TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 6, pp. 2239-2258, December 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

WEAK HARDY SPACES $H^{p,\infty}$ ON SPACES OF HOMOGENEOUS TYPE AND THEIR APPLICATIONS

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Abstract. In this paper, we introduce weak Hardy spaces $H^{p,\infty}$ on spaces of homogeneous type. We establish an atomic decomposition characterization of these spaces, show the boundedness of fractional integral operators and provide an $H^{p,\infty}$ interpolation theorem. Applications to the Nagel-Stein's singular integral operators and fractional integral operators are also discussed.

1. Introduction and Main Results

The theory of weak Hardy spaces is very important in harmonic analysis since it can sharpen the endpoint weak type estimate for variant important operators (see, for example, [9]). The weak Hardy spaces were first studies in [8] as special Hardy-Lorentz spaces which are the intermediate spaces between two Hardy spaces. R. Fefferman and Soria [9] established an atomic decomposition of the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$. The atomic decompositions of the weak Hardy spaces $H^{p,\infty}$ on homogeneous groups were given by Liu in [15]. Ding and Lan [5] developed the theory of weak Hardy spaces associated to expansive dilations on $\mathbb{R}^n$. The weak Hardy space $H^{1,\infty}$ on spaces of homogeneous type was recently studied in [6].

In this paper, we shall study the theory of weak Hardy spaces $H^{p,\infty}$ on the homogeneous space having some reverse doubling property (see Definition 1.1 below). More precisely, we will first give an atomic decomposition characterization of $H^{p,\infty}$. Then we use this characterization to derive the $(H^{p,\infty}, L^{q,\infty})$ and $(H^{p,\infty}, H^{q,\infty})$ boundedness of fractional integral operators and prove an $H^{p,\infty}$ interpolation theorem. Finally, applications to the boundedness of Nagel-Stein's singular integral operators and fractional integral operators in $H^{p,\infty}$ are discussed. We remark that our theory is so general

Received February 14, 2012, accepted April 11, 2012.
Communicated by Yong-Sheng Han.
2010 *Mathematics Subject Classification*: Primary: 42B35; Secondary: 42B20, 42B25.
Key words and phrases: Weak Hardy spaces, Maximal functions, Singular integral, Space of homogeneous type.
The research was supported by NSF of China (Grant: 11101423, 11171345) and the Fundamental Research Funds for the Central Universities of China (Grant: 2009QS12).

that it covers the results in [5, 6, 15] as special cases and can be applied to more variant different settings such as Euclidean spaces with A_∞-weights, Ahlfors n-regular metric measure spaces (see, for example, [12]), Lie groups of polynomial growth (see, for instance, [1, 24, 25]) and Carnot-Carathéodory spaces with doubling measure (see [18, 19, 20, 21]).

Before giving the main results, let us recall some definitions and notions first. The following notion of spaces of homogeneous type was introduced by Coifman and Weiss in [3], see also [4].

Definition 1.1. Let $(\mathcal{X}, d)$ be a metric space with a regular Borel measure μ such that all balls defined by d have finite and positive measures. The quasi-metric satisfies the following triangle inequality,

$$d(x,z) \leq \tau(d(x,y) + d(y,z)). \tag{1.1}$$

For any $x \in \mathcal{X}$ and $r > 0$, set $B(x,r) = \{y \in \mathcal{X} : d(x,y) < r\}$. (X, d, μ) is called a space of homogeneous type (or a homogenous space) if there exists a constant $C_1 \geq 1$ such that for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu(B(x,2r)) \leq C_1 \mu(B(x,r)). \tag{1.2}$$

We also assume throughout that μ satisfies the following reverse doubling condition that there exist constants $1 < \kappa \leq n < \infty$, $0 < C_2 \leq 1$ and $C_3 > 1$ such that for all $x \in \mathcal{X}$, $0 < r < \text{diam}(\mathcal{X})/2$ and $1 \leq s < \text{diam}(\mathcal{X})/(2r)$,

$$C_2 s^\kappa \mu(B(x,r)) \leq \mu(B(x,sr)) \leq C_3 s^n \mu(B(x,r)), \tag{1.3}$$

where $\text{diam}(\mathcal{X}) = \sup_{x,y\in\mathcal{X}} d(x,y)$. The least possible value of n in (1.3) is called the dimension of $\mathcal{X}$, which is still denoted by n. We use dx to denote $d\mu(x)$ for simplicity.

Throughout this paper, we always assume that $\mathcal{X}$ is a homogeneous space with the reverse doubling condition (1.3) and $\mu(\mathcal{X}) = \infty$. We remark that all the examples mentioned above fall under the scope of the current setting.

Denote by C a positive constant independent of main parameters involved, which may vary at different occurrences. Constants with subscripts do not change through the whole paper. Let $A \lesssim B$ denote $A \leq CB$ and let $A \approx B$ mean $A \lesssim B$ and $B \lesssim A$. Denote $V(x,y) = \mu(B(x, d(x,y)))$ and $V_{2^j}(x) = \mu(B(x, 2^j))$. It is easy to see $V(x,y) \approx V(y,x)$.

Now we briefly recall the notions that we need to define the weak Hardy spaces $H^{p,\infty}(\mathcal{X})$ (see [6] and the references therein).

Definition 1.2. Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is said to be an approximation of the identity of order $(\epsilon_1, \epsilon_2, \epsilon_3)$ (in short, $(\epsilon_1, \epsilon_2, \epsilon_3)$- AOTI), if there exists a constant $C_4 > 0$ such that for all $k \in \mathbb{Z}$ and all x, x', y and $y' \in \mathcal{X}$, $S_k(x,y)$, the integral kernel of S_k is a function from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying

(i) $|S_k(x,y)| \le C_4 \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_2}}{(2^{-k}+d(x,y))^{\epsilon_2}}$;

(ii) $|S_k(x,y) - S_k(x',y)| \le C_4 \frac{d(x,x')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}} \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_2}}{(2^{-k}+d(x,y))^{\epsilon_2}}$ for $d(x,x') \le (2^{-k}+d(x,y))/2$

(iii) Property (ii) holds with x and y interchanged;

(iv) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]| \le C_4 \frac{d(x,x')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}} \frac{d(y,y')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}}$ $\times \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_3}}{(2^{-k}+d(x,y))^{\epsilon_3}}$ for $d(x,x') \le (2^{-k}+d(x,y))/3$ and $d(y,y') \le (2^{-k}+d(x,y))/3$;

(v) $\int_{\mathcal{X}} S_k(x,y)dy = \int_{\mathcal{X}} S_k(x,y)dx = 1$.

Using the size condition (i), it is not hard to show

$$\int_{\mathcal{X}} |S_k(x,y)|dy \le C, \quad \int_{\mathcal{X}} |S_k(x,y)|dx \le C. \tag{1.4}$$

The test functions are defined as follows.

Definition 1.3. Let $x_1 \in \mathcal{X}, r \in (0,\infty), \beta \in (0,1]$ and $\gamma \in (0,\infty)$. A function φ on $\mathcal{X}$ is said to be a test function of type (x_1,r,β,γ) if

(i) $|\varphi(x)| \le C \frac{1}{\mu(B(x,r+d(x,x_1)))} \left(\frac{r}{r+d(x_1,x)}\right)^{\gamma}$ for all $x \in \mathcal{X}$;

(ii) $|\varphi(x)-\varphi(y)| \le C \left(\frac{d(x,y)}{r+d(x_1,x)}\right)^{\beta} \frac{1}{\mu(B(x,r+d(x,x_1)))} \left(\frac{r}{r+d(x_1,x)}\right)^{\gamma}$ for all $x,y \in \mathcal{X}$ satisfying $d(x,y) \le (r+d(x_1,x))/2$.

