1 w(x)f(x)\overline{\phi(x)}\,dx \tag{3.4}$$

for each $\phi \in V$.

The left-hand side of (3.4) defines a quadratic form $B(u,\phi)$ on $V \times V$ which is continuous and coercive on $V \times V$. Hence, by the Lax-Milgram theorem (cf. [22, Lemma 2.2.1, p.26]), equation (3.4) has a unique weak solution $u \in V$ for all $f \in X$ $(\subseteq V^*)$. Let

$$D(\widetilde{B}) := \{u \in V \mid \phi \to B(u,\phi) \text{ is continuous on } V \\ \text{with respect to } \|.\|_X\}.$$

Since V is dense in X, for each $u \in D(\widetilde{B})$ the mapping $\phi \in V \to B(u,\phi)$ can be extended to a continuous functional on X and, consequently, there exists a unique element in X, say $\widetilde{B}u$, such that $B(u,\phi) = (\widetilde{B}u,\phi)$. By [22, Theorems 2.2.2 and 2.2.3, pp.28-29], $\widetilde{B}$ is an isomorphism from $D(\widetilde{B})$ onto X, so that for all $f \in X$ there exists a unique $u \in D(\widetilde{B})$ satisfying (3.4). On the other hand, if we observe that (3.4) holds for all $\phi \in C_o^\infty(0,1)$ it is deduced that the derivative $\frac{d}{dx}(e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} u')$ in the sense of distributions fulfils

$$\int_0^1 w(x)u(x)\overline{\phi(x)}\,dx - \int_0^1 w(x)w(x)^{-1}\frac{d}{dx}(e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} u')(x)\overline{\phi(x)}\,dx \\ = \int_0^1 w(x)f(x)\overline{\phi(x)}\,dx,$$

and $w^{-1}\frac{d}{dx}(e^{\int_{\frac{1}{2}}^x \frac{\beta(t)}{\alpha(t)}\,dt} u') \in X$. Therefore $\widetilde{B}u = f$ means just that the solution u belongs in fact to the required space $D(A)$ with $(I-A)u = f$.

The proof is now complete. ■

Since a nonpositive self-adjoint operator acting in a space X such that $R(I-A) = X$ is sectorial in $\{z \in \mathbb{C} \mid |\arg z| < \pi\}$, according to [14, Theorem 5.4, p.34], we obtain the following result which extends [2, Theorem 1.2] to the case $\beta \neq 0$ (in the one-dimensional case).

Theorem 3.2. *Under the same assumptions of Lemma* 3.1*, the operator* $Au := \alpha u'' + \beta u'$ *with domain* $D(A)$ *as before, generates a uniformly bounded semigroup on* $L^2_w(0,1)$*, which is analytic in the right half plane.*

Remark 3.3. The self-adjointness of A could be proven in an alternative way as in Metafune and Pallara [18]. To this purpose, we want to notice that, if the assumption $\frac{\beta}{\alpha} \in L^1(0,1)$ is replaced by $e^{-\int_{\frac{1}{2}}^{x} \frac{\beta(t)}{\alpha(t)}\,dt} \in L^1(0,1)$, and we change $H_o^1(0,1)$ to

$$\{u \in X | \int_0^1 e^{\int_{\frac{1}{2}}^{x} \frac{\beta(t)}{\alpha(t)}\,dt} |u'(x)|^2\,dx < \infty,\, u(0) = u(1) = 0\},$$

our approach furnishes an alternative method to prove Proposition 3.5 in [18].

We also observe that if $\frac{\beta}{\alpha} \in L^1(0,1)$ and $w \in L^1(0,1)$, then we have a regular Sturm-Liouville operator and it is already known from [7] that

$$D(A) = \{u \in AC[0,1] |\, u' \in AC[0,1],\, \alpha u'' + \beta u' \in L_w^2(0,1),\, u(0) = u(1) = 0\}$$

is a domain entailing that $(A, D(A))$ is self-adjoint and nonpositive on $L_w^2(0,1)$.

On the other hand, if

$$w \notin L^1(0, \frac{1}{2}),\, w \notin L^1(\frac{1}{2}, 1)$$

and

$$\int_0^1 w(x)|u(x)|^2\,dx < +\infty,\; \int_0^1 w(x)|w(x)^{-1}(e^{\int_{\frac{1}{2}}^{x} \frac{\beta(t)}{\alpha(t)}\,dt} u')'|^2\,dx < +\infty,$$

then [18, Lemma 3.3] implies that $u(x) \to 0$ as $x \to 0, 1$ and hence the operator A coincides with the maximal operator A_M whose domain is given by

$$\begin{aligned} D(A_M) = &\{u \in L_w^2(0,1) |\, u, u' \in AC_{\rm loc}(0,1), \\ &\alpha u'' + \beta u' = w^{-1}(e^{\int_{\frac{1}{2}}^{x} \frac{\beta(t)}{\alpha(t)}\,dt} u') \in L_w^2(0,1)\}. \end{aligned}$$

Example. Let $\alpha(x) := x^j(1-x)^j$, $\beta(x) := x^k(1-x)^k$, where $j, k \geq 0$ and $x \in [0,1]$. Then Theorem 3.2 applies when $k > j - 1$.

