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ON THE SEMI-LINEAR WAVE EQUATIONS (I)

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Abstract. In this work we consider the triviality of the solutions of the initial-boundary value problems for some semi-linear wave equations.

Introduction

In this paper we work with the triviality of the solutions of the initial-boundary value problems for the semi-linear wave equation

$$\text{(SL)} \qquad \begin{cases} \square\, u + f(u) = 0 \quad \text{in } [0,T) \times \Omega,\ u = 0 \text{ on } [0,T] \times \partial\Omega\,, \\ u(0,\cdot) = u_0 \in H_0^1(\Omega),\ \dot{u}(0,\cdot) = u_1 \in L^2(\Omega), \end{cases}$$

where $0 < T \leq \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain on which the divergent theorem is applicable and $(L^2(\Omega), \|\cdot\|_2)$, $(H_0^1(\Omega), \|\cdot\|_{1,2})$ are the usual spaces of Lebesgue and Sobolev. Further, we employ the following abridgements:

$$\cdot := \partial/\partial t,\ \nabla := (\partial/\partial x_1, \cdots, \partial/\partial x_n),\ Du := (\dot{u}, \nabla u),\ \square := \partial^2/\partial t^2 - \triangle,$$

$$|Du|^2 := \dot{u}^2 + |\nabla u|^2, \quad A_u(t) := \|Du\|_2^2(t) = \int_\Omega |Du\,(t,x)|^2 dx,$$

$$a_u(t) := \int_\Omega u(t,x)^2 dx,\ F(s) := \int_0^s f(r)dr,$$

$$E_u(t) := \int_\Omega \left(|Du(t,x)|^2 + 2F(u(t,x))\right) dx.$$

For a Banach space X and $0 < T \leq \infty$, we set

$$C^k(0,T,X) := \text{space of } C^k\text{-functions } [0,T) \to X,$$

$$H1 := C^0(0,T,H_0^1(\Omega)) \cap C^1(0,T,L^2(\Omega)),$$

$$H2 := C^1(0,T,H_0^1(\Omega)) \cap C^2(0,T,L^2(\Omega)).$$

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Jörgens [2] proved the existence for global solutions of IBVP (SL) for $\Omega = \mathbb{R}^3$ and a non-linearity of the form $f(u) = ug(u^2)$. Jörgens′ result permits the treatment for example for the wave equation $\Box u + u^3 = 0$. Browder [1] generalized Jörgens′ result to $\Omega = \mathbb{R}^n$. Von Wahl and Heinz [10] extended Browder′s result. Li [7] proved the non-existence of global solutions of IBVP (SL) under the assumptions $E(0) < 0$, $a'(0) > 0$ and

$$\eta f(\eta) - 2(1+2\alpha)\, F(\eta) \leq 2\alpha C_\Omega^2 \eta^2 \quad \forall \eta \in \mathbb{R}.$$

The result of Li [7] allows treatment of IBVP (SL) for $f(u) = u^p$, $p \in [1, n/n-2]$. In this case Li [6] showed the uniqueness of the solutions. For further contributions to the theme "blow-up", see John [3] and Racke [8]. In [7] we have an interesting result which says that the solutions of IBVP (SL) must be the trivial function if $u_0 \equiv 0 \equiv u_1 \equiv f(0)$ and

$$\eta f(\eta) + 2F(\eta) \geq -k|\eta|^p \quad \forall \eta \in \mathbb{R}.$$

The proof is based on the following Lemma 1.

1. Fundamental Lemmas

Lemma 1. *Suppose that $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^2-function with $b(0) = 0 = b'(0)$ and there exist two constants k_1, k_2 such that*

$$b^{''}(t) + k_1 b^{'}(t) + k_2 b(t) \leq 0. \tag{1}$$

Then $b \equiv 0$.

Proof of Lemma 1. We prove this lemma under two instances: $k_1 = 0$ and $k_1 \neq 0$.

1) For $k_1 = 0$ and $k_2 \geq 0$, by the positivity of b and the inequality (1) we find

$$b^{''}(t) \leq 0 \qquad \forall t \geq 0$$

and

$$b^{'}(t) \leq b^{'}(0) = 0 \qquad \forall t \geq 0. \tag{2}$$

Integrating the inequality (2) from 0 to t we obtain the inequality $b(t) \leq 0$. Herewith it follows automatically the assertion of Lemma 1 for the case $k_1 = 0$ and $k_2 \geq 0$. We show next the case $k_1 = 0$ and $k_2 \leq 0$.

We set $k_2 := -r^2$. Then, by the inequality (1) we get

$$b^{''}(t) - r^2 b(t) \leq 0 \qquad \forall t \geq 0. \tag{3}$$

Multiplying inequality (3) with exp (rt), we then arrive at the following estimate

$$\frac{d}{dt}\Big(e^{rt}\cdot b^{'}(t)-re^{rt}\cdot b(t)\Big)=e^{rt}\cdot\Big(b^{''}(t)-r^2b(t)\Big)\leq 0 \qquad \forall t\geq 0,$$

and so

$$e^{rt}\cdot b^{'}(t)-re^{rt}\cdot b(t)\leq b^{'}(0)-rb(0)=0 \qquad \forall t\geq 0.$$

This means however $b^{'}(t)-rb(t)\leq 0$. From the Gronwall inequality it follows at once the assertion of Lemma 1 for the case $k_1=0$.