We denote by $\mathcal{G}(x_1,r,\beta,\gamma)$ the set of all test functions of type (x_1,r,β,γ). If $\varphi \in \mathcal{G}(x_1,r,\beta,\gamma)$ we define its norm by $\|\varphi\|_{\mathcal{G}(x_1,r,\beta,\gamma)} := \inf\{C : \ (i) \ and \ (ii) \ hold\}$. The space $\mathcal{G}(x_1,r,\beta,\gamma)$ is called to be the space of test functions.

Throughout the whole paper, we fix $x_1 \in \mathcal{X}$. Let $\mathcal{G}(\beta,\gamma) = \mathcal{G}(x_1,1,\beta,\gamma)$. It is easy to see that for any $x_2 \in \mathcal{X}$ and $r > 0$, we have $\mathcal{G}(x_2,r,\beta,\gamma)$ with equivalent norms. For any given $\epsilon \in (0,1]$, let $\mathcal{G}_0^{\epsilon}(\beta,\gamma)$ be the completion of the space $\mathcal{G}(\epsilon,\epsilon)$ in $\mathcal{G}(\beta,\gamma)$ when $\beta,\gamma \in (0,\epsilon]$. Moreover, $\varphi \in \mathcal{G}_0^{\epsilon}(\beta,\gamma)$ if and only if $\varphi \in \mathcal{G}(\beta,\gamma)$ and there exists $\{\phi_i\}_{i\in\mathbb{N}}$ such that $\|\varphi-\phi_i\|_{\mathcal{G}(\beta,\gamma)} \to 0$ as $i \to \infty$. If $\varphi \in \mathcal{G}_0^{\epsilon}(\beta,\gamma)$, define $\|\varphi\|_{\mathcal{G}_0^{\epsilon}(\beta,\gamma)} = \|\varphi\|_{\mathcal{G}(\beta,\gamma)}$. Obviously, $\mathcal{G}_0^{\epsilon}(\beta,\gamma)$ is a Banach space. The notation $(\mathcal{G}_0^{\epsilon}(\beta,\gamma))'$ denotes the dual space of $\mathcal{G}_0^{\epsilon}(\beta,\gamma)$.

Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $0 < \epsilon < \min\{\epsilon_1,\epsilon_2\}$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI. For $f \in (\mathcal{G}_0^{\epsilon}(\beta,\gamma))'$ and $\beta,\gamma \in (0,\epsilon)$, the non-tangential maximal operator $\mathcal{M}_\sigma$ is defined by

$$\mathcal{M}_\sigma(f)(x) := \sup_{k\in\mathbb{Z}} \sup_{d(x,y)\le\sigma 2^{-k}} |S_k(f)(y)|.$$

The grand maximal function is defined by

$$f^*(x) := \sup\left\{|\langle f, \varphi\rangle| : \varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma), \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \le 1 \text{ for some } r > 0\right\}.$$

The radial maximal operator $\mathcal{M}_0$ is defined by

$$\mathcal{M}_0 f(x) := \sup_{k\in\mathbb{Z}} |S_k(f)(x)|.$$

For more details and results about harmonic analysis on spaces of homogeneous type, we refer the readers to [3, 4, 11].

Now we introduce the weak Hardy spaces $H^{p,\infty}$ on the spaces of homogeneous type.

Definition 1.4. Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon \in (0, \min\{\epsilon_1, \epsilon_2\})$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3) - \text{AOTI}$. Let $p \in (n/(n+1), 1]$, $\sigma \in (0,\infty)$ and $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with some $\beta, \gamma \in (0,\epsilon)$. The weak Hardy spaces $H^{p,\infty}$ on homogeneous spaces $\mathcal{X}$ is defined by

$$H^{p,\infty}(\mathcal{X}) = \{f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))' : \mathcal{M}_\sigma f \in L^{p,\infty}(\mathcal{X})\}.$$

The $H^{p,\infty}(\mathcal{X})$ quasi-norm of f is defined by

$$\|f\|_{H^{p,\infty}(\mathcal{X})} := \|\mathcal{M}_\sigma f\|_{L^{p,\infty}(\mathcal{X})}.$$

Remark 1.1. (i) For any $p \in (n/(n+1), 1]$, we always assume that $\epsilon \in (n(1/p - 1), 1)$ so that $p \in (n/(n+\epsilon), 1)$ and the weak Hardy spaces are defined via some $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with $n(1/p-1) < \beta, \gamma < \epsilon$ and some $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI $\{S_k\}_{k\in\mathbb{Z}}$ with $\min\{\epsilon_1,\epsilon_2\} > \epsilon,\ \epsilon_1 \le 1$ and $\epsilon_3 > 0$.
(ii) Both the Hardy spaces $H^p(\mathcal{X})$ and the weak Hardy spaces $H^{p,\infty}(\mathcal{X})$ can equivalently be defined via Littlewood-Paley functions, radial maximal functions, nontangential maximal functions and grand maximal functions (see [6] and the references therein).

Our first result is the following

Theorem 1.1. *Let $p \in (n/(n+1), 1]$. Given $f \in H^{p,\infty}(\mathcal{X})$, there exists a sequence of bounded functions $\{f_k\}_{k=-\infty}^{\infty}$ with the following properties:*

(a) $f - \sum_{|k|\le N} f_k \to 0$ *in* $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ *and* $\|f_k\|_{L^\infty(\mathcal{X})} \le C2^k$.

(b) *Each f_k may be further decomposed as $f_k = \sum_{i=1}^\infty h_i^k$ in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$, where the h_i^k satisfies:*

  (i) *h_i^k is supported in a ball B_i^k with $\{B_i^k\}$ having bounded overlap for each k;*

  (ii) $\int_{B_i^k} h_i^k(x)dx = 0$;

  (iii) $\|h_i^k\|_{L^\infty} \le C2^k$ *and* $\sum_i \mu(B_i^k) \le C_0 2^{-kp}$ *with the constant* $C_0 \approx \|f\|^p_{H^{p,\infty}(\mathcal{X})}$.

Conversely, if f is a distribution satisfying (a) *and* (b) (i)-(iii)*, then* $f \in H^{p,\infty}(\mathcal{X})$ *and* $\|f\|^p_{H^{p,\infty}(\mathcal{X})} \le cC_0$ (*where c is some absolute constant*).

As an application of Theorem 1.1, we shall prove the following interpolation theorem.