4. Some Examples on the Unbounded Interval $(0, +\infty)$

In this section we shall show that the technique of Section 3 works in the infinite interval $(0, +\infty)$ too. To this aim, we shall introduce the space with weight $L_w^2(0, +\infty)$, where w is suitably chosen depending on the coefficients of the considered operator.

Example 1. Let us consider the linear Kompaneets equation

$$\frac{\partial u}{\partial t} = \frac{1}{\beta(x)} \frac{\partial}{\partial x}[\alpha(x)(\frac{\partial u}{\partial x} + k(x)u)], \quad t > 0,\; x > 0,$$

with initial condition

$$u(0,x) = u_0(x), \qquad x > 0,$$

and boundary conditions

$$\lim_{x\to 0, x\to +\infty} \alpha(x)[\frac{\partial u}{\partial x}(t,x) + k(x)u(t,x)] = 0, \quad t > 0.$$

For this equation, in the general nonlinear case, we refer to J. A. Goldstein in [15], while for the linear case we quote K. Wang in [25].

We shall prove an analyticity result, extending Wang's theorem [25, p.568] in that weaker regularity will be assumed regarding α, β.

More precisely, we make the following assumptions :

(4.1) $\alpha \in C(0,+\infty)$, $\beta \in L^\infty_{\text{loc}}|(0,+\infty)$, $\alpha(x) > 0$ and $\beta(x) > 0$ for all $x \in (0,+\infty)$, $\alpha(x) = O(x^j)$ as $x \to 0$ for some $j \geq 1$;

(4.2) $k \in C(0,+\infty)$;

(4.3) if $\gamma(x) := e^{\int_1^x k(t)\,dt}$, $x \in (0,+\infty)$, then

$$\int_0^\infty \frac{\beta(t)}{\gamma(t)}\,dt < +\infty;$$

(4.4) $\inf_{(0,+\infty)} \dfrac{\gamma(x)}{\beta(x)} > 0$, $\inf_{(0,+\infty)} \dfrac{\gamma(x)}{\alpha(x)} > 0$.

To begin with our analysis, we observe that the differential operator

$$Wu := \frac{1}{\beta}[\alpha(u' + ku)]'$$

is formally expressed by means of

$$Wu = \frac{1}{\beta}(\frac{\alpha}{\gamma}(\gamma u)')',$$

where $'$ denotes the derivative.

Let us introduce the weighted-L^2 space $(X, \langle\cdot,\cdot\rangle)$ by

$$X := \{u : (0,+\infty) \to \mathbb{C} |\, u\,\text{measurable}, \int_0^\infty \beta(x)\gamma(x)|u(x)|^2\,dx < +\infty\}.$$

Endowed with the inner product

$$\langle u, v\rangle := \int_0^\infty \beta(x)\gamma(x)u(x)\overline{v}(x)\,dx, \quad u, v \in X,$$

the pair $(X, \langle\cdot,\cdot\rangle)$ is a Hilbert space. Let V be the Hilbert space

$$V := \{u \in C^1(0,\infty) \cap X | \int_0^\infty \frac{\alpha(x)}{\gamma(x)} |(\gamma \cdot u)'(x)|^2 \, dx < +\infty, \\ \int_0^\infty |\frac{\gamma(x)}{\beta(x)}| (\frac{\alpha}{\gamma}(\gamma \cdot u)')'(x)|^2 \, dx < +\infty\}$$

with the inner product $\langle\cdot,\cdot\rangle_V$ given by

$$\langle u, v\rangle_V := \langle u, v\rangle + \int_0^\infty \frac{\alpha(x)}{\gamma(x)} (\gamma \cdot u)'(x)(\gamma \cdot \overline{v})'(x) \, dx \\ + \int_0^\infty \frac{\gamma(x)}{\beta(x)} (\frac{\alpha}{\gamma}(\gamma \cdot u)')'(x)(\frac{\alpha}{\gamma}(\gamma \cdot \overline{v})')'(x) \, dx$$

for $u, v \in V$. That is, $u \in C^1(0,\infty)$ belongs to V if and only if the following conditions hold

$$\int_0^\infty \frac{\beta(x)}{\gamma(x)} |(\gamma u)(x)|^2 \, dx < +\infty,$$

$$\int_0^\infty \frac{\gamma(x)}{\alpha(x)} \left|\frac{\alpha(x)}{\gamma(x)}(\gamma u)'(x)\right|^2 dx < +\infty,$$

$$\int_0^\infty \frac{\gamma(x)}{\beta(x)} \left|(\frac{\alpha}{\gamma}(\gamma u)')'(x)\right|^2 dx < +\infty.$$