2) Suppose now $k_1\neq 0$. Through the transformation

$$B(t):=e^{kt}\cdot b(t), \quad k:=k_1/2,$$

the inequality (1) is transformed into

$$B^{''}(t)-\left(k^2-k_2\right)B(t)=e^{kt}\left(b^{''}(t)+2kb^{'}(t)+k_2b(t)\right)\leq 0 \quad \forall t\geq 0.$$

From the first step it follows the statement of Lemma 1. ∎

We need the arguments below about the differentiability of a and A. The following Lemma 2 is important for us. We show Lemma 2 through the elementary knowledge about the integral calculus in Sobolev space.

Lemma 2. *Suppose that $u\in H1$. Then a_u is continuously differentiable, $a_u^{'}$ and A_u are weakly differentiable in $(0,T)$ if $\dot{u}$ is absolutely continuous. Further we have*

$$a_u^{'}(t)=2\int_\Omega u(t,x)\dot{u}(t,x)dx \quad \forall t\in(0,T), \tag{4}$$

where $\dot{u}$ means the first partial derivative in time t of u in the sense of $L^2(\Omega)$. If $u\in H2$ then $a_u\in C^2(0,T)$ and $A_u\in C^1(0,T)$. Furthermore we get

$$a_u^{''}(t)=2\int_\Omega\left(\dot{u}(t,x)^2+u(t,x)\ddot{u}(t,x)\right)dx \quad \text{in } L^2(\Omega) \quad \forall t\in(0,T) \tag{5}$$

and

$$A_u^{'}(t)=2\int_\Omega(\nabla\dot{u}(t,x)\cdot\nabla u(t,x)+\dot{u}(t,x)\ddot{u}(t,x))dx \ \text{ in } L^2(\Omega) \ \forall t\in(0,T). \tag{6}$$

Proof. We shall show the uniqueness of the solutions of the initial-boundary value problem for the semi-linear wave equation (SL). We can write a and A instead of a_u and A_u for convenience. Then we show first the differentiability of a and the continuity of $a^{'}$ for $u \in H1$. Secondly, we show the absolute continuity of $a^{'}$ for $u \in H1$. Then it follows automatically that $a \in C^2(0,T)$ for u in $H2$. Thirdly, we show the absolute continuity of A for u in $H1$. Then A is differentiable automatically for u in $H2$. At the end we show the continuity of $A^{'}$ for $u \in H2$.

1) Continuous differentiability of a in $(0,T)$ for u in $H1$.

From the definition of $a(t)$ we get

$$\begin{aligned}
\lim_{h\to 0} \frac{a(t+h)-a(t)}{h} &= \lim_{h\to 0} \frac{\int_\Omega \left(u(t+h,x)^2 - u(t,x)^2\right) dx}{h} \\
&= \lim_h \frac{\int_\Omega (u(t+h,x)+u(t,x))(u(t+h,x)-u(t,x))dx}{h} \\
&\leq 2(\|u(t+h)\|_2 + \|u(t)\|_2) \cdot \frac{\|u(t+h)-u(t)\|_2}{h}.
\end{aligned}$$

By the differentiability of u in $L^2(\Omega)$ it follows the existence of the above limit. Using the definition of $\dot{u}$, we get

$$\lim_{h\to 0} \frac{\|u(t+h)-u(t)\|_2}{h} = \dot{u}(t) \mid \text{in } L^2(\Omega) \text{ sense.} \tag{7}$$

This means that the identity (7) is valid in $L^2(\Omega)$, that is,

$$\lim_{h\to 0} \frac{\|u(t+h)-u(t)\|_2}{h} = \|\dot{u}(t)\|_2$$

and

$$\lim_{h\to 0} \left\| \frac{u(t+h)-u(t)}{h} - \dot{u}(t) \right\|_2 = 0.$$

By the definition of $a(t)$ we have therefore

$$\begin{aligned}
&\limsup_{h\to 0} \left| \frac{a(t+h)-a(t)}{h} - 2\int_\Omega u(t,x)\dot{u}(t,x)dx \right| \\
&= \limsup_{h\to 0} \left| \int_\Omega \left(\begin{array}{c} \dfrac{u(t+h,x)-u(t,x)}{h}(u(t+h,x)+u(t,x)) \\ -2u(t,x)\dot{u}(t,x) \end{array} \right) dx \right|.
\end{aligned} \tag{8}$$

We divide the identity (8) into two parts and obtain

$$\limsup_{h\to 0}\left|\frac{a(t+h)-a(t)}{h}-2\int_\Omega u(t,x)\dot u(t,x)dx\right|$$
$$= \limsup_{h\to 0}\left|\begin{array}{c}\int_\Omega\left(\left(\frac{u(t+h,x)-u(t,x)}{h}-\dot u(t,x)\right)\cdot(u(t+h,x)+u(t,x))dx\right)\\ +\int_\Omega \dot u(t,x)(u(t+h,x)-u(t,x))dx.\end{array}\right|. \tag{9}$$

From (9) it follows the following estimate

$$\begin{aligned}
&\limsup_{h\to 0}\left|\frac{a(t+h)-a(t)}{h}-2\int_\Omega u(t,x)\dot u(t,x)dx\right|\\
&\le \limsup_{h\to 0}\left|\int_\Omega\left(\left(\frac{u(t+h,x)-u(t,x)}{h}-\dot u(t,x)\right)\cdot(u(t+h,x)+u(t,x))\right)dx\right|\\
&\quad+\limsup_{h\to 0}\left|\int_\Omega \dot u(t,x)(u(t+h,x)-u(t,x))dx\right|\\
&\le \limsup_{h\to 0}\|u(t+h)+u(t)\|_2\cdot\left\|\frac{u(t+h)-u(t)}{h}-\dot u(t)\right\|_2\\
&\quad+\limsup_{h\to 0}\|\dot u(t)\|_2\|u(t+h)-u(t)\|_2=0.
\end{aligned}\tag{10}$$