Theorem 1.2. *Let $n/(n+1) < q < p \le 1 < p_0 < \infty$. Suppose T is a subadditive operator, and T is bounded both on $L^{p_0}(\mathcal{X})$ and the Hardy space $H^q(\mathcal{X})$. Then T is bounded on $H^{p,\infty}(\mathcal{X})$.*

For the $(H^{p,\infty}(\mathcal{X}), L^{q,\infty}(\mathcal{X}))$ boundedness of fractional integral operators, we have

Theorem 1.3. *Let $T_\alpha f(x) = \int_{\mathcal{X}} K_\alpha(x,y)f(y)dy$ be an bounded operator from $L^{p_0}(\mathcal{X})$ to $L^{q_0}(\mathcal{X})$ for some $1 < p_0 < q_0 < \infty$ satisfying $\frac{1}{p_0} - \frac{1}{q_0} = \alpha$ and $0 < \alpha < 1$. If K has the following regularity in the second variable: there exists constants C, $\varepsilon > 0$ such that for all $x, y, y' \in \mathcal{X}$ with $d(y,y') \le d(x,y)/2$ and $x \ne y$,*

$$|K_\alpha(x,y) - K_\alpha(x,y')| \le C\frac{d(y,y')^\varepsilon}{V(x,y)^{1-\alpha}d(x,y)^\varepsilon}. \tag{1.5}$$

Then for $0 < \alpha < \min\{\varepsilon, \epsilon_1\}/n$ and for all p, q with $n/(n+1) < p < q \le 1$ and $1/p - 1/q = \alpha$, T_α is bounded from $H^{p,\infty}(\mathcal{X})$ to $L^{q,\infty}(\mathcal{X})$. Moreover, there exists a constant C, independent of f and λ, such that for each $\lambda > 0$,

$$\mu(\{x : |T_\alpha f(x)| > \lambda\}) \le C\left(\frac{\|f\|_{H^{p,\infty}(\mathcal{X})}}{\lambda}\right)^q.$$

The $(H^{p,\infty}, H^{q,\infty})$ boundedness of fractional integrals is given below.

Theorem 1.4. *Under the same hypothesis as in Theorem 1.3, if T_α further satisfies the following cancellation condition: for any function $\phi \in L^{p_0}(\mathcal{X})$ with compact support satisfying $\int_{\mathcal{X}} \phi = 0$, we have*

$$\int_{\mathcal{X}} (T_\alpha \phi)(x)dx = 0. \tag{1.6}$$

Then for $0 < \alpha < \min\{\varepsilon, \epsilon_1\}/n$ and for all p, q with $n/(n+1) < p < q \le 1$ and $1/p - 1/q = \alpha$, T_α is bounded from $H^{p,\infty}(\mathcal{X})$ to $H^{q,\infty}(\mathcal{X})$. Moreover, there exists a constant C, independent of f and λ, such that for each $\lambda > 0$,

$$\mu(\{x : |(\mathcal{M}_0 T_\alpha f)(x)| > \lambda\}) \le C\left(\frac{\|f\|_{H^{p,\infty}(\mathcal{X})}}{\lambda}\right)^q.$$

Remark 1.2. (i) The condition (1.6) is a natural condition to guarantee the $(H^p(\mathcal{X}), H^q(\mathcal{X}))$ boundedness of fractional integral operator T_α. Indeed, if ϕ is an (p, p_0) atom, then (1.6) is just the required cancelation condition for $T(\phi)$ to be a "molecule". In

the classical case when $\mathcal{X} = \mathbb{R}^n$, the condition (1.6) is satisfied for the Riesz potential operator (see, for instance, [16, Chapter 3]).
(ii) In Theorems 1.3 and 1.4, if $d(y, y') \leq d(x, y)/c$ for some $c > 1$, (1.5) holds, then the conclusions of Theorems 1.3 and 1.4 remain true.
(iii) When $p = 1$, the result of Theorem 1.3 was obtained in [6]. Thus Theorem 1.3 can be regarded as an extension of the result in [6]. The conclusion of Theorem 1.4 is new.

The following of the paper is arranged as follows. Section 2 is devoted to some preliminary lemmas. The proofs of the theorems are presented in Section 3. Finally, in Section 4, we give applications to the Nagel-Stein's singular integrals and fractional integrals.

2. Some Lemmas

In this section, we give some lemmas that will be used to prove the theorems.

The following result was independently founded by Stein-Taibleson-Weiss [23] and by Kalton [13].

Lemma 2.1. *Let g_k be a sequence of measurable functions and let $0 < r < 1$. Assume that $|\{|g_k| > \lambda\}| \leq C/\lambda^r$ with C independent of k and λ. Then, for every numerical sequence $\{c_k\}$ in l^r we have*

$$\left| \left\{ x : \left| \sum_k c_k g_k(x) \right| > \lambda \right\} \right| \leq \frac{2-r}{1-r} \frac{C}{\lambda^r} \sum_k |c_k|^r.$$

The following lemma is the Whitney decomposition theorem in homogeneous spaces $\mathcal{X}$ ([22]).

Lemma 2.2. *Let Ω be an open proper subset of $\mathcal{X}$ and let $d(x) = \inf\{d(x, y) : y \notin \Omega\}$. Let $r(x) = d(x)/30$. Then there exist a positive number L depending on C_3, τ, n, but independent of Ω, and a sequence $\{x_k\}_k$ such that if we denote $r(x_k)$ by r_k, then*

(i) *$B(x_k, r_k/4)$ are pairwise disjoint;*

(ii) *$\cup_k B(x_k, r_k) = \Omega$;*

(iii) *for every given k, $B(x_k, 15r_k) \subset \Omega$;*

(iv) *for every given k, $x \in B(x_k, 15r_k)$ implies that $15r_k < d(x) < 45r_k$;*

(v) *for every given k, there exists a $y_k \notin \Omega$ such that $d(x_k, y_k) < 45r_k$;*

(vi) *$\{B(x_k, 13\tau^2 r_k)\}_{k=1}^{\infty}$ have bounded overlap, that is, for every given k, the number of balls $B(x_i, 13\tau^2 r_i)$ whose intersections with the ball $B(x_k, 13\tau^2 r_k)$ are non-empty is at most L.*

From Lemma 2.2, we can construct the following partition of unity in homogenous spaces $\mathcal{X}$.

Lemma 2.3. *Let Ω be an open subset of $\mathcal{X}$ with finite measure. Consider the sequence $\{x_k\}_k$ and $\{r_k\}_k$ given in Lemma 2.2. Then there exist non-negative functions $\{\varphi_k\}_k$ satisfying:*

(i) *for any given k, $0 \leq \varphi_k \leq 1$, supp $\varphi_k \subset B(x_k, 2r_k)$ and $\sum_k \varphi_k = \chi_\Omega$;*

(ii) *for any given k and $x \in B(x_k, r_k)$, $\varphi_k(x) \geq 1/C$, where C is a positive constant depending only on C_3, but independent of Ω;*

(iii) *there exists a positive constant C independent of Ω such that for all k and all $\epsilon \in (0, 1]$,*

$$\|\varphi_k\|_{\mathcal{G}(x_k, r_k, \epsilon, \epsilon)} \leq C\mu(B(x_k, r_k)).$$

In this case, we say that $\{\varphi_k\}_k$ are "bump functions" associated with $\{B(x_k, r_k)\}_k$.

