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NONLINEAR MEAN ERGODIC THEOREMS II

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Abstract. The purpose of this paper is to improve the previous results due to the author.

Introduction

Throughout this paper, let C be a nonempty subset of a real Hilbert space H and $T : C \to C$ be a (nonlinear) mapping. The purpose of this paper is to improve the results in [4]. We emphasize that the closedness and convexity of C and the asymptotic nonexpansivity of T are not assumed in this paper.

It is known that if $\{x_n\}$ is a bounded sequence in H, then there exists a unique element y in H such that $\overline{\lim}_{n\to\infty}\|x_n - y\| < \overline{\lim}_{n\to\infty}\|x_n - z\|$ for every $z \in H\backslash\{y\}$. The element y is called the *asymptotic center* of $\{x_n\}$ (see [2]).

Definition 0.1. A sequence $\{x_n\}$ in H is said to be *strongly* (resp. *weakly*) *almost-convergent* to an element x in H if $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} x_{i+k} = x$ (resp. w - $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} x_{i+k} = x$) uniformly in $k = 0, 1, 2, \ldots$, where lim(resp. w - lim) denotes the strong (resp. weak) limit.

The set of *fixed points* of T will be denoted by $F(T)$.

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We consider the following conditions in this section.

Condition (α_2). For every $u, v \in C$ and integer $k \geq 0$, there exists a $\delta_k(u,v) \geq 0$ with $\lim_{k\to\infty} \delta_k(u,v) = 0$ such that

$$(\alpha_2) \qquad \begin{aligned} \|T^k u - T^k v\|^p \leq & a_k\|u-v\|^p + c[a_k\|u\|^p - \|T^k u\|^p \\ & + a_k\|v\|^p - \|T^k v\|^p] + \delta_k(u,v), \end{aligned}$$

where a_k, c and p are nonnegative constants independent of u and v such that $\lim_{k\to\infty} a_k = 1$ and $p \geq 1$.

Condition (β_1). For every bounded set $B \subset C, v \in C$ and integer $k \geq 0$, there exists a $\delta_k(B,v) \geq 0$ with $\lim_{k\to\infty} \delta_k(B,v) = 0$ such that

$$(\beta_1) \qquad \begin{aligned} \|T^k u + T^k v\|^p \leq & a_k\|u+v\|^p + c[a_k\|u\|^p - \|T^k u\|^p \\ & + a_k\|v\|^p - \|T^k v\|^p] + \delta_k(B,v) \quad \text{for } u \in B, \end{aligned}$$

where a_k, c and p are nonnegative constants independent of B and v such that $\lim_{k\to\infty} a_k = 1$ and $p \geq 1$.

Condition (β_3). For every bounded set $B \subset C, v \in C$ and integer $k \geq 0$, there exists a $\delta_k(B,v) \geq 0$ with $\lim_{k\to\infty} \delta_k(B,v) = 0$ such that

$$(\beta_3) \qquad \begin{aligned} \|u - v\|^p \leq & a_k\|T^k u - T^k v\|^p + c[a_k\|T^k u\|^p - \|u\|^p \\ & + a_k\|T^k v\|^p - \|v\|^p] + \delta_k(B,v) \quad \text{for } u \in B, \end{aligned}$$

where a_k, c and p are the same constants as in (β_1).

It is easy to see that T satisfies

$$(1.1) \qquad \varliminf_{k\to\infty} \|T^k u - T^k v\| \leq \|u - v\| \quad \text{for every } u, v \in C$$

if and only if T satisfies condition (α_2) with $c = 0$ and $p = 1$. (1.1) is a condition of asymptotically nonexpansive type. This condition (1.1) has been considered in [5]. Clearly, condition (a_1) in [4] implies condition (β_1) above, and conditions (a_2) and (a_3) in [4] imply conditions (α_2) and (β_3) above, respectively. Therefore, Theorem 1.1 improves [4, Theorem 1.1], and Theorems 1.2 and 1.3 improve [4, Theorems 1.2 and 1.3], respectively.