In view of (4.4), any element u of V has the property that

$$\frac{\alpha}{\gamma}(\gamma u)' \in H^1(0,+\infty),$$

so that $\lim\limits_{x\to+\infty} \frac{\alpha(x)}{\gamma(x)}(\gamma u)'(x)$ exists and equals zero.

Moreover, there exists

$$\lim_{x\to 0^+} \frac{\alpha(x)}{\gamma(x)}(\gamma u)'(x) = \lambda \in \mathbb{C}.$$

On the other hand, if $\lambda \neq 0$, then necessarily

$$\int_0^1 \frac{\gamma(x)}{\alpha(x)} \, dx < +\infty,$$

contradicting (4.1).

Hence, any element u of V satisfies the boundary conditions

$$\lim_{x\to 0^+, x\to+\infty} \frac{\alpha(x)}{\gamma(x)}(\gamma u)'(x) = 0.$$

Our next goal is to show that the operator $(A, D(A))$, where $D(A) := V$, and

$$Au := Wu, \qquad u \in D(A),$$

is symmetric (with respect to the inner product $\langle \cdot, \cdot \rangle$) and nonpositive. Let u, v be given in V. Then

$$\begin{aligned}\langle Au, v\rangle &= \int_0^{+\infty} \gamma(x)\frac{d}{dx}(\frac{\alpha}{\gamma}\frac{d}{dx}(\gamma\cdot u))(x)\overline{v}(x)\,dx \\ &= [\frac{\alpha(x)}{\gamma(x)}\frac{d}{dx}(\gamma\cdot u)(x)(\gamma(x)\overline{v}(x))]_0^{+\infty} \\ &\quad - \int_0^{+\infty}\frac{\alpha(x)}{\gamma(x)}\frac{d}{dx}(\gamma\cdot u)(x)\frac{d}{dx}(\gamma\cdot\overline{v})(x)\,dx.\end{aligned}$$

We know that the above two integrals converge and thus there exist both limits

$$\lim_{x\to 0^+}\frac{\alpha(x)}{\gamma(x)}\frac{d}{dx}(\gamma\cdot u)(x)(\gamma(x)\overline{v}(x)),\ \lim_{x\to+\infty}\frac{\alpha(x)}{\gamma(x)}\frac{d}{dx}(\gamma\cdot u)(x)(\gamma(x)\overline{v}(x)).$$

We show that they vanish.

Let $\lim\limits_{x\to 0^+}\frac{\alpha(x)}{\gamma(x)}\frac{d}{dx}(\gamma\cdot u)(x)(\gamma(x)\overline{v}(x)) = \mu \neq 0$. Since $u \in V$, we have

$$\begin{aligned}|\frac{\alpha(x)}{\gamma(x)}(\gamma\cdot u)'(x)| &= |\int_0^x \frac{d}{dt}(\frac{\alpha}{\gamma}(\gamma\cdot u)')(t)\,dt| \\ &\le [\int_0^x \frac{\beta(t)}{\gamma(t)}\,dt]^{\frac12}(\int_0^x\frac{\gamma(t)}{\beta(t)}|(\frac{\alpha}{\gamma}(\gamma\cdot u)')'(t)|^2\,dt)^{\frac12} \\ &\le (\int_0^x\frac{\beta(t)}{\gamma(t)}\,dt)^{\frac12}\|u\|_V \\ &\le c(\int_0^x\frac{\beta(t)}{\gamma(t)}\,dt)^{\frac12}.\end{aligned}$$

Hence, there is a positive constant c_1 such that

$$|\gamma(x)v(x)|^2 \ge \frac{c_1}{\displaystyle\int_0^x\frac{\beta(t)}{\gamma(t)}\,dt},$$

where $x \in (0,\delta)$ for a suitable $\delta > 0$. This implies that

$$\begin{aligned}\int_0^\delta \frac{\beta(x)}{\gamma(x)}|\gamma(x)v(x)|^2\,dx &\ge c_1\int_0^\delta \frac{\frac{\beta(x)}{\gamma(x)}}{\int_0^x\frac{\beta(t)}{\gamma(t)}\,dt}\,dx \\ &= c_1[\log\int_0^x\frac{\beta(t)}{\gamma(t)}\,dt]_0^\delta \\ &= +\infty,\end{aligned}$$

a contradiction because of (4.3).