Lemma 2.4. *Let $\delta > 0$, $0 < a \leq 1$, $0 < \alpha < a/n$, then there is a constant C depending only on C_1 and C_3 such that*

$$\int_{d(x,y)\geq\delta} \frac{1}{V(x,y)^{1-\alpha} d(x,y)^a} dy \leq C\mu(B(x,\delta))^\alpha \delta^{-a}.$$

Indeed, by (1.1) and (1.3),

$$\begin{aligned}
&\int_{d(x,y)\geq\delta} \frac{1}{V(x,y)^{1-\alpha} d(x,y)^a} dy \\
=&\sum_{j=0}^{\infty} \int_{2^j\delta\leq d(x,y)<2^{j+1}\delta} \frac{1}{V(x,y)^{1-\alpha} d(x,y)^a} dy \\
\leq&\sum_{j=0}^{\infty} \frac{1}{[\mu(B(x,2^j\delta))]^{1-\alpha}\,(2^j\delta)^a} \cdot \mu(B(x, 2^{j+1}\delta)) \\
\leq&C_1C_3 \sum_{j=0}^{\infty} 2^{-j(a-\alpha n)} (\mu(B(x,\delta)))^\alpha \delta^{-a} \\
\leq&C(\mu(B(x,\delta)))^\alpha \delta^{-a}.
\end{aligned}$$

3. Proof of the Theorems

This section is devoted to the proofs of the theorems.

3.1. Proof of Theorems 1.1

For $k \in \mathbb{Z}$, we set $\Omega_k = \{x \in \mathcal{X} : f^*(x) > 2^k\}$. Then for any $k \in \mathbb{Z}$, Ω_k is a proper open subset of $\mathcal{X}$ with $\mu(\Omega_k) \leq C2^{-kp}\|f\|^p_{H^{p,\infty}(\mathcal{X})} < \infty$. Let

$\{B_i^k\}_{i=1}^\infty = \{B(x_i^k, r_i^k)\}_{i=1}^\infty$ be the Whitney decomposition of Ω_k, and let φ_i^k be the "bump functions" associated to B_i^k in the sense of Lemmas 2.2 and 2.3. For each $k \in \mathbb{Z}$, define $d_k(x) = \inf\{d(x,y) : \ y \notin \Omega_k\}$. Denote

$$m_i^k = \frac{1}{\int_{\mathcal{X}} \varphi_i^k} \int_{\mathcal{X}} f \varphi_i^k.$$

We decompose f as

$$f(x) = \left(f(x)\chi_{\Omega_k^c}(x) + \sum_{i=1}^\infty m_i^k \varphi_i^k(x) \right) + \sum_{i=1}^\infty (f(x) - m_i^k)\varphi_i^k(x),$$

where and in what follows, we use A^c to denote the complement of the set A in $\mathcal{X}$. Denote

$$g_k(x) := \left(f(x)\chi_{\Omega_k^c}(x) + \sum_{i=1}^\infty m_i^k \varphi_i^k(x) \right).$$

Clearly,

$$|f(x)\chi_{\Omega_k^c}(x)| \le Cf^*(x)\chi_{\Omega_k^c}(x) \le C2^k. \tag{3.1}$$

By (v) in Lemma 2.2, there exist $y_k \in \Omega_k^c$

$$|m_i^k| \le Cf^*(y_k) \le C2^k. \tag{3.2}$$

Thus $|g_k(x)| \le C2^k$ for all $x \in \mathcal{X}$. Therefore, we have the uniform convergence,

$$\lim_{k \to -\infty} g_k(x) = 0. \tag{3.3}$$

On the other hand, noticing that $\mu(\Omega_k) = O(2^{-kp}) \to 0$, as $k \to \infty$, we obtain

$$\lim_{k \to \infty} g_k(x) = f(x), \ a.e.. \tag{3.4}$$

By (3.3) and (3.4), we can write

$$f = \sum_{k=-\infty}^\infty g_{k+1} - g_k := \sum_{k=-\infty}^\infty f_k, \quad \text{a.e..}$$

One can check

$$\begin{aligned} f_k =& \sum_{i=1}^\infty \left[(f - m_i^k)\varphi_i^k - \sum_{j=1}^\infty (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1} \right] \\ &+ \sum_{j=1}^\infty \left[\sum_{i=1}^\infty (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1} - (f - m_j^{k+1})\varphi_j^{k+1} \right], \end{aligned}$$

where all the series converges in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and

$$m_{ij}^{k+1} = \frac{1}{\int \varphi_i^k \varphi_j^{k+1}} \int f\varphi_i^k\varphi_j^{k+1}.$$

Let $\beta_i^k = (f-m_i^k)\varphi_i^k - \sum_{j=1}^\infty (f-m_{ij}^{k+1})\varphi_i^k\varphi_j^{k+1}$ and $\gamma_j^{k+1} = \sum_{i=1}^\infty (f-m_{ij}^{k+1})\varphi_i^k\varphi_j^{k+1}$ $-(f-m_j^{k+1})\varphi_j^{k+1}$. Denote $\widetilde{B}_i^k := B(x_i^k, 13\tau^2 r_i^k)$, where τ is the constant appearing in the triangle inequality (1.1). Then by Lemma 2.2 (vi), we know that, for each $k \in \mathbb{Z}$, $\{\widetilde{B}_i^k\}_i$ has bounded overlap. Clearly, $\mathrm{supp}\beta_i^k \subset B(x_i^k, 2r_i^k) \subset \widetilde{B}_i^k$. Now we claim that for each $j \in \mathbb{Z}$, there exists an $i \in \mathbb{Z}$ such that $\mathrm{supp}\gamma_j^k \subset \widetilde{B}_i^k$. Indeed,

$$B(x_j^{k+1}, 2r_j^{k+1}) \subset \Omega_{k+1} \subset \Omega_k = \bigcup_{k=1}^\infty B(x_i^k, r_i^k).$$

Thus there exists $B(x_i^k, r_i^k) = B(x_{i_j}^k, r_{i_j}^k)$ such that $B(x_i^k, r_i^k) \cap B(x_j^{k+1}, 2r_j^{k+1}) \neq \emptyset$. Then for any $x \in B(x_j^{k+1}, 2r_j^{k+1})$ and any $y \in B(x_i^k, r_i^k) \cap B(x_j^{k+1}, 2r_j^{k+1})$, by Lemma 2.2 (iv) and $d_{k+1}(y) \le d_k(y)$,

$$\begin{aligned} d(x, x_i^k) &\le \tau^2[d(x, x_j^{k+1}) + d(x_j^{k+1}, y) + d(y, x_i^k)] \le \tau^2[4r_j^{k+1} + r_i^k] \\ &\le \tau^2[(4/15)d_k(y) + r_i^k] \le 13\tau^2 r_i^k. \end{aligned}$$

Therefore

$$\mathrm{supp}\gamma_j^k \subset B(x_j^{k+1}, 2r_j^{k+1}) \subset \widetilde{B}_i^k,$$

which verifies the claim. Denote $\widetilde{\gamma}_i^k = \gamma_j^k$ so that $\mathrm{supp}\widetilde{\gamma}_i^k \subset \widetilde{B}_i^k$.

Next, by (3.1), (3.2) and noticing that $\{\widetilde{B}_j^{k+1}\}_{j=1}^\infty$ have bounded overlap, we have

$$\begin{aligned} |\beta_i^k| &= |(f-m_i^k)\varphi_i^k - \sum_{j=1}^\infty (f-m_{ij}^{k+1})\varphi_i^k\varphi_j^{k+1}| \\ &\le |f\varphi_i^k \chi_{\Omega_{k+1}^c}| + |m_i^k|\varphi_i^k + \sum_{j=1}^\infty |m_{ij}^{k+1}|\varphi_i^k\varphi_j^{k+1} \\ &\le C2^k. \end{aligned}$$

Similarly, $|\widetilde{\gamma}_j^k| \le C2^k$. Obviously,

$$\int_{\mathcal{X}} \beta_i^k(x)dx = 0 = \int_{\mathcal{X}} \widetilde{\gamma}_i^k(x)dx.$$

Define $h_i^k = \beta_i^k + \widetilde{\gamma}_i^k$, then $f_k = \sum_{i=1}^\infty h_i^k$ and the convergence in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ can be verified as in [5]. Thus, conclusions (a), (i) and (ii) of (b) in Theorem 1.1 have been proved to hold.