Now we show the continuity of $a^{'}$. By using the estimate (10), we obtain

$$\begin{aligned}
a^{'}(t+h)-a^{'}(t) &= 2\int_\Omega(u(t+h,x)\dot u(t+h,x)-u(t,x)\dot u(t,x))dx\\
&= 2\int_\Omega u(t+h,x)(\dot u(t+h,x)-\dot u(t,x))dx\\
&\quad +2\int_\Omega \dot u(t,x)(u(t+h,x)-u(t,x))dx.
\end{aligned}\tag{11}$$

By the Hölder inequality and (11), we reach at the estimate

$$\begin{aligned}
&|a^{'}(t+h)-a^{'}(t)|\\
&\le 2\|u(t+h)\|_2\|\dot u(t+h)-\dot u(t)\|_2+2\|\dot u(t)\|_2\|u(t+h)-u(t)\|_2.
\end{aligned}\tag{12}$$

From the continuity of u and $\dot u$ and the boundedness of $u(t+h)$ and $\dot u(t)$ in $L^2(\Omega)$ follows the continuity of $a^{'}$. Hence a is continuously differentiable in $(0,T)$. By the same argument we can prove the existence of the limit of $a^{'}$ at $t=0$.

2) We prove now the absolute continuity of $a^{'}(t)$ in $(0,T)$ for those u in $H1$ for which $\dot u$ is absolutely continuous.

From the absolute continuity of $\dot{u}$ and the inequality (12) it follows at once the absolute continuity of $a^{'}$. With the help of the famous theorem for absolute continuity in reflexive Banach spaces, one sees the weak differentiability of $a^{'}$ in $(0,T)$. Further we get

$$\begin{aligned}
&\limsup\nolimits_{h\to 0}\left|\frac{a^{'}(t+h)-a^{'}(t)}{h}-2\int_\Omega\left(\dot{u}(t,x)^2+u(t,x)\ddot{u}(t,x)\right)dx\right| \\
&= \limsup\nolimits_{h\to 0} 2\left|\int_\Omega\left(\begin{array}{c}\dfrac{u(t+h,x)\dot{u}(t+h,x)-u(t,x)\dot{u}(t,x)}{h}\\ \\ -\dot{u}(t,x)^2-u(t,x)\ddot{u}(t,x)\end{array}\right)dx\right|.
\end{aligned} \tag{13}$$

We divide the equation (13) into three parts and obtain

$$\begin{aligned}
&\limsup\nolimits_{h\to 0}\left|\frac{a^{'}(t+h)-a^{'}(t)}{h}-2\int_\Omega\left(\dot{u}(t,x)^2+u(t,x)\ddot{u}(t,x)\right)dx\right| \\
&= \limsup\nolimits_{h\to 0} 2\left|\begin{array}{c}\displaystyle\int_\Omega\left(\frac{u(t+h,x)-u(t,x)}{h}-\dot{u}(t,x)\right)\dot{u}(t+h,x)dx\\ \displaystyle+\int_\Omega\left(\frac{\dot{u}(t+h,x)-\dot{u}(t,x)}{h}-\ddot{u}(t,x)\right)u(t,x)dx\\ \displaystyle+\int_\Omega(\dot{u}(t+h,x)-\dot{u}(t,x))\dot{u}(t,x)dx.\end{array}\right|.
\end{aligned} \tag{14}$$

By (14) we obtain the following estimate

$$\begin{aligned}
&\limsup\nolimits_{h\to 0}\left|\frac{a(t+h)-a(t)}{h}-2\int_\Omega\left(\dot{u}(t,x)^2+u(t,x)\ddot{u}(t,x)\right)dx\right| \\
&\leq \limsup\nolimits_{h\to 0}\|\dot{u}(t+h)\|_2\left\|\frac{u(t+h)-u(t)}{h}-\dot{u}(t)\right\|_2 \\
&\quad+\limsup\nolimits_{h\to 0}\|\dot{u}(t)\|_2\|\dot{u}(t+h)-\dot{u}(t)\|_2 \\
&\quad+\lim\nolimits_{h\to 0}\|u(t)\|_2\left\|\frac{\dot{u}(t+h)-\dot{u}(t)}{h}-\ddot{u}(t)\right\|_2.
\end{aligned}$$

Now we prove the continuity of $a^{''}$. Through the same arguments we have got the following estimate

$$a^{''}(t+h)-a^{''}(t)=2\int_{\Omega}\left[\left(\begin{array}{c}\dot{u}(t+h,x)^2-\dot{u}(t,x)^2\\ +u(t+h,x)\ddot{u}(t+h,x)-u(t,x)\ddot{u}(t,x)\end{array}\right)\right]dx$$

$$(15)\quad =2\int_{\Omega}[\dot{u}(t+h,x)(\dot{u}(t+h,x)-\dot{u}(t,x))+\dot{u}(t,x)(\dot{u}(t+h,x)-\dot{u}(t,x))]dx$$
$$+2\int_{\Omega}[(\ddot{u}(t+h,x)-\ddot{u}(t,x))u(t+h,x)+(u(t+h,x)-u(t,x))\ddot{u}(t,x)]dx.$$

From the Hölder inequality it follows the boundedness of $|a^{''}(t+h)-a^{''}(t)|$ with the bound

$$2\|\dot{u}(t+h)\|_2\|\dot{u}(t+h)-\dot{u}(t)\|_2+2\|\dot{u}(t)\|_2\|\dot{u}(t+h)-\dot{u}(t)\|_2$$
$$+2\|u(t+h)\|_2\|\ddot{u}(t+h)-\ddot{u}(t)\|_2+2\|\ddot{u}(t)\|_2\|u(t+h)-u(t)\|_2.$$

Because of the continuity of $u, \dot{u}$ and $\ddot{u}$ and the boundedness of $\dot{u}(t+h)$, $\dot{u}(t)$, $u(t+h)$ and $\ddot{u}(t)$ in $L^2(\Omega)$, by (15) it follows the continuity of $a^{''}$. Hence $a^{'}$ is continuously differentiable in $(0,T)$ for u in $H2$.