Analogously, if $\lim_{x\to+\infty} \frac{\alpha(x)}{\gamma(x)} \frac{d}{dx}(\gamma \cdot u)(x)(\gamma(x)\overline{v}(x)) = \mu \neq 0$, from

$$\begin{aligned}|\frac{\alpha(x)}{\gamma(x)}(\gamma \cdot u)'(x)| &= |-\int_x^\infty \frac{d}{dt}(\frac{\alpha}{\gamma}(\gamma \cdot u)')(t)\,dt| \\ &\leq (\int_x^\infty \frac{\beta(t)}{\gamma(t)}\,dt)^{\frac{1}{2}} \|u\|_V,\end{aligned}$$

we can repeat the same argument, taking into account (4.3) again, because

$$\frac{\beta(x)}{\gamma(x)}|\gamma(x)v(x)|^2 \geq \frac{c_1 \frac{\beta(x)}{\gamma(x)}}{\int_x^\infty \frac{\beta(t)}{\gamma(t)}\,dt},$$

with $x \in [\delta, \infty)$ for a suitable $\delta \geq 1$, implies that

$$\int_\delta^\infty \frac{\beta(x)}{\gamma(x)}|\gamma(x)v(x)|^2\,dx \geq -c_1 \int_\delta^\infty \frac{f'(x)}{f(x)}\,dx = -c_1[\log f(x)]_\delta^\infty,$$

where $f(x) := \int_x^\infty \frac{\beta(t)}{\gamma(t)}\,dt \to 0$, as $x \to +\infty$. This cannot happen if $v \in X$.

Therefore

$$\langle Au, v\rangle = -\int_0^\infty \frac{\alpha(x)}{\gamma(x)} \frac{d}{dx}(\gamma \cdot u)(x) \frac{d}{dx}(\gamma \cdot \overline{v})(x)\,dx.$$

A second integration by parts yields

$$\langle Au, v\rangle = \langle v, Au\rangle \quad \text{for } u, v \in V.$$

Moreover, $\langle Au, u\rangle \leq 0$ for all $u \in V$.

Now, let us consider the sesquilinear form $b(u, v)$ defined on $V_1 \times V_1$ by

$$b(u, v) := \langle u, v\rangle + \int_0^\infty \frac{\alpha(x)}{\gamma(x)} \frac{d}{dx}(\gamma \cdot u)(x) \frac{d}{dx}(\gamma \cdot \overline{v})(x)\,dx,$$

where $u, v \in V_1$ and V_1 is the Hilbert space given by

$$V_1 := \{f : (0, \infty) \to \mathbb{C} |\, f \in X, \int_0^\infty \frac{\alpha(x)}{\gamma(x)} |\frac{d}{dx}(\gamma \cdot f)(x)|^2\,dx < \infty\}$$

equipped with the inner product

$$\langle u, v\rangle_{V_1} := b(u, v).$$

Hence, for all $u \in D(A)$ we have

$$\langle (I - A)u, u\rangle = b(u, u) = \|u\|_{V_1}^2.$$

Using similar arguments as in [22, p.25], we observe that $V_1 \hookrightarrow X \hookrightarrow V_1^*$ and $I-A$ can be extended as a continuous and coercive operator (still denoted by $I-A$) from V_1 to V_1^*. Thus the Lax-Milgram theorem and similar remarks as in the proof of Lemma 3.1 imply that

$$X \subseteq V_1^* \subseteq R(I-A).$$

Therefore, $(A, D(A))$ is self-adjoint.

This completes the proof that $(A, D(A))$ generates an analytic semigroup on X.

Remark 4.1. If we take

$$k(x) := k_o > 0,\ \alpha(x) := x^j,\ \beta(x) := x^s$$

with $j \geq 1$ and $s > 0$, then all assumptions (4.1)-(4.4) are satisfied.