Finally, since $f \in H^{p,\infty}$ and $\{B_i^k\}$ have the bounded overlap, by (1.2),

$$\sum_{i=1}^{\infty} \mu(\widetilde{B}_i^k) \lesssim \sum_{i=1}^{\infty} \mu(B_i^k) \lesssim \mu(\Omega_k) \lesssim 2^{-kp} \|f\|_{H^{p,\infty}(\mathcal{X})}^p,$$

which verifies (iii) of (b). Thus we finish the construction of the atomic decomposition.

For the converse, we fix $\alpha > 0$, and choose k_0 so that $2^{k_0} \le \alpha < 2^{k_0+1}$. Write

$$f = \sum_{k=-\infty}^{k_0-1} f_k + \sum_{k=k_0}^{\infty} f_k = F_1 + F_2.$$

Now since

$$\mathcal{M}_0(F_1)(x) \le \sum_{k=-\infty}^{k_0-1} \mathcal{M}_0(f_k)(x) \le C \sum_{k=-\infty}^{k_0-1} 2^k \le C_3\alpha,$$

and $\mu(\{x \in \mathcal{X}) : \mathcal{M}_0(F_1)(x) > C_3\alpha\}) = 0$, we have

$$\mu(\{x \in \mathcal{X} : \mathcal{M}_0(f)(x) > (C_3+1)\alpha\}) \le \mu(\{x \in \mathcal{X} : \mathcal{M}_0(F_2)(x) > \alpha\}).$$

Set

$$A_{k_0} = \bigcup_{k=k_0}^{\infty} \bigcup_{i\ge 1} 3\tau B_i^k,$$

where $3\tau B_i^k$ denotes the ball with radii of $3r_i^k$ centered at x_i^k. By (1.3),

$$\mu(A_{k_0}) \le (3\tau)^n C_0 2^{-k_0} \le C/\alpha^p.$$

Therefore it suffices to verify

$$I = \mu(\{x \notin A_{k_0} : \mathcal{M}_0(F_2)(x) > \alpha\}) \le C/\alpha^p. \tag{3.5}$$

Note that for $x \notin 3\tau B_i^k$ and $y \in B_i^k$, we have

$$d(x,y) \ge \frac{1}{\tau} d(x, x_i^k) - d(y, x_i^k) \ge 2d(y, x_i^k).$$

Thus

$$|S_j(x,y) - S_j(x, x_i^k)| \lesssim \frac{d(y, x_i^k)^{\epsilon_1}}{d(x,y)^{\epsilon_1} V(x,y)}.$$

Hence by the cancellation condition of h_{i_k},

$$\begin{aligned}
\mathcal{M}_0(h_i^k)(x) =& \sup_j \left| \int [S_j(x,y) - S_j(x, x_i^k)] h_i^k(y) dy \right| \\
\le& C2^k \frac{\mu(B_i^k) d(y, x_i^k)^{\epsilon_1}}{V(x,y) d(x,y)^{\epsilon_1}} \\
\le& C2^k \frac{\mu(B_i^k)(r_i^k)^{\epsilon_1}}{\mu(B(x_i^k, d(x, x_i^k))) d(x, x_i^k)^{\epsilon_1}}.
\end{aligned}$$

By (1.4)

$$\mu(B(x_i^k, d(x, x_i^k))) \leq \left(\frac{d(x, x_i^k)}{r_i^k}\right)^n \mu(B_i^k).$$

Therefore,

$$\mathcal{M}_0(h_i^k)(x) \leq C 2^k \frac{\mu(B_i^k)^{1+\frac{\epsilon_1}{n}}}{V(x, x_i^k)^{1+\frac{\epsilon_1}{n}}}.$$

Now applying lemma 2.1 with $g_{ki} = V(x, x_i^k)^{-1-\frac{\epsilon_1}{n}}$, $r = (1 + \frac{\epsilon_1}{n})^{-1}$, and $c_{ki} = 2^k \mu(B_i^k)^{1+\frac{\epsilon_1}{n}}$, we obtain

$$I \leq \frac{C_{\epsilon_1, n}}{\alpha^r} \sum_{k \geq k_0} \sum_i 2^{kr} \mu(B_i^k) \leq C_0 \frac{C_{\epsilon_1, n}}{\alpha^r} \sum_{k \geq k_0} 2^{kr} 2^{-kp}.$$

Now since $p > n/(n+\epsilon) > r$ (see Remark 1.1), the last series converges and bounded by

$$C_0 \frac{C_{\epsilon_1, n}}{\alpha^q} 2^{-k_0(p-r)} = C/\alpha^p,$$

where C is independent of α. This complete the proof of Theorem 1.1. ■

3.2. Proof of Theorem 1.2

For every $f \in H^{p,\infty}(\mathcal{X})$, and $\lambda > 0$, we need to prove that

$$\mu(\{x \in \mathcal{X} : (Tf)^*(x) > \lambda\}) \leq C \lambda^{-p} \|f\|^p_{H^{p,\infty}(\mathcal{X})},$$

with constant C independent of f and λ.

Pick $k_0 \in \mathbb{Z}$, such that $2^{k_0} \leq \lambda < 2^{k_0+1}$. By the atomic decomposition of $H^{p,\infty}(\mathcal{X})$, write f as $f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k := F_1 + F_2$. Noticing that $p_0 > 1$, we have

$$\begin{aligned}
\|F_1\|_{L^{p_0}(\mathcal{X})} &\leq C \sum_{k=-\infty}^{k_0} \|f_k\|_{L^{p_0}(\mathcal{X})} \\
&\leq C \sum_{k=-\infty}^{k_0} 2^k \left(\sum_i \mu(B_i^k)\right)^{1/p_0} \\
&\leq C \|f\|^{p/p_0}_{H^{p,\infty}(\mathcal{X})} \sum_{k=-\infty}^{k_0} 2^{k(1-p/p_0)} \\
&\leq C \|f\|^{p/p_0}_{H^{p,\infty}(\mathcal{X})} 2^{k_0(1-p/p_0)}.
\end{aligned}$$

Then

$$\begin{aligned}\mu(\{x \in \mathcal{X} : (TF_1)^*(x) > \lambda\}) &\leq \lambda^{-p_0} \|(TF_1)^*\|^{p_0}_{L^{p_0}(\mathcal{X})} \\ &\leq C\lambda^{-p_0} \|TF_1\|^{p_0}_{L^{p_0}(\mathcal{X})} \\ &\leq C\lambda^{-p_0} \|F_1\|^{p_0}_{L^{p_0}(\mathcal{X})} \\ &\leq C\lambda^{-p_0} \|f\|^p_{H^{p,\infty}(\mathcal{X})} 2^{k_0(p_0-p)} \\ &\leq C\lambda^{-p_0} \|f\|^p_{H^{p,\infty}(\mathcal{X})} \lambda^{p_0-p} \\ &= C\lambda^{-p} \|f\|^p_{H^{p,\infty}(\mathcal{X})}.\end{aligned}$$