Theorem 1.1. *Suppose condition* (β_1) *holds. Then for every* $x \in C, \{T^n x\}$ *is strongly almost-convergent to its asymptotic center.*

Proof. Let $x \in C$ and n be a nonnegative integer. By condition (β_1) with $B = \{T^n x\}$ (singleton) and $v = u = T^n x$, we get $\|T^{k+n}x\|^p \leq a_k\|T^n x\|^p + (1/(2^p+2c))\delta_k(\{T^n x\}, T^n x)$ for $k \geq 0$. Letting $k \to \infty$, we have $\overline{\lim}_{k\to\infty}\|T^k x\| \leq \|T^n x\|$, which implies

$$\{\|T^n x\|\} \text{ is convergent.} \tag{1.2}$$

Let $n > m \geq 0$. By condition (β_1) with $B = \{T^\ell x; \ell \geq 0\}, u = T^{m+i}x, v = T^m x$ and $k = n - m$, we have

$$\begin{aligned}\|T^{n+i}x + T^n x\|^p \leq & a_{n-m}\|T^{m+i}x + T^m x\|^p + c[a_{n-m}\|T^{m+i}x\|^p - \|T^{n+i}x\|^p \\ & + a_{n-m}\|T^m x\|^p - \|T^n x\|^p] + \delta_{n-m}(B, T^m x) \\ \leq & \|T^{m+i}x + T^m x\|^p + [(2M)^p + 2cM^p]|a_{n-m} - 1| \\ & + c(\|T^{m+i}x\|^p - \|T^{n+i}x\|^p + \|T^m x\|^p - \|T^n x\|^p) \\ & + \delta_{n-m}(B, T^m x) \text{ for } i \geq 0,\end{aligned}$$

where $M = \sup_{\ell \geq 0} \|T^\ell x\|$. Combining this with (1.2) we obtain

$$\lim_{m\to\infty} \lim_{n\to\infty} \sup_{i\geq 0} \left[\|T^{n+i}x + T^n x\|^p - \|T^{m+i}x + T^m x\|^p\right] \leq 0,$$

which implies

$$\lim_{m\to\infty} \lim_{n\to\infty} \sup_{i\geq 0} \left[\|T^{n+i}x + T^n x\|^2 - \|T^{m+i}x + T^m x\|^2\right] \leq 0.$$

Therefore by [4, Proposition 1.5(I)], $\{T^n x\}$ is strongly almost-convergent to its asymptotic center. ■

Theorem 1.2. *Suppose condition (α_2) holds. If either $F(T) \neq 0$ or $c > 0$ in (α_2), and if $x \in C$ satisfies*

$$\lim_{m\to\infty} \lim_{n\to\infty} \sup_{i\geq 0} \left[\|T^{m+i}x - T^n x\|^2 - \|T^{n+i}x - T^n x\|^2\right] \leq 0, \tag{1.3}$$

then $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.

Proof. We first consider the case when $c > 0$ in (α_2). Let $n \geq 0$ be arbitrarily fixed. By condition (α_2) with $u = v = T^n x$, we have $\|T^{k+n}x\|^p \leq a_k\|T^n x\|^p + \delta_k(T^n x, T^n x)/2c$ for $k \geq 0$. Letting $k \to \infty$, $\overline{\lim}_{k\to\infty}\|T^k x\| \leq \|T^n x\|$, which implies that $\{\|T^n x\|\}$ is convergent. By virtue of [4, Proposition 1.5(II)], $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.

Next, let $F(T) \neq \emptyset$ and $c = 0$ in (α_2), i.e., for every $u, v \in C$ and integer $k \geq 0$ there exists a $\delta_k(u, v) \geq 0$ such that

$$\|T^k u - T^k v\|^p \leq a_k\|u - v\|^p + \delta_k(u, v), \tag{1.4}$$

where a_k and p are nonnegative constants independent of u and v such that $\lim_{k\to\infty} a_k = 1$ and $p \geq 1$. Take an $f \in F(T)$. Let $n \geq 0$ be arbitrarily fixed. By (1.4) with $u = T^n x$ and $v = f$, we have $\|T^{k+n}x - f\|^p \leq a_k\|T^n x - f\|^p + \delta_k(T^n x, f)$ for $k \geq 0$. Letting $k \to \infty$, we get $\overline{\lim}_{k\to\infty}\|T^k x - f\| \leq \|T^n x - f\|$ and hence $\{\|T^n x - f\|\}$ is convergent. Using [4, Proposition 1.5(II)] again, we obtain the conclusion. ∎

Remarks. 1) We see that if T satisfies condition (α_2), then $\{\|T^{n+i}x - T^n y\|\}$ is convergent for every $x, y \in C$ and $i \geq 0$. 2) Suppose T satisfies condition (α_2) and the following

Condition (α_1). For every $u, v \in C$ and integer $k \geq 0$, there exists a $\delta_k(u,v) \geq 0$ with $\lim_{k\to\infty}\delta_k(u,v) = 0$ such that

$$(\alpha_1)\qquad \begin{aligned}\|T^k u + T^k v\|^q \leq{}& a_k\|u+v\|^q + d[a_k\|u\|^q - \|T^k u\|^q \\ &+ a_k\|v\|^q - \|T^k v\|^q] + \delta_k(u,v),\end{aligned}$$

where a_k, d and q are nonnegative constants independent of u and v such that $\lim_{k\to\infty} a_k = 1$ and $q \geq 1$.