3) Analogously, if $u\in H2$, then $A\in C^1(0,T)$. Further, by the definition of A we have

$$\frac{A(t+h)-A(t)}{h}=\int_{\Omega}\left(\frac{\dot{u}(t+h,x)^2-\dot{u}(t,x)^2}{h}+\frac{|\nabla u(t+h,x)|^2-|\nabla u(t,x)|^2}{h}\right)dx$$

and

$$\left|\frac{A(t+h)-A(t)}{h}-2\int_{\Omega}[\nabla\dot{u}(t,x)\cdot\nabla u(t,x)+\dot{u}(t,x)\ddot{u}(t,x)]dx\right|$$

$$(16)\quad =\left|\begin{array}{c}\int_{\Omega}\left(\left(\frac{\dot{u}(t+h,x)-\dot{u}(t,x)}{h}-\ddot{u}(t,x)\right)\cdot(\dot{u}(t+h,x)+\dot{u}(t,x))dx\right)\\ +\int_{\Omega}\left(\begin{array}{c}\ddot{u}(t,x)(\dot{u}(t+h,x)-\dot{u}(t,x))+\nabla\dot{u}(t,x)\\ \cdot\nabla(u(t+h,x)-u(t,x))\end{array}\right)dx\\ +\int_{\Omega}\left(\begin{array}{c}\nabla\left(\frac{u(t+h,x)-u(t,x)}{h}-\dot{u}(t,x)\right)\\ \cdot\nabla(u(t+h,x)+u(t,x))dx.\end{array}\right)\end{array}\right|.$$

From (16) it follows the estimate

$$\begin{aligned}
&\left|\frac{A(t+h)-A(t)}{h} - 2\int_\Omega [\nabla \dot{u}(t,x)\cdot\nabla u(t,x) + \dot{u}(t,x)\ddot{u}(t,x)]dx\right| \\
&\le \left\|\frac{\dot{u}(t+h)-\dot{u}(t)}{h} - \ddot{u}(t)\right\|_2 (\|\dot{u}(t+h)\|_2 + \|\dot{u}(t)\|_2) \\
&\quad + \|\ddot{u}(t)\|_2\|\dot{u}(t+h)-\dot{u}(t)\|_2 + |\nabla\dot{u}(t)\|_2\|\nabla(u(t+h)-u(t))\|_2 \\
&\quad + \left\|\nabla\left(\frac{u(t+h)-u(t)}{h} - \dot{u}(t)\right)\right\|_2 \|\nabla(u(t+h)+u(t))\|_2.
\end{aligned}$$

Hence the assertions about A in Lemma 2 follow. ∎

Definition 3. $u \in H1$ is a *weak solution* of the problem (SL) if

$$\begin{aligned}
&\int_0^t \int_\Omega (\nabla u(s,x)\cdot\nabla\varphi(s,x) - \dot{u}(s,x)\dot{\varphi}(s,x) + f(u(s,x))\varphi(s,x))dxds \\
&\qquad = \int_\Omega (\dot{u}(0,x)\varphi(0,x) - \dot{u}(t,x)\varphi(t,x))dx \quad \forall\varphi\in H1.
\end{aligned}$$

Remark 3. This definition is resulted through the multiplication with φ and integration from 0 to t. From this we obtain the following Lemma 4.

Lemma 4. *Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a continuously differentiable function with the property that $vf(v)$ and $F(v)$ are both in $L^1(\Omega)$ for each v in $H_0^1(\Omega)$; $f(v)\in W^{1,1}(0,T,L^2(\Omega))$ for each v in $H2$; $f(u): H2\to H1$ is local Lipschitz, i.e., there exists a function $M(\|u\|_{H2},\|v\|_{H2})$ such that*

$$\|f(u)-f(v)\|_{H1} \le M(\|u\|_{H2},\|v\|_{H2})\|u-v\|_{H2} \quad \forall u,v\in H2, \tag{17}$$

where $M(\|u\|_{H2},\|v\|_{H2})$ is bounded if $\|u\|_{H2}$ and $\|v\|_{H2}$ are all bounded. Suppose $u_0\in H^2(\Omega)\cap H_0^1(\Omega)$ and $u_1\in H_0^1(\Omega)$. Then the problem (SL) *has exactly one solution $u\in H1$ and $a\in C^1(0,T)$, $a'(t)$ is differentiable almost everywhere in $(0,T)$ and*

$$a''(t) = 2\int_\Omega \left(\dot{u}(t,x)^2 - |\nabla u(t,x)|^2 - u(t,x)f(u(t,x))\right)dx \ \textit{a.e. in } (0,T), \tag{18}$$

$$A'(t) = -2\int_\Omega \dot{u}(t,x)f(u(t,x))dx \quad \textit{a.e. in } (0,T). \tag{19}$$

In order to facilitate the flow of the argument, we will postpone its proof to the end of the paper. We have the following main results.

2. Main Results

The initial-boundary value problem for the damping wave equation

$$\Box u + g(\dot{u}) = 0$$

has a solution in H1, provided $g : \mathbb{R} \to \mathbb{R}^+$ is in C^1 such that $g^{'}(t) \geq 0\ \forall t \geq 0$ (see [9] p. 29). The triviality of the solutions of wave equation of this type will be discussed later in another work. Using the method here one can consider also the wave equation

$$\Box u + f(u, \dot{u}) = 0$$

with the constraint

$$\xi \cdot f(\eta, \xi) \geq 0$$

or

$$f(\eta, \xi)^2 \leq k\left(|\eta|^p + \xi^2\right) \forall(\eta, \xi) \in \mathbb{R}^2, 2 \leq p \leq 2n/n - 2.$$

We have the following main results.