Thus, to finish the proof of Theorem 1.2, it suffices to show that

$$\mu(\{x \in \mathcal{X} : (TF_2)^*(x) > \lambda\}) \leq C\lambda^{-p} \|f\|^p_{H^{p,\infty}}. \tag{3.6}$$

It is easy to see that for some constant C, $C^{-1}2^{-k}\mu(B_i^k)^{-1/q}h_i^k$ is an $H^{q,\infty}$ atom. Then $f_k \in H^q(\mathcal{X})$, and

$$\|f_k\|^q_{H^q(\mathcal{X})} \leq C \sum_i 2^{kq} \mu(B_i^k) \leq C2^{k(q-p)} \|f\|^p_{H^{p,\infty}(\mathcal{X})}.$$

Since T is bounded on $H^q(\mathcal{X})$, by the grand maximal function characterization of $H^q(\mathcal{X})$ (see Remark 1.1),

$$\mu(\{x \in \mathcal{X} : (Tf_k)^*(x) > \lambda\}) \leq C\lambda^{-q} \|Tf_k\|^q_{H^q(\mathcal{X})} \leq C\lambda^{-q} \|f_k\|^q_{H^q(\mathcal{X})}.$$

Consequently,

$$\mu(\{x \in \mathcal{X} : \left[T(f_k/\|f_k\|_{H^q(\mathcal{X})})\right]^*(x) > \lambda\}) \leq C\lambda^{-q}.$$

Noting that $(TF_2)^*(x) \leq \sum_{k=k_0}^{\infty} (Tf_k)^*(x)$. Then applying Lemma 2.1, we obtain

$$\begin{aligned}&\mu(\{x \in \mathcal{X} : (TF_2)^*(x) > \lambda\}) \\ &\leq \mu(\{x \in \mathcal{X} : \sum_{k=k_0+1}^{\infty} \|f_k\|_{H^q(\mathcal{X})} \cdot [T(f_k/\|f_k\|_{H^q(\mathcal{X})})]^*(x) > \lambda\}) \\ &\leq \frac{2-q}{1-q} \frac{1}{\lambda^q} \sum_{k=k_0+1}^{\infty} \|f_k\|^q_{H^q(\mathcal{X})} \\ &\leq \frac{C\|f\|^p_{H^{p,\infty}(\mathcal{X})}}{\lambda^q} \sum_{k=k_0}^{\infty} 2^{k(q-p)} \\ &\leq C2^{k_0(q-p)} \|f\|^p_{H^{p,\infty}(\mathcal{X})}/\lambda^q \\ &\leq C\lambda^{-p} \|f\|^p_{H^{p,\infty}(\mathcal{X})},\end{aligned}$$

which verifies (3.6). This completes the proof of Theorem 1.2. ∎

3.3. Proof of Theorem 1.3

Fix λ. Set $\eta = \lambda^{q/p}\|f\|^{1-q/p}_{H^{p,\infty}(\mathcal{X})}$. Take $\bar{k}_0 \in \mathbb{Z}$ such that $2^{\bar{k}_0} \le \eta < 2^{\bar{k}_0+1}$. Split f into two parts

$$f = \sum_{k=-\infty}^{\bar{k}_0} f_k + \sum_{k=\bar{k}_0+1}^{\infty} f_k := F_3 + F_4.$$

From the atomic decomposition of f, it follows that

$$\begin{aligned}\|F_3\|_{L^{p_0}(\mathcal{X})} \le& \sum_{k=-\infty}^{\bar{k}_0} \|f_k\|_{L^{p_0}(\mathcal{X})}\\ \le& C\sum_{k=-\infty}^{\bar{k}_0} 2^k \left(\sum_i \mu(B_i^k)\right)^{1/p_0}\\ \le& C\|f\|^{p/p_0}_{H^{p,\infty}(\mathcal{X})} \sum_{k=-\infty}^{\bar{k}_0} 2^{k(1-p/p_0)}\\ \le& C\|f\|^{p/p_0}_{H^{p,\infty}(\mathcal{X})}\eta^{1-p/p_0}\\ =& C\lambda^{1-q/q_0}\|f\|^{q/q_0}_{H^{p,\infty}(\mathcal{X})}.\end{aligned} \tag{3.7}$$

By the $L^{p_0}(\mathcal{X}) - L^{q_0}(\mathcal{X})$ boundedness of T_α and (3.7),

$$\begin{aligned}&\mu(\{x \in \mathcal{X} : |T_\alpha F_3(x)| > \lambda\})\\ &\le c\lambda^{-q_0}\|T_\alpha F_3\|^{q_0}_{L^{q_0}(\mathcal{X})}\\ &\le c\lambda^{-q_0}\|F_3\|^{q_0}_{L^{p_0}(\mathcal{X})}\\ &\le c\lambda^{-q_0}\left(\lambda^{1-q/q_0}\|f\|^{q/q_0}_{H^{p,\infty}(\mathcal{X})}\right)^{q_0}\\ &= C\left(\frac{\|f\|_{H^{p,\infty}}}{\lambda}\right)^q.\end{aligned}$$

Let $\widehat{B_i^k} = 3\tau B_i^k$ and $E_{\bar{k}_0} = \bigcup_{k=\bar{k}_0+1}^{\infty}\bigcup_i \widehat{B_i^k}$. By Theorem 1.1,

$$\begin{aligned}\mu(E_{\bar{k}_0}) &\le C\sum_{k=\bar{k}_0+1}^{\infty}\sum_i \mu(B_i^k) \le C\|f\|^p_{H^{p,\infty}(\mathcal{X})}\sum_{k=\bar{k}_0+1}^{\infty} 2^{-kp}\\ &\le C\|f\|^p_{H^{p,\infty}(\mathcal{X})}\eta^{-p} \le C\left(\frac{\|f\|^p_{H^{p,\infty}}}{\lambda}\right)^q.\end{aligned} \tag{3.8}$$

Thus, to finish the proof, it suffices to show

$$\mu(\{x \in E^c_{\bar{k}_0} : |T_\alpha F_4(x)| > \lambda\}) \le C\left(\frac{\|f\|_{H^{p,\infty}}}{\lambda}\right)^q. \tag{3.9}$$

By the use of cancellation condition of h_i^k, Minkowski's inequality and (1.5),

$$
\begin{aligned}
&\mu(\{x \in E_{\bar{k}_0}^c : |T_\alpha F_4(x)| > \lambda\}) \\
&\leq \lambda^{-1} \int_{E_{\bar{k}_0}^c} |T_\alpha F_4(x)| dx \\
&\leq \lambda^{-1} \sum_{k=\bar{k}_0+1}^{\infty} \sum_{i=0}^{\infty} \int_{B_i^k} |h_i^k(y)| \int_{E_{\bar{k}_0}^c} |K(x,y) - K(x,x_i^k)| dx dy.
\end{aligned}
$$