Then we see that for every $x, y \in C$

$$(*)\qquad \lim_{n\to\infty}\|T^{n+i}x - T^n y\| \text{ exists uniformly in } i \geq 0.$$

(This is an extension of [1, Theorem 2.3].) Clearly, $(*)$ with $y = x$ satisfies (1.3). So, in this case, for every $x \in C$, $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.

Theorem 1.3. *Suppose condition* (β_3) *holds.*

(I) *If* $x \in C$ *and* $\{\|T^n x\|\}$ *is convergent, then* $\{T^n x\}$ *is strongly almost-convergent to its asymptotic center.*

(II) *If either* $F(T) \neq \emptyset$ *or* $c > 0$ *in* (β_3)*, then for every* $x \in C$*, either* $\lim_{n\to\infty}\|T^n x\| = \infty$ *or* $\{T^n x\}$ *is strongly almost-convergent to its asymptotic center.*

Proof. (I) Set $B = \{T^n x; n \geq 0\}$. Let $n > m \geq 0$. By condition (β_3) with $u = T^{m+i}x, v = T^m x$ and $k = n - m$, we have

$$(1.5)\qquad \begin{aligned}\|T^{m+i}x - T^m x\|^p \leq{}& \|T^{n+i}x - T^n x\|^p + [(2M)^p + 2cM^p]|a_{n-m} - 1| \\ &+ c[\|T^{n+i}x\|^p - \|T^{m+i}x\|^p + \|T^n x\|^p - \|T^m x\|^p] \\ &+ \delta_{n-m}(B, T^m x)\end{aligned}$$

for $i \geq 0$, where $M = \sup_{\ell \geq 0} \|T^\ell x\|$. Since $\{\|T^n x\|\}$ is convergent, we see from (1.5) that

$$\lim_{m\to\infty} \lim_{n\to\infty} \sup_{i\geq 0} \left[\|T^{m+i}x - T^m x\|^2 - \|T^{n+i}x - T^n x\|^2 \right] \leq 0.$$

By virtue of [4, Proposition 1.5(II)], $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.

(II) Let $x \in C$ and suppose $\underline{\lim}_{n\to\infty}\|T^n x\| < \infty$. We first consider the case when $c > 0$ in (β_3). Let $n \geq 0$ be arbitrarily fixed. By condition (β_3) with $B = \{T^n x\}$ (singleton) and $u = v = T^n x$, we have $\|T^n x\|^p \leq a_k\|T^{k+n}x\|^p + \delta_k(\{T^n x\}, T^n x)/2c$ for $k \geq 0$. Letting $k \to \infty$, we obtain $\|T^n x\| \leq \underline{\lim}_{k\to\infty}\|T^k x\|$, which implies that $\{\|T^n x\|\}$ is convergent. Therefore by part (I), $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.

Next, let $F(T) \neq \emptyset$ and $c = 0$ in (β_3), i.e., for every bounded set $B \subset C$, $v \in C$ and integer $k \geq 0$, there exists a $\delta_k(B, v) \geq 0$ with $\lim_{k\to\infty} \delta_k(B, v) = 0$ such that

$$\|u - v\|^p \leq a_k\|T^k u - T^k v\|^p + \delta_k(B, v) \quad \text{for } u \in B, \tag{1.6}$$

where a_k and p are nonnegative constants independent of B and v such that $p \geq 1$ and $\lim_{k\to\infty} a_k = 1$. Take an $f \in F(T)$ and let $n \geq 0$ be arbitrarily fixed. Using (1.6) with $B = \{T^n x\}, u = T^n x$ and $v = f$, we have $\|T^n x - f\|^p \leq a_k\|T^{k+n}x - f\|^p + \delta_k(\{T^n x\}, f)$ for $k \geq 0$. This implies that $\{\|T^n x - f\|\}$ is convergent.