Theorem 5. *The initial-boundary value problem* [IBVP] *for the wave equation* (SL) *has exactly one global solution* $u \equiv 0$ *in* $H2$ *provided* $u_0 \equiv 0 \equiv u_1$ *and*

$$f(\eta)^2 \leq k\left(\eta^2 + |\eta|^p\right) \quad \forall \eta \in \mathbb{R}, \quad p \in [2, 2n/(n-2)]. \tag{20}$$

Theorem 6. *Suppose that* $u \in H1$ *is a weak solution of the* IBVP *for the semi-linear wave equation* (SL). *Then we have* $u \equiv 0$ *and* $f(0) = 0$ *if* $u_0 \equiv 0 \equiv u_1$ *and there exists a positive constant* k *such that*

$$F(\eta) \geq -k\eta^2 \quad \forall\, \eta \in \mathbb{R}. \tag{21}$$

Theorem 7. *Suppose that* $\Omega := B_{r_2}(0) - \overline{B_{r_1}(0)},\ r_2 > r_1 > 0,$ *is an annulus in* $\mathbb{R}^n$ *and there exists two positive constants* $k > 0$ *and* $p > 1$ *such that*

$$\eta f(\eta) + 2F(\eta) \geq -k|\eta|^p \quad \forall \eta \in \mathbb{R}. \tag{22}$$

Then the IBVP *for* (SL) *has exactly one global radial solution* $u \equiv 0$ *in* $H2$ *if* f *is local Lipschitz and*

$$u_0 \equiv 0 \equiv u_1 \equiv f(0).$$

Remark 7. We here require no upper-bound on p. $f(u) = -mu + ku^q$ is a typical example for this condition.

3. Proof of Theorem 5

It is evident that $u \equiv 0$ is a solution of the IBVP for the semi-linear wave equation (SL). Suppose that $0 \neq u \in H2$ is another solution of (SL). Since the continuity of A and $u_0 = 0 = u_1$, it follows that $A(0) = 0$ and the following supremum exists

$$t_1 := \sup\{t \geq 0 : A(t) \leq 1\}. \tag{23}$$

By (20) and Lemma 4 we find the inequality

$$A^{'}(t) \leq A(t) + k\left(A(t) + \int_\Omega |u|^p(t,x)dx\right). \tag{24}$$

For $2 \leq p \leq 2n/(n-2)$ we have the Sobolev inequality

$$\|u\|_p^p(t) \leq k_1\|Du\|_2^p(t) = k_1 A(t)^{\frac{p}{2}}. \tag{25}$$

Through (23) (24) and (25) we arrive at

$$A^{'}(t) \leq (1+k)A(t) + k \cdot k_1 A(t)^{\frac{p}{2}} \leq k_2 A(t) \quad \text{for } t \text{ in } [0, t_1] \tag{26}$$

since $A(t) \leq 1 \; \forall t \in [0, t_1]$, where $k_2 := 1 + k + 2^{-1}pkk_1$.

Multiply the inequality (26) with $\exp(-k_2 t)$. Then it brings to

$$(\exp(-k_2 t) \cdot A(t))^{'} = \exp(-k_2 t)(A^{'}(t) - k_2 A(t)) \leq 0 \quad \forall t \in [0, t_1]. \tag{27}$$

By (27) we have found

$$A(t) \leq A(0) \cdot \exp(k_2 t) = 0 \qquad \forall t \in [0, t_1].$$

Hence we get $u \equiv 0$ in $[0, t_1]$. Repeating the above process, we reach at $u \equiv 0$ in $\mathbb{R}^+$. This contradicts $u \neq 0$. ■

Corollary 5. *Theorem* 5 *is true particularly for the well-defined functions*

$$f(u) = u^{p/2} + u^{q/2}, u^{p/2} - u^{q/2},\; p, q \in [2, 2n/n-2]$$

or under the assumption

$$f(\eta)^2 \leq \sum_{i=1}^m k_i|\eta|^{p_i} \quad \forall \eta \in \mathbb{R},$$

where k_i = positive constants, $p_i \in [2, 2n/n-2]$.

Application of Theorem 5. *For a local Lipschitz function f there exists exactly one solution $u \in H1$ of the* IBVP *for the wave equation* (SL), *provided $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$.*

The existence and uniqueness of solutions are well known but our proof for uniqueness is short. We now show it below. Suppose that u and v are two solutions of (SL). Then

$$\Box(u-v)+f(u)-f(v)=0,(u-v)(0,\cdot)=0=(\dot{u}-\dot{v})(0,\cdot).$$

We set

$$g(u-v):=f(u)-f(v).$$

Then we have

$$\begin{aligned}\|g(u-v)\|_2(t) &= \|f(u)-f(v)\|_2(t)\leq M(\|u\|_{1,2},\|v\|_{1,2})\|u-v\|_{1,2}(t)\\ &\leq M\left(\|u\|_{1,2},\ \|v\|_{1,2}\right)\|D(u-v)\|_2(t).\end{aligned}$$

Like the proof in Theorem 5 we get $\|D(u-v)\|_2(t)\equiv 0$ and the uniqueness of the solutions of this problem (SL) follows. ■