Note that if $x \in E_{\bar{k}_0}^c$ and $y \in B_i^k$, then by (1.1),

$$
d(x,y) \geq \frac{1}{\tau} d(x,x_i^k) - d(x_i^k,y) \geq 2d(x_i^k,y).
$$

Thus by Lemma 2.4, we have

$$
\begin{aligned}
&\mu(\{x \in E_{\bar{k}_0}^c : |T_\alpha F_4(x)| > \lambda\}) \\
\leq& C\lambda^{-1} \sum_{k=\bar{k}_0+1}^{\infty} \sum_{i=0}^{\infty} \int_{B_i^k} |h_i^k(y)| \int_{E_{\bar{k}_0}^c} \frac{d(y,x_i^k)^\varepsilon}{V(x,y)^{1-\alpha} d(x,y)^\varepsilon} dx dy \\
\leq& C\lambda^{-1} \sum_{k=\bar{k}_0+1}^{\infty} 2^k \sum_{i=0}^{\infty} \mu(B_i^k)^{1+\alpha} \\
\leq& C\lambda^{-1} \sum_{k=\bar{k}_0+1}^{\infty} 2^k \left(\sum_{i=0}^{\infty} \mu(B_i^k) \right)^{1+\alpha} \\
\leq& C\lambda^{-1} \|f\|_{H^{p,\infty}}^{p(1+\alpha)} \sum_{k=\bar{k}_0+1}^{\infty} 2^{k[1-p(1+\alpha)]} \\
\leq& C\lambda^{-1} \|f\|_{H^{p,\infty}}^{p(1+\alpha)} \eta^{1-p(1+\alpha)} \\
=& C\lambda^{-1} \|f\|_{H^{p,\infty}}^{p(1+\alpha)} \left(\lambda^\theta \|f\|_{H^{p,\infty}}^{1-\theta} \right)^{1-p(1+\alpha)} \\
=& C \left(\frac{\|f\|_{H^{p,\infty}}}{\lambda} \right)^q,
\end{aligned}
$$

which gives (3.9). Thus the proof of Theorem 1.3 is completed. ■

3.4. Proof of Theorem 1.4

Since $f \in H^{p,\infty}(\mathcal{X})$, $\mathcal{M}_0(f) \in L^{p,\infty}(\mathcal{X})$. To prove the theorem, it suffices to show

$$
\mu(\{x \in \mathcal{X} : \mathcal{M}_0(T_\alpha f)(x) > \lambda\}) \leq C(\|f\|_{H^{p,\infty}(\mathcal{X})}/\lambda)^q, \quad \text{for any } \lambda > 0. \tag{3.10}
$$

Let k_0, η, F_3 and F_4 be defined as in the proof of Theorem 1.3. For any $\lambda > 0$, applying the same argument as in the proof of (3.7), we can get

$$\|F_3\|_{L^{p_0}(\mathcal{X})} \lesssim \lambda^{1-\frac{q}{q_0}} \|f\|_{H^{p,\infty}(\mathcal{X})}^{\frac{q}{q_0}}.$$

By the $L^{q_0}(\mathcal{X})$ boundedness of $\mathcal{M}_0$, and the (L^{p_0}, L^{q_0}) boundedness of T_α, we have

$$\begin{aligned}\mu(\{x \in \mathcal{X} : \mathcal{M}_0(T_\alpha F_3)(x) > \lambda\}) &\le C(\|\mathcal{M}_0(T_\alpha F_3)\|_{L^{q_0}}/\lambda)^{q_0} \\ &\le C(\|F_3\|_{L^{p_0}}/\lambda)^{q_0} \le C(\|f\|_{H^{p,\infty}}/\lambda)^{q}.\end{aligned}$$

Denote $\overline{B_i^k} = B(x_i^k, 4r_i^k)$ and $E = \cup_{k=k_0}^{\infty} \cup_i \overline{B_i^k}$, then similar argument as in the proof of (3.8) yields, $\mu(E) \le C(\|f\|_{H^{p,\infty}}/\lambda)^q$. Thus, to finish the proof, it remains to be verified that

$$\mu(\{x \in E^c : \mathcal{M}_0(T_\alpha F_4)(x) > \lambda\}) \le C(\|f\|_{H^{p,\infty}(\mathcal{X})}/\lambda)^q. \tag{3.11}$$

Since $h_i^k \in L^{p_0}(\mathcal{X})$ has compact support with $\int_{\mathcal{X}} h_i^k(x)dx = 0$, by the cancellation condition (1.6),

$$\int_{\mathcal{X}} T_\alpha(h_i^k)(x)dx = 0. \tag{3.12}$$

Thus, for any $m \in \mathbb{Z}$ and for any $x \in E^c$, by the cancellation condition for h_i^k and (3.12), we have

$$\begin{aligned}&|S_m(T_\alpha h_i^k)(x)| \\ =&\left|\int_{\mathcal{X}} (T_\alpha h_i^k)(y)[S_m(x,y) - S_m(x,x_i^k)]dy\right| \\ \le& \int_{\widetilde{B_i^k}} |(T_\alpha h_i^k)(y)| \cdot |S_m(x,y) - S_m(x,x_i^k)|dy \\ &+ \int_{2r_i^k \le d(y,x_i^k) < d(x,x_i^k)/2} \left(\int_{B_i^k} |h_i^k(v)||K_\alpha(y,v) - K_\alpha(y,x_i^k)|dv\right) \\ &\quad \cdot |S_m(x,y) - S_m(x,x_i^k)|dy \\ &+ \int_{d(y,x_i^k) \ge d(x,x_i^k)/2} \left(\int_{B_i^k} |h_i^k(v)||K_\alpha(y,v) - K_\alpha(y,x_i^k)|dv\right) \\ &\quad (|S_m(x,y)| + |S_m(x,x_i^k)|)dy \\ :=& J_1 + J_2 + J_3.\end{aligned}$$

We first give the estimate for J_1. By the definition of S_m,

$$
\begin{aligned}
J_1 &\le \frac{C}{V(x,x_i^k)d(x,x_i^k)^{\epsilon_1}} \int_{\widetilde{B_i^k}} (T_\alpha h_i^k)(y) d(y,x_i^k)^{\epsilon_1} dy \\
&\le \frac{C(r_i^k)^{\epsilon_1}}{V(x,x_i^k)d(x,x_i^k)^{\epsilon_1}} \|T_\alpha h_i^k\|_{L^{q_1}(\mathcal{X})} \mu(B_i^k)^{1-\frac{1}{q_1}} \\
&\le C2^k \mu(B_i^k)^{1+\alpha} \frac{(r_i^k)^{\epsilon_1}}{V(x,x_i^k)d(x,x_i^k)^{\epsilon_1}}.
\end{aligned} \tag{3.13}
$$

Next, we estimate J_2. Since $d(y,x_i^k) \ge 2r_i^k > 2d(v,x_i^k)$, by (1.5) and Definition 1.2,

$$
\begin{aligned}
J_2 &\le C2^k \mu(B_i^k) \int_{2r_i^k \le d(y,x_i^k) < d(x,x_i^k)/2} \frac{1}{V(y,x_i^k)^{1-\alpha}} \cdot \frac{(r_i^k)^{\epsilon_1}}{V(x,x_i^k)d(x,x_i^k)^{\epsilon_1}} dy \\
&\le \frac{C2^k \mu(B_i^k)(r_i^k)^{\epsilon_1}}{V(x,x_i^k)^{1-\alpha} d(x,x_i^k)^{\epsilon_1}}.
\end{aligned} \tag{3.14}
$$

As for J_3, noting that $x \in E^c$ and $v \in B_i^k$ imply $d(y,x_i^k) > d(x,x_i^k)/2 \ge 2d(v,x_i^k)$. By (1.5),

$$
\begin{aligned}
J_3 \le& C \int_{d(y,x_i^k) \ge d(x,x_i^k)/2} \left(\int_{B_i^k} \frac{|h_i^k(v)| d(v,x_i^k)^\varepsilon}{V(y,x_i^k)^{1-\alpha} d(y,x_i^k)^\varepsilon} dv \right) \\
&\cdot \left(|S_m(x,y)| + \frac{1}{V(x,x_i^k)} \right) dy \\
\le& \frac{C2^k \mu(B_i^k)(r_i^k)^\varepsilon}{V(x,x_i^k)^{1-\alpha} d(x,x_i^k)^\varepsilon} \|S_m(x,\cdot)\|_{L^1(\mathcal{X})} \\
&+ \frac{C2^k \mu(B_i^k)(r_i^k)^\varepsilon}{V(x,x_i^k)} \int_{d(y,x_i^k) \ge d(x,x_i^k)/2} \frac{1}{V(y,x_i^k)^{1-\alpha} d(y,x_i^k)^\varepsilon} dy \\
\le& \frac{C2^k \mu(B_i^k)(r_i^k)^\varepsilon}{V(x,x_i^k)^{1-\alpha} d(x,x_i^k)^\varepsilon},
\end{aligned} \tag{3.15}
$$

where in the last inequality, we use (1.4) and Lemma 2.4.