Example 2. Let us consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = x\dfrac{\partial^2 u}{\partial x^2}(t,x) + a\dfrac{\partial u}{\partial x}(t,x), \quad t, x > 0, \\ u(0,x) = u_0(x), \quad x > 0, \end{cases}$$

with $a \in \mathbb{R}$ and the related operator

$$A_1 u := xu'' + au', \qquad x \in (0, +\infty).$$

A basic study of this operator, or the related operator $(xu)'' + au'$, which is of great importance in the theory of probability, was done by W. Feller in [13] and by H. Brezis, W. Rosenkrantz and B. Singer in [3]. These authors [3] showed that A_1 with domain

$$D(A_1) := \{u \in C^2(\mathbb{R}_+) \cap B_0^1 \,|\, A_1 u \in B_0,\ \lim_{x\to 0^+} xu''(x) = 0\},$$

where

$$B_0^k := \{f : [0,+\infty) \to \mathbb{C} \,|\, f^{(j)} \text{ continuous and bounded on } [0,+\infty), \lim_{x\to+\infty} f^{(j)}(x) = 0,\ j = 0,1,..,k\} \quad (k \in \mathbb{N})$$

and

$$B_0 := \{f : [0,+\infty) \to \mathbb{C} \,|\, f \text{ continuous and bounded on } [0,+\infty), \lim_{x\to+\infty} f(x) = 0\},$$

generates a C_0-semigroup of contractions in B_0, provided that $a > 0$.

More recently, in [1] S. Angenent established the analyticity of the semigroup generated by

$$A_2 u := u'' + \frac{b-1}{x} u',\ b \in \mathbb{C},\ \operatorname{Re} b + m > 0,$$

on

$$E_m := \{f \in \mathcal{D}'(0,\infty)|\ f^{(m)} \text{bounded and uniformly continuous on } [0,+\infty),\ f^{(2k+1)}(0) = 0,\ 1 \le 2k+1 \le m\}$$

with domain

$$D(A_2) := E_m \cap E_{m+2}.$$

It is easily seen (e.g. in [3]) that a change of variable transforms an operator like A_2 into an operator A_1. In what follows, we shall treat A_1 in an L^2-space with weight. To accomplish this, we observe that, formally,

$$A_1 u = x^{1-a} \frac{d}{dx}(x^a \frac{du}{dx})$$

but the assumptions required in the treatment of Example 1 fail here, provided that

$$\alpha(x) := x^a, \quad \beta(x) := x^{a-1}, \quad k(x) :\equiv 0.$$

Then, we introduce $Y := D(A_1)$ as the completion of $C_0^\infty(0,\infty)$ with respect to the inner product

$$\langle u, v\rangle_Y := \int_0^\infty x^{a-1} u(x)\overline{v}(x)\,dx + \int_0^\infty x^a u'(x)\overline{v}'(x)\,dx + \int_0^\infty x^{1-a} \frac{d}{dx}(x^a \frac{du(x)}{dx}) \frac{d}{dx}(x^a \frac{d\overline{v}(x)}{dx})\,dx$$

for $u, v \in C_0^\infty(0,\infty)$. Let us denote by W the completion of $C_0^\infty(0,\infty)$ with respect to the inner product

$$\langle u, v\rangle_W := \int_0^\infty x^{a-1} u(x)\overline{v}(x)\,dx + \int_0^\infty x^a u'(x)\overline{v}'(x)\,dx,$$

where $u, v \in C_0^\infty(0,\infty)$ and by X the space

$$X := \{u : (0,+\infty) \to \mathbb{C}|\ u \,\text{measurable}, \int_0^\infty x^{a-1}|u(x)|^2\,dx < \infty\}$$

equipped with the inner product

$$\langle u, v\rangle := \int_0^\infty x^{a-1} u(x)\overline{v}(x)\,dx, \quad u, v \in X,$$

and the related norm $\|.\|$.

Let

$$b(u,v) := \langle u,v\rangle_W, \quad u,v \in W.$$

Then, similar arguments to those used in Example 1 show that the operator $\widetilde{B}$ associated to $b(u,v)$ is an isomorphism from W to its dual space W^* (with respect to the inner product in X). Moreover, if $\widetilde{A}$ is the operator associated to the sesquilinear form $a(u,v)$ on $W\times W$ given by

$$a(u,v) := \int_0^\infty x^a u'(x)\overline{v}'(x)\,dx,$$

and A is the part of $\widetilde{A}$ in X, then $I-A=B$, where B is the part of $\widetilde{B}$ in X.

Hence, $I-A$ is an isomorphism from $D(A)$ onto X.

But $D(A)=D(A_1)$ and $Au=A_1u$ for all $u\in C_0^\infty(0,\infty)$. On the other hand, the equalities

$$\begin{aligned}
\langle A_1u,v\rangle &= \int_0^\infty \frac{d}{dx}(x^a u'(x))\overline{v}(x)\,dx\\
&= -\int_0^\infty x^a u'(x)\overline{v}'(x)\,dx\\
&= \int_0^\infty u(x)\frac{d}{dx}(x^a\overline{v}'(x))\,dx\\
&= \langle u, A_1v\rangle
\end{aligned}$$

for all $u,v\in D(A_1)$, show that A_1 is symmetric on X.