4. Proof of Theorem 6

We choose

$$\varphi_\varepsilon(s,\cdot)=\frac{u(s+\varepsilon,\cdot)-u(s,\cdot)}{\varepsilon}\in H1$$

in Definition 3 and let $\varepsilon\to 0$. Then we can get the estimate

$$A(t)=-2\int_\Omega F(u(t,x))dx\leq 2ka(t).$$

With the help of Poincaré inequality it follows

$$(28)\ C_\Omega^2\int_\Omega u(t,x)^2dx+\int_\Omega \dot{u}(t,x)^2dx\leq\int_\Omega\left(|\nabla u(t,x)|^2+\dot{u}(t,x)^2\right)dx:=A(t).$$

By (28) we have therefore

$$\begin{aligned}C_\Omega a^{'}(t) &= 2C_\Omega\int_\Omega u(t,x)\dot{u}(t,x)dx\leq C_\Omega^2a(t)+\int_\Omega\dot{u}(t,x)^2dx\\ &\leq A(t)\leq 2ka(t)\quad\forall t\geq 0.\end{aligned}$$

So it brings to

$$a^{'}(t)-2kC_\Omega^{-1}a(t)\leq 0\qquad\forall t\geq 0. \tag{29}$$

Multiplying the inequality (29) with $\exp(-rt),\ r:=2kC_\Omega^{-1}$, we obtain

$$\left(e^{-rt}a(t)\right)^{'}=e^{-rt}(a^{'}(t)-ra(t))\leq 0\qquad\forall t\geq 0,$$

herewith

$$a(t) \le a(0)\exp(rt) = 0 \qquad \forall t \ge 0 \tag{30}$$

since $u_0 \equiv 0 \equiv u_1$. By (30) it follows immediately that $u \equiv 0$ and $f(0) = 0$. ■

Corollary 6. *Theorem* 6 *is valid particularly for monotonic increasing function* f *with* $f(0) = 0$. *For instance,* $f(u) = u^{2p-1},\ -1 + \exp u$.

5. Proof of Theorem 7

It is clear that $u \equiv 0$ is a radial solution of IBVP for the wave equation (SL).

1) Suppose that $0 \ne u(t, |x|) = u(t, r),\ r = |x|$, is another radial solution in $H2$ of (SL). We set

$$u(t,r) = v(t,r)r^{(1-n)/2},\ v_r(t,r) := \frac{\partial v(t,r)}{\partial r}.$$

Then we have

$$\begin{aligned}
\ddot{u}(t,r) &= \ddot{v}(t,r)r^{(1-n)/2},\ r_i := \frac{\partial r}{\partial x_i} = \frac{x_i}{r},\\
u_i(t,r) := \frac{\partial u(t,r)}{\partial x_i} &= v_r(t,r)\ r^{\frac{1-n}{2}-1}x_i + \frac{1-n}{2}r^{\frac{1-n}{2}-2}x_i v(t,r)
\end{aligned}$$

and

$$\begin{aligned}
u_{ii} &= v_{rr}r^{\frac{1-n}{2}-2}x_i^2 + v_r\left(r^{\frac{1-n}{2}-1} - \frac{1+n}{2}r^{\frac{1-n}{2}-3}x_i^2 + \frac{1-n}{2}r^{\frac{1-n}{2}-3}x_i^2\right)\\
&\quad + \frac{1-n}{2}v\left(-\frac{3+n}{2}r^{\frac{1-n}{2}-4}x_i^2 + r^{\frac{1-n}{2}-2}\right)\\
&= v_{rr}r^{\frac{1-n}{2}-2}x_i^2 + v_r\left(r^{\frac{1-n}{2}-1} - nr^{\frac{1-n}{2}-3}x_i^2\right)\\
&\quad + \frac{1-n}{2}v\left(-\frac{3+n}{2}r^{\frac{1-n}{2}-4}x_i^2 + r^{\frac{1-n}{2}-2}\right).
\end{aligned}$$

From this it follows

$$\begin{aligned}
\triangle u &= v_{rr}r^{\frac{1-n}{2}} + v_r\left(nr^{\frac{1-n}{2}-1} - nr^{\frac{1-n}{2}-3}r^2\right)\\
&\quad + \frac{1-n}{2}v\left(-\frac{3+n}{2}r^{\frac{1-n}{2}-4}r^2 + nr^{\frac{1-n}{2}-2}\right)\\
&= v_{rr}r^{\frac{1-n}{2}} - \frac{(n-1)(n-3)}{4r^2}r^{\frac{1-n}{2}}.
\end{aligned}$$

We set

$$G(v) = \int_0^v g(s)ds, \ g(v) = \frac{(n-1)(n-3)}{4r^2}v + r^{\frac{n-1}{2}} f\left(r^{\frac{n-1}{2}}v\right).$$

Then the equation (SL) will be transformed into the following equation

$$\text{(RG)} \quad \begin{aligned} &\ddot{v} - \partial^2 v/\partial r^2 + g(v) = 0 \quad \text{ in } [0,T) \times (r_1, r_2), \\ &v(0,r) = u_0 r^{(n-1)/2} \equiv 0 \equiv \dot{v}(0,r), v(t,r_1) \equiv 0 \equiv v(t,r_2). \end{aligned}$$

Choose $\eta = r^{(1-n)/2}v$ in (22), then we get

$$\text{(31)} \quad \begin{aligned} \eta f(\eta) + 2F(\eta) &= r^{-(n-1)/2} v f\left(r^{-(n-1)/2}v\right) + 2F\left(r^{-(n-1)/2}v\right) \\ &\geq -kr^{-(n-1)p/2}|v|^p. \end{aligned}$$