Let $n > m \geq 0$. By (1.6) with $B = \{T^\ell x; \ell \geq 0\}, u = T^{m+i}x, v = T^m x$ and $k = n - m$, we have

$$\|T^{m+i}x - T^m x\|^p \leq \|T^{n+i}x - T^n x\|^p + |a_{n-m} - 1|(2M)^p + \delta_{n-m}(B, T^m x)$$

for $i \geq 0$, where $M = \sup_{\ell\geq 0} \|T^\ell x\|$, which implies

$$\lim_{m\to\infty} \lim_{n\to\infty} \sup_{i\geq 0} \left[\|T^{m+i}x - T^m x\|^2 - \|T^{n+i}x - T^n x\|^2 \right] \leq 0.$$

It follows from [4, Proposition 1.5(II)] that $\{T^n x\}$ is strongly almost-convergent to its asymptotic center. ∎

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We consider the following conditions in this section.

Condition (β_2). For every bounded set $B \subset C, v \in C$ and integer $k \geq 0$, there exists a $\delta_k(B,v) \geq 0$ with $\lim_{k\to\infty} \delta_k(B,v) = 0$ such that

$$(\beta_2)\qquad \begin{aligned} \|T^k u - T^k v\|^p \leq & a_k\|u-v\|^p + c[a_k\|u\|^p - \|T^k u\|^p \\ & + a_k\|v\|^p - \|T^k v\|^p] + \delta_k(B,v) \quad \text{for } u \in B, \end{aligned}$$

where a_k, c and p are the same constants as in condition (β_1).

Condition (β_4). For every bounded set $B \subset C, v \in C$ and integer $k \geq 0$, there exists a $\delta_k(B,v) \geq 0$ with $\lim_{k\to\infty} \delta_k(B,v) = 0$ such that

$$(\beta_4)\qquad \begin{aligned} \|u+v\|^p \leq & a_k\|T^k u + T^k v\|^p + c[a_k\|T^k u\|^p - \|u\|^p \\ & + a_k\|T^k v\|^p - \|v\|^p] + \delta_k(B,v) \quad \text{for } u \in B, \end{aligned}$$

where a_k, c and p are the same constants as in condition (β_1).

It is easy to see that T satisfies

$$\lim_{k\to\infty} \sup_{u\in B}(\|T^k u - T^k v\| - \|u-v\|) \leq 0 \tag{2.1}$$

for every bounded set $B \subset C$ and $v \in C$ if and only if T satisfies condition (β_2) with $c = 0$ and $p = 1$. (2.1) is a condition of asymptotically nonexpansive type and this kind of condition has been introduced in [3]. Clearly, conditions (a_2) and (a_4) in [4] imply conditions (β_2) and (β_4) above, respectively. Therefore, the following Theorems 2.1 and 2.2 improve [4, Theorems 2.1 and 2.2], respectively.

Theorem 2.1. *Suppose condition* (β_2) *holds. If either* $F(T) \neq \emptyset$ *or* $c > 0$ *in* (β_2), *then for every* $x \in C, \{T^n x\}$ *is weakly almost-convergent to its asymptotic center.*

Proof. Let $x \in C$. We first consider the case when $c > 0$ in (β_2). Let $n \geq 0$ be arbitrarily fixed . By condition (β_2) with $B = \{T^n x\}$ and $u = v = T^n x$, we have $\|T^{k+n}x\|^p \leq a_k\|T^n x\|^p + \delta_k(\{T^n x\}, T^n x)/2c$ for $k \geq 0$, which implies that $\{\|T^n x\|\}$ is convergent. Using condition (β_2) with $B = \{T^\ell x; \ell \geq 0\}, u = T^{m+i}x, v = T^m x$ and $k = n - m$, we see that if $n > m \geq 0$, then

$$\begin{aligned} \|T^{n+i}x - T^n x\|^p \leq & \|T^{m+i}x - T^m x\|^p + [(2M)^p + 2cM^p]|a_{n-m} - 1| \\ & + c(\|T^{m+i}x\|^p - \|T^{n+i}x\|^p + \|T^m x\|^p - \|T^n x\|^p) \\ & + \delta_{n-m}(B, T^m x) \end{aligned}$$

for $i \geq 0$, where $M = \sup_{\ell \geq 0} \|T^\ell x\|$. Since $\{\|T^n x\|\}$ is convergent, the above inequality shows that

$$\lim_{m\to\infty} \lim_{n\to\infty} \sup_{i\geq 0} \left[\|T^{n+i}x - T^n x\|^p - \|T^{m+i}x - T^m x\|^p \right] \leq 0$$

and then

$$\lim_{m\to\infty} \lim_{n\to\infty} \sup_{i\geq 0} \left[\|T^{n+i}x - T^n x\|^2 - \|T^{m+i}x - T^m x\|^2 \right] \leq 0$$

so that

$$\lim_{m\to\infty} \lim_{n\to\infty} \lim_{i\to\infty} \left[\|T^{n+i}x - T^n x\|^2 - \|T^{m+i}x - T^m x\|^2 \right] \leq 0. \tag{2.2}$$