Therefore, since A_1 is nonpositive, we conclude that $(A_1,D(A_1))$ is self-adjoint and nonpositive, hence it generates an analytic semigroup on X. Moreover, if we apply Hardy's inequality, which says that

$$\int_0^\infty t^{-\sigma}|u(t)|^p\,dt \le \left(\frac{p}{|\sigma-1|}\right)^p\int_0^\infty t^{-\sigma+p}|u'(t)|^p\,dt$$

whenever $1<p<\infty$, $\sigma\neq 1$ and $u\in C_0^\infty(0,\infty)$ (cf. inequality (11) in [23, p. 262]), we deduce that

$$\begin{aligned}
&a<1 \quad \text{implies} \quad \exists \lim_{x\to 0^+} u(x)\in\mathbb{C},\\
&a\le 0 \quad \text{implies} \quad \exists \lim_{x\to 0^+} u(x)=0=\lim_{x\to+\infty} x^a u'(x),\\
&a\ge 1 \quad \text{implies} \quad \exists \lim_{x\to+\infty} u(x)=0=\lim_{x\to 0^+} x^a u'(x),
\end{aligned}$$

for all $u\in D(A_1)$.

Further, if $a \neq 1$, by virtue of Hardy's inequality again, for any $u \in D(A_1)$ we have

$$\int_0^\infty x^{-(1+a)} |x^a u'(x)|^2 \, dx < \infty,$$

so that

$$\int_0^\infty x^{a-1} |u'(x)|^2 \, dx < \infty.$$

Since

$$\int_0^\infty x^{1-a} |x^a u''(x) + a x^{a-1} u'(x)|^2 \, dx < \infty,$$

we deduce that

$$\int_0^\infty x^{1+a} |u''(x)|^2 \, dx < +\infty$$

for all $u \in D(A_1)$.

Therefore, if $a > 1$, then necessarily $\lim_{x \to +\infty} u'(x) = 0$ too.

Final remarks. Now we want to discuss a special class of examples to illustrate why we focus on analyticity and not on other aspects of positive semigroups (such as dominant eigenvalues, irreducibility, etc.).

Let $X = C[0, +\infty] := \{u \in C[0, +\infty) | \lim_{x \to +\infty} u(x) < \infty\}$ and denote by $X_0 = C_0(0, +\infty)$, the space of all continuous (real) functions on $(0, +\infty)$ that vanish at both 0 and ∞. Let $c \in \mathbb{R}$ and for $\alpha \in \mathbb{R}$ let

$$A_\alpha u(x) := x^\alpha u''(x) + c x^{\alpha - 1} u'(x).$$

Let

$$\begin{aligned} D(B_\alpha) &:= \{u \in X \cap C^2(0, +\infty) | \, A_\alpha u \in X_0\}, \\ D(C_\alpha) &:= \{u \in X_0 \cap C^2(0, +\infty) | \, A_\alpha u \in X_0\}, \end{aligned}$$

and let B_α (resp. C_α) be the restriction of A_α to $D(B_\alpha)$ (resp. $D(C_\alpha)$).

Then B_α comes equipped with the Wentzell boundary conditions, that is, for $u \in D(B_\alpha)$, $B_\alpha u$ vanishes on the spatial boundary. The same is true for C_α but in this case we may equivalently view the boundary condition as being the homogeneous Dirichlet one. To see this, consider the resolvent problem

$$\lambda u - A_\alpha u = h$$

for $\lambda > 0$. When $h \in X_0$, then $u \in X_0$ if and only if $A_\alpha u \in X_0$, i.e., the Dirichlet and Wentzell boundary conditions are equivalent for C_α. We will show below that C_α has no eigenvalue for $\alpha \neq 0$. (Of course, $B_\alpha 1 = 0$, but $1 \in X \setminus X_0$; that is why we are focusing on C_α and not on B_α.) The unusual spectral behavior of C_α is caused by the singular behavior of the coefficients of A_α at the spatial endpoints.

The positive operators arising from Markov processes are generators of analytic semigroups when they are uniformly elliptic. These semigroups are analytic on lots of spaces, including $C[a,b]$ and $L^p(a,b)$. The question of the analyticity of positive semigroups generated by nonuniformly elliptic operators is in general very difficult to answer and the focus of this paper is to answer this question affirmatively in some cases. But let us now return to B_α and C_α.