From (31) it follows therefore

$$\begin{aligned} vg(v) + 2G(v) &= \frac{(n-1)(n-3)}{4r^2}v^2 + r^{\frac{n-1}{2}} v f\left(r^{-\frac{n-1}{2}}v\right) \\ &+ \frac{(n-1)(n-3)}{4r^2}v^2 + 2r^{\frac{n-1}{2}} \int_0^v f\left(r^{-\frac{n-1}{2}}s\right) ds \\ \text{(32)} \quad &= \frac{(n-1)(n-3)}{2r^2}v^2 + r^{\frac{n-1}{2}} v f\left(r^{-\frac{n-1}{2}}v\right) + 2r^{\frac{n-1}{2}} r^{\frac{n-1}{2}} \int_0^{r^{-\frac{n-1}{2}}v} f(s)ds \\ &= \frac{(n-1)(n-3)}{2r^2}v^2 + r^{\frac{n-1}{2}} \left(v f\left(r^{-\frac{n-1}{2}}v\right) + 2r^{\frac{n-1}{2}} F\left(r^{-\frac{n-1}{2}}v\right)\right) \\ &\geq \frac{(n-1)(n-3)}{2r^2}v^2 - kr^{-(n-1)(p/2-1)}|v|^p \geq -k_1(v^2 + |v|^p), \end{aligned}$$

with $k_1 := \max\left\{-2^{-1}(n-1)(n-3)r^{-2} + kr^{(1-n)(p/2-1)} : r \in [r_1, r_2]\right\}$.

2) Because $v \in H2$ is a solution of (RG), we have found that

$$\begin{aligned} &\int_{r_1}^{r_2} (\dot{v}(t,s)\ddot{v}(t,s) - \dot{v}(t,s)v_{rr}(t,s) + \dot{v}(t,s)g(v(t,s)))ds = 0, \\ &\int_{r_1}^{r_2} (\dot{v}(t,s)\ddot{v}(t,s) + \dot{v}_r(t,s)v_r(t,s) + \dot{v}(t,s)g(v(t,s)))ds \\ &\quad = \dot{v}(t,r_2)v_r(t,r_2) - \dot{v}(t,r_1)v_r(t,r_1) = 0, \end{aligned}$$

$$\text{(33)} \quad \frac{d}{dt}\int_{r_1}^{r_2} \left(\dot{v}(t,s)^2 + v_r(t,s)^2 + 2G(v(t,s))\right) ds = 0.$$

By (33) we therefore reach the following identity

$$
\begin{aligned}
&\int_{r_1}^{r_2} \left(\dot v(t,s)^2 + v_r(t,s)^2 + 2G(v(t,s))\right) ds \\
&= \int_{r_1}^{r_2} \left(\dot v(0,s)^2 + v_r(0,s)^2 + 2G(v(0,s))\right) ds \\
&= 2\int_{r_1}^{r_2} G(v(0,s))ds = 0
\end{aligned} \tag{34}
$$

by $v(0,r) = 0 = \dot v(0,r)$. Multiplying the equation (RG) with v and integrating from r_1 to r_2, we get

$$\frac{1}{2}\frac{d}{dt}\int_{r_1}^{r_2} v^2(t,s)ds = \int_{r_1}^{r_2} \left(\dot v(t,s)^2 - v_r(t,s)^2 - v(t,s)g(v(t,s))\right) ds,$$

$$2\int_{r_1}^{r_2} v_r(t,s)^2 ds = -\frac{\hat a''(t)}{2} - \int_{r_1}^{r_2} (v(t,s)g(v(t,s)) + 2G(v(t,s)))ds, \tag{35}$$

where $\hat a(t) := \int_{r_1}^{r_2} v(t,r)^2 dr$ and $v_r := \dfrac{\partial v(t,r)}{\partial r}$.
By (32) and (35) we obtain

$$2\int_{r_1}^{r_2} v_r(t,s)^2 ds \le -\frac{\hat a''(t)}{2} + k_1\int_{r_1}^{r_2} \left(v(t,s)^2 + |v(t,s)|^p\right) ds. \tag{36}$$

By the Sobolev embedding $H_0^1(r_1,r_2) \subset C_0^0(r_1,r_2) \subset L^p(r_1,r_2)$, there exists a positive constant k_2 such that

$$\int_{r_1}^{r_2} |v(t,s)|^p ds \le k_2\left(\int_{r_1}^{r_2} \left(v(t,s)^2 + v_r(t,s)^2\right) ds\right)^{\frac{p}{2}}. \tag{37}$$

By the inequalities (36) and (37) we have found that

$$2\int_{r_1}^{r_2} v_r(t,s)^2 ds \le -\frac{\hat a''(t)}{2} + k_1\int_{r_1}^{r_2} v(t,s)^2 ds + k_1k_2\left(\int_{r_1}^{r_2} \left(v(t,s)^2 + v_r(t,s)^2\right) ds\right)^{\frac{p}{2}}$$

and

$$
\begin{aligned}
&\hat a''(t) + 4\int_{r_1}^{r_2} v_r(t,s)^2 ds \\
\le\ & 2k_1\int_{r_1}^{r_2} v(t,s)^2 ds + 2k_1k_2\left(\int_{r_1}^{r_2} \left(v(t,s)^2 + v_r(t,s)^2\right) ds\right)^{\frac{p}{2}}.
\end{aligned} \tag{38}
$$

This yields the estimate

$$\text{(39)}\quad \hat{a}''(t) + 4\int_{r_1}^{r_2} v_r(t,s)^2 ds \le 2k_1\hat{a}(t) + k_3\left(\hat{a}(t)^{\frac{p}{2}} + \left(\int_{r_1}^{r_2} v_r(t,s)^2 ds\right)^{\frac{p}{2}}\right),$$

where $k_3 := 2^{\frac{p}{2}}k_1k_2$.