By [4, Proposition 2.3] with $x_n = T^n x, \{T^n x\}$ is weakly almost-convergent to its asymptotic center.

Next, let $F(T) \neq \emptyset$ and $c = 0$ in (β_2), i.e., for every bounded set $B \subset C$, $v \in C$ and integer $k \geq 0$, there exists a $\delta_k(B, v) \geq 0$ wih $\lim_{k\to\infty} \delta_k(B, v) = 0$ such that

$$\|T^k u - T^k v\|^p \leq a_k \|u - v\|^p + \delta_k(B, v) \quad \text{for } u \in B, \tag{2.3}$$

where a_k and p are nonnegative constants independent of B and v such that $\lim_{k\to\infty} a_k = 1$ and $p \geq 1$. Since (2.3) implies (1.4), we see from the proof of Theorem 1.2 that $\{\|T^n x - f\|\}$ is convergent, where $f \in F(T)$. Using (2.3) with $B = \{T^\ell x; \ell \geq 0\}, u = T^{m+i}x, v = T^m x$ and $k = n - m$, we have that if $n > m \geq 0$, then

$$\|T^{n+i}x - T^n x\|^p \leq \|T^{m+i}x - T^m x\|^p + (2M)^p |a_{n-m} - 1| + \delta_{n-m}(B, T^m x)$$

for $i \geq 0$, where $M = \sup_{\ell \geq 0} \|T^\ell x\|$. This implies (2.2). Therefore, using [4, Proposition 2.3] with $x_n = T^n x - f$, we see that $\{T^n x - f\}$ is weakly almost-convergent to its asymptotic center z, so that $\{T^n x\}$ is weakly almost-convergent to its asymptotic center $z + f$. ■

Theorem 2.2. *Suppose condition* (β_4) *holds. Then for every* $x \in C$*, either* $\lim_{n\to\infty} \|T^n x\| = \infty$ *or* $\{T^n x\}$ *is weakly almost-convergent to its asymptotic center.*

Proof. Let $x \in C$, and suppose $\underline{\lim}_{n\to\infty} \|T^n x\| < \infty$. By condition (β_4) with $B = \{T^n x\}$ and $v = u = T^n x$, we have

$$\|T^n x\|^p \leq a_k \|T^{k+n}x\|^p + \delta_k(\{T^n x\}, T^n x)/(2^p + 2c) \quad \text{for } k, n \geq 0,$$

which implies that $\{\|T^n x\|\}$ is convergent.

Let $n > m \geq 0$. By condition (β_4) with $B = \{T^\ell x; \ell \geq 0\}, u = T^{m+i}x, v = T^m x$ and $k = n - m$, we have

$$\begin{aligned}\|T^{m+i}x + T^m x\|^p \leq & \|T^{n+i}x + T^n x\|^p + [(2M)^p + 2cM^p]|a_{n-m} - 1| \\ & + c(\|T^{n+i}x\|^p - \|T^{m+i}x\|^p + \|T^n x\|^p - \|T^m x\|^p) \\ & + \delta_{n-m}(B, T^m x)\end{aligned}$$

for $i \geq 0$, where $M = \sup_{\ell \geq 0} \|T^\ell x\|$. Combining this with the convergence of $\{\|T^n x\|\}$, we obtain

$$\lim_{m\to\infty} \lim_{n\to\infty} \sup_{i\geq 0} \left[\|T^{m+i}x + T^m x\|^p - \|T^{n+i}x + T^n x\|^p\right] \leq 0$$

and a fortiori

$$\lim_{m\to\infty} \lim_{n\to\infty} \lim_{i\to\infty} \left[\|T^{m+i}x + T^m x\|^2 - \|T^{n+i}x + T^n x\|^2\right] \leq 0.$$

So, it follows from [4, Proposition 2.3] that $\{T^n x\}$ is weakly almost-convergent to its asymptotic center. ∎

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