For $\lambda > 0$, let $U(\lambda)$ be the scaling transformation defined by $U(\lambda)f(x) := f(\lambda x)$. Then $U(\lambda)$ is an isometric isomorphism on both X and X_0 and $(U(\lambda))^{-1} = U(1/\lambda)$. It is easy to see that

$$U(\lambda)^{-1}A_\alpha U(\lambda) = \lambda^{2-\alpha}A_\alpha$$

for all $\lambda > 0, \alpha \in \mathbb{R}$. Consequently, B_α (and C_α too) is isometrically equivalent to any positive multiple of itself if $\alpha \neq 2$. Thus if μ is a nonzero eigenvalue of either B_α or C_α (and if $\alpha \neq 2$), then so is $\lambda\mu$ for all positive λ. Now consider $\mu = 0:$ the general solution of

$$x^\alpha u''(x) + cx^{\alpha-1}u'(x) = 0$$

is given by $u(x) = c_1 + c_2x^{1-c}$ unless $c = 1$; when it is, $u(x) = c_1 + c_2 \log x$. Then 0 is not an eigenvalue of C_α. Finally consider the case of $\alpha = 2$. To solve

$$x^2u''(x) + cxu'(x) = \mu u(x),$$

we seek solutions of the form $u(x) = x^r$ and we get

$$r = \frac{1}{2}(1 - c \pm \sqrt{(c-1)^2 + 4\mu}),$$

interpreting x^{a+ib} as $x^a(\cos(b\log x)+i\sin(b\log x))$. In all cases, it easily follows that C_2 has no eigenvalues and B_2 has no nonzero eigenvalues.

The function ψ defined by $\psi(t) = \tan(\frac{\pi}{2}t)$ is a diffeomorphism from $[0,1]$ onto $[0,\infty]$.

Let $Vf := f \circ \psi$ for $f \in X$ and let

$$D_\alpha := VB_\alpha V^{-1},$$
$$E_\alpha := VC_\alpha V^{-1}.$$

Then D_α (resp. E_α) is the restriction of $VA_\alpha V^{-1}v := \beta_2 v'' + \beta_1 v'$ to $C[0,1]$ (resp. $C_o(0,1)$) with Wentzell (resp. homogeneous Dirichlet) boundary conditions. Here

$$\beta_2(t) := \frac{4}{\pi^2}[\tan(\frac{\pi}{2}t)]^\alpha\{1 + \tan^2(\frac{\pi}{2}t)\}^{-2},$$

$$\beta_1(t) := \frac{2}{\pi}[\tan 2(\frac{\pi}{2}t)]^{\alpha-1}[(c-2)\tan^2(\frac{\pi}{2}t) + c]\{1 + \tan^2(\frac{\pi}{2}t)\}^{-2}.$$

Near $t = 0$,

$$\beta_2(t) \approx \text{const.}\, t^{\alpha}, \quad \beta_1(t) \approx \text{const.}\, t^{\alpha-1};$$

while near $t = 1$,

$$\beta_2(t) \approx \text{const.}\,[\tan(\frac{\pi}{2}t)]^{\alpha-4},$$

$$\beta_1(t) \approx \text{const.}\,[\tan(\frac{\pi}{2}t)]^{\alpha-3} \quad \text{if} \quad c \neq 2,$$

$$\beta_1(t) \approx \text{const.}\,[\tan(\frac{\pi}{2}t)]^{\alpha-5} \quad \text{if} \quad c = 2.$$

It follows that for $c \leq 2$,

$$\frac{\beta_1(t)}{\beta_2(t)} \approx \text{const.}\, t \quad \text{near} \quad t = 0$$

and

$$\frac{\beta_1(t)}{\beta_2(t)} \approx k \tan(\frac{\pi}{2}t) \quad \text{near} \quad t = 1,$$

where $k > 0$ (resp. $k = 0,\ k < 0$) if $c > 2$ (resp. $c = 2,\ c < 2$).

It follows that

$$W(t) := e^{-\int_{\frac{1}{2}}^{t} \frac{\beta_1(s)}{\beta_2(s)}\, ds} \in L^1(0,1),$$

and so D_α is densely defined, m-dissipative and generates a positive C_0-semigroup on $C[0,1]$ by the theorem of Clément and Timmermans [6]. Since $e^{tD_\alpha}(C_0(0,1)) \subset C_0(0,1)$, it follows that the same is true for E_α on $C_0(0,1)$. Consequently, B_α on X and C_α on X_0 generate positive C_0- semigroups.

Concerning analyticity of the semigroup generated by C_α, we point out that very recent results by Campiti and Metafune [4] assure that $A_\alpha$with domain

$$D_1(C_\alpha) := \{u \in D(C_\alpha) |\, x^{\frac{\alpha}{2}} u',\, x^{\alpha} u'' \ \text{ bounded at } \ +\infty\}$$

generates an analytic semigroup on $C[0,+\infty]$, provided that $0 < \alpha < 2$ and $c < 1$.