3) By using $v(0,r) = 0 = \dot{v}(0,r)$, we get $v_r(0,r) = 0 = \dot{v}_r(0,r)$, $\hat{a}(0) = 0$ and the following supremum exists for $p > 2$

$$\text{(40)}\qquad t_2 := \sup\left\{\bar{t} : \int_{r_1}^{r_2} v_r(t,s)^2 ds \le k_4,\ \hat{a}(t) \le 1,\ t \le \bar{t}\right\},$$

$$k_4 := \min\left\{1, (4/k_3)^{2/(p-2)}\right\}.$$

Through the definition (40), in $[0,t_2]$ we find that

$$\text{(41)}\quad \begin{aligned} & k_3\left(\int_{r_1}^{r_2} v_r(t,s)^2 ds\right)^{\frac{p}{2}} = k_3\int_{r_1}^{r_2} v_r(t,s)^2 ds \left(\int_{r_1}^{r_2} v_r(t,s)^2 ds\right)^{\frac{p}{2}-1} \\ & \le k_3\int_{r_1}^{r_2} v_r(t,s)^2 ds \cdot k_4^{\frac{p}{2}-1} \\ & \le k_3 \cdot \left[\left(\frac{4}{k_3}\right)^{\frac{2}{p-2}}\right]^{\frac{p}{2}-1} \cdot \int_{r_1}^{r_2} v_r(t,s)^2 ds \\ & \le 4\int_{r_1}^{r_2} v_r(t,s)^2 ds. \end{aligned}$$

From the inequalities (38), (39) and (41) we obtain

$$\hat{a}''(t) \le 2k_1\hat{a}(t) + k_3\hat{a}(t)^{\frac{p}{2}} \le \left(2k_1 + \frac{pk_3}{2}\right)\hat{a}(t) \qquad \forall t \in [0,t_2),$$

since $\hat{a}(t) \le 1$ in $[0,t_2]$. By $\hat{a}(0) = 0 = \hat{a}'(0)$ and Lemma 1 it follows

$$\hat{a}(t) \equiv 0 \qquad \forall t \in [0,t_2].$$

Note that t_2 must be ∞, for otherwise it will produce a contradiction to the definition of t_2. For $p \in [1,2]$, from Lemma 1 and the inequality (39) it follows that $\hat{a}(t) \equiv 0$. So $u \equiv 0$. This is however a contradiction to the assumption $u \ne 0$. ∎

6. Proof of Lemma 4

We show the existence and uniqueness of the solutions of the IBVP (SL) through an elementary method.

By [5, p. 95; 4, p. 31], we suppose that $v_1 \in H2$ is the existing solution of the IBVP for the wave equation

$$\begin{aligned} \Box v_1 + f(u_0) &= 0 \text{ in } [0,T) \times \Omega, v_1(t,x) = 0 \text{ on } [0,T] \times \partial\Omega, \\ v_1(0,\cdot) &= u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \dot{v}_1(0,\cdot) = u_1 \in H_0^1(\Omega). \end{aligned}$$

Then we have $v_1(t) \in H^2(\Omega)$ for each $t \in [0,T]$ and we get also

$$\begin{aligned} E'_{v_1}(t) &= \frac{d}{dt} \int_\Omega \left(\dot{v}_1(t,x)^2 + |\nabla v_1(t,x)|^2 + 2v_1(t,x) f(u_0(x)) \right) dx \\ &= 2 \int_\Omega [\dot{v}_1(t,x)(\dot{v}_1(t,x) + f(u_0(x))) + \nabla v_1(t,x) \cdot \nabla \dot{v}_1(t,x)] dx = 0. \end{aligned}$$

Using Lemma 2 we have therefore the identity

$$\begin{aligned} \frac{1}{2} a''_{v_1}(t) &= \int_\Omega \left(\dot{v}_1(t,x)^2 + v_1(t,x) \ddot{v}_1(t,x) \right) dx \\ &= \int_\Omega \left(\dot{v}_1(t,x)^2 - |\nabla v_1(t,x)|^2 - v_1(t,x) f(u_0(x)) \right) dx. \end{aligned}$$

Suppose that $v_{m+1} := Sv_m \in H2$ is the existing solution of the IBVP for the wave equation

$$\Box v_{m+1} + f(v_m) = 0 \text{ in } [0,T) \times \Omega,\ v_{m+1}(t,x) = 0 \text{ on } [0,T] \times \partial\Omega,$$

$$v_{m+1}(0,\cdot) = u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \dot{v}_{m+1}(0,\cdot) = u_1 \in H_0^1(\Omega).$$

Then we have found that $v_{m+1}(t) \in H^2(\Omega)$ for each $t \in [0,T]$ and

$$\begin{aligned} E'_{v_{m+1}}(t) &= \frac{d}{dt} \int_\Omega \Big(\dot{v}_{m+1}(t,x)^2 + |\nabla v_{m+1}(t,x)|^2 \\ &\quad + 2v_{m+1}(t,x) f(v_m(t,x)) \Big) dx = 0. \end{aligned}$$

By Lemma 2 once more we obtain therefore the identity

$$\begin{aligned} \frac{1}{2} a''_{v_{m+1}}(t) &= \int_\Omega \Big(\dot{v}_{m+1}(t,x)^2 - |\nabla v_{m+1}(t,x)|^2 \\ &\quad - v_{m+1}(t,x) f(v_m(t,x)) \Big) dx. \end{aligned}$$

After some long computations one can verify the following statements.

(1) v_m is uniformly bounded in $H1$ and in $H2$.

(2) v_m is a Cauchy sequence in $H1$ and in $H2$.

Hence we reach at the assertions of Lemma 4. ∎

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