On the other hand, for $\alpha = 2$, B_α generates an analytic semigroup on $C[0,+\infty]$ by virtue of [12, Theorem 1.2].

References

1. S. Angenent, Local existence and regularity for a class of degenerate parabolic equations, *Math. Ann.* **280** (1988), 465-482.
2. V. Barbu, A. Favini, and S. Romanelli, Degenerate evolution equations and regularity of their associated semigroups, *Funkcial. Ekvac.* **39** (1996), 421-448.

3. H. Brezis, W. Rosenkrantz, and B. Singer, On a degenerate elliptic-parabolic equation occurring in the theory of probability, *Comm. Pure Appl. Math.* **24** (1971), 395-416.

4. M. Campiti and G. Metafune, Ventcel's boundary conditions and analytic semigroups, *Arch. Math.* **70** (1998), 377-390.

5. M. Campiti, G. Metafune, and D. Pallara, Degenerate self-adjoint evolution equations on the unit interval, *Semigroup Forum* **57** (1998), 1-36.

6. Ph. Clément and C. A. Timmermans, On C_0-semigroups generated by differential operators satisfying Ventcel's boundary conditions, *Indag. Math.* **89** (1986), 379-387.

7. W. Everitt, M. K. Kuong, and A. Zettl, Differential operators and quadratic inequalities with a degenerate weight, *J. Math. Anal. Appl.* **98** (1984), 378-399.

8. A. Favini and A. Yagi, Multivalued linear operators and degenerate evolution equations, *Ann. Mat. Pura Appl.* (4) **163** (1993), 353-384.

9. A. Favini and A. Yagi, *Degenerate Parabolic Equations* (in preparation).

10. A. Favini, J. A. Goldstein, and S. Romanelli, An analytic semigroup associated to a degenerate evolution equation, in: *Stochastic Processes and Functional Analysis* (J. A. Goldstein, N. E. Gretsky, and J. Uhl, eds.), M. Dekker, New York, 1996, pp.85-100.

11. A. Favini and S. Romanelli, Analytic semigroups on $C[0,1]$ generated by some classes of second order differential operators, *Semigroup Forum* **56** (1998), 362-372.

12. A. Favini and S. Romanelli, Degenerate second order operators as generators of analytic semigroups on $C[0,+\infty]$ or on $L^p_{\alpha^{-\frac{1}{2}}}(0,+\infty)$, in: *Approximation and Optimization*, Proceedings of International Conference on Approximation and Optimization (Romania) (ICAOR) Cluj-Napoca, July 29 - August 1, 1996 (D. Stancu, G. Coman, W. W. Breckner and P. Blaga eds.), Vol. II, Transilvania Press, 1997, pp. 93-100.

13. W. Feller, Two singular diffusion problems, *Ann. Math.* **54** (1951), 173-182.

14. J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Univ. Press, Oxford, New York, 1985.

15. J. A. Goldstein, The Kompaneets equation, in: *Differential Equations in Banach Spaces* (G. Dore, A. Favini, E. Obrecht and A. Venni eds.), M. Dekker, New York, 1993, pp.115-123.

16. T. Kato, *Perturbation Theory for Linear Operators*, Springer Verlag, Berlin - Heidelberg - New York, 1966.

17. A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, PNLDE 16 Birkhäuser Verlag, Basel - Boston - Berlin, 1995.

18. G. Metafune and D. Pallara, Trace formulas for some singular differential operators, Tübinger Berichte zur Funktionalanalysis, Heft 6 (Jahrgang 1996/97), 176-197.

19. R. Nagel (ed.), *One-Parameter Semigroups of Positive Operators*, Lecture Notes in Math. 1184, Springer Verlag, Berlin - Heidelberg - New York - Tokyo, 1986.

20. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer Verlag, Berlin - Heidelberg - Tokyo, 1983.

21. H. B. Stewart, Generation of analytic semigroups by strongly elliptic operators, *Trans. Amer. Math. Soc.* **199** (1974), 141-162.

22. H. Tanabe, *Equations of Evolution*, Monographs and Studies in Mathematics 6, Pitman, London-San Francisco-Melbourne, 1979.

23. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Math. Library 18, North-Holland Publ. Co., Amsterdam-New York-Oxford, 1978.

24. V. Vespri, Analytic semigroups, degenerate elliptic operators and applications to nonlinear Cauchy problems, *Ann. Mat. Pura Appl.* (4) **155** (1989), 353-388.

25. K. Wang, The linear Kompaneets equation, *J. Math. Anal. Appl.* **198** (1996), 552-570.

26. K. Yosida, *Functional Analysis*, Springer Verlag, Berlin-Heidelberg-New York, 1974.

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