Combining the estimates in (3.13), (3.14) and (3.15) yields

$$
|S_m(T_\alpha h_i^k)(x)| \le \frac{C2^k \mu(B_i^k)(r_i^k)^{\epsilon'}}{V(x,x_i^k)^{1-\alpha} d(x,x_i^k)^{\epsilon'}} \le \frac{C2^k \mu(B_i^k)^{1+\frac{\epsilon'}{n}}}{V(x,x_i^k)^{1+\frac{\epsilon'}{n}-\alpha}}, \quad \text{for any } x \in E^c,
$$

where $\epsilon' = \min\{\epsilon_1, \varepsilon\}$. Thus,

$$
\mathcal{M}_0(T_\alpha h_i^k)(x) \le \frac{C2^k \mu(B_i^k)^{1+\frac{\epsilon'}{n}}}{V(x,x_i^k)^{1+\frac{\epsilon'}{n}-\alpha}}, \quad \text{for any } x \in E^c.
$$

Denote $C_i^k = C2^k\mu(B_i^k)^{1+\frac{\epsilon'}{n}}$ and $g_i^k(x) = V(x, x_i^k)^{-(1+\frac{\epsilon'}{n}-\alpha)}$. Then

$$\mu(\{x \in E^c : g_i^k(x) > \lambda\}) = \mu(\{x \in E^c : V(x, x_i^k)^{-(1+\frac{\epsilon'}{n}-\alpha)} > \lambda\}) \lesssim \lambda^{-\frac{1}{1+\frac{\epsilon'}{n}-\alpha}}.$$

Since $\alpha < \epsilon'/n$, applying Lemma 2.1 with $r = 1/(1 + \frac{\epsilon'}{n} - \alpha)$, we obtain

$$\begin{aligned}\mu(\{x \in E^c : \mathcal{M}_\sigma(T_\alpha F_4)(x) > \lambda\}) &\le \mu(\{x \in E^c : \sum_{k=k_0}^{\infty}\sum_i C_i^k g_i^k(x) > \lambda\}) \\ &\le C\lambda^{-r}\sum_{k=k_0}^{\infty}\sum_i (2^k\mu(B_i^k)^{1+\frac{\epsilon'}{n}})^r \\ &\le C\lambda^{-r}\sum_{k=k_0}^{\infty} 2^{kr}\left[\sum_i \mu(B_i^k)\right]^{1+\frac{\epsilon'}{n}r} \\ &\le C(\|f\|_{H^{p,\infty}(\mathcal{X})}/\lambda)^q.\end{aligned}$$

This complete the proof of Theorem 1.4. ∎

4. Applications to the Nagel-stein Singular Integrals and Fractional Integrals

In this section, we give applications of the theorems. Let M be a boundary of an unbounded model domain of polynomial type in $\mathbb{C}^2$, which appears in estimates for solutions of the Kohn-Laplacian; see [2, 17, 14, 19, 20]. More precisely, let $\Omega = \{(z, w) \in \mathbb{C}^2 : \Im m[w] > P(z)\}$, where P is a real, subharmonic, non-harmonic polynomial of degree m. Then $M = \partial\Omega$ can be identified with $\mathbb{C} \times \mathbb{R}$. There are real vector fields $\{X_1, X_2\}$ and their commutators of orders $\le m$ span the tangent space at each point. If we endow M with the control distance d and the Lebesgue measure μ, then M is a space of homogenous type and μ satisfy (1.3) with $n = m + 2$ and $\kappa = 4$ (see [18]).

In [19], Nagel and Stein considered a singular integral operator $\tilde{T}$ on M. The operator $\tilde{T}$ initially is given as a map from $C_0^\infty(M)$ to $C^\infty(M)$, whose distribution kernel $\tilde{K}(x, y)$ is C^∞ away from the diagonal of $M \times M$ and the following four properties are supposed to hold:

(I-1) If φ , ψ $\in C_c^\infty(M)$ have disjoint supports, then

$$\langle \tilde{T}\varphi, \psi\rangle = \int_{M\times M} \tilde{K}(x, y)\,\varphi(y)\,\psi(x)\,dx\,dy.$$

(I-2) If φ is a normalized bump function associated to a ball of radius r, then $|\partial_X^a \tilde{T}\varphi| \lesssim r^{-a}$. More precisely, for each integer $a \ge 0$, there is another integer $b \ge 0$ and a constant $M_{a,b}$ so that whenever φ is a C^∞ function supported in a ball $B(x_0, r)$, then

$$\sup_{x\in M} r^a|(\partial_X^a \tilde{T}\varphi)(x)| \le M_{a,b} \sup_{c\le b} \sup_{x\in B(x_0,r)} r^c|\partial_X^c(\varphi)|.$$

(I-3) If $x \neq y$, then for every $a \geq 0$,

$$|\partial_{X,Y}^a \tilde{K}(x,y)| \lesssim d(x,y)^{-a}V(x,y)^{-1}.$$

(I-4) Properties (I-1) through (I-3) also hold with x and y interchanged. That is, these properties also hold for the adjoint operator $\tilde{T}^t$ defined by

$$\langle \widetilde{T}^t\varphi, \psi\rangle = \langle \widetilde{T}\psi, \varphi\rangle.$$

Nagel and Stein [19] proved the $L^p(M)\,(1 < p < \infty)$ boundedness of $\tilde{T}$. The boundedness of $\tilde{T}$ in Hardy spaces $H^q\,(n/(n+\epsilon_1) < q \leq 1)$ was given in [10]. Recently, Ding and the second author [6] proved $(H^{1,\infty}(M), L^{1,\infty}(M))$ boundedness of $\tilde{T}$.

By the results of [19] and [10], applying Theorem 1.3, we obtain

Theorem 4.1. *The Nagel-Stein singular integral operators $\tilde{T}$ are bounded from $H^{p,\infty}(M)$ to itself for $p \in (n/(n+\epsilon_1), 1]$.*

For $0 < \alpha < 1$, we also consider corresponding fractional integral operator $\tilde{T}_\alpha$, which is given by

$$\tilde{T}_\alpha(f)(x) = \int_M \tilde{K}_\alpha(x,y)f(y)dy,$$

where the kernel $\tilde{K}_\alpha(x,y)$ satisfies

$$|\partial_{X,Y}^a \tilde{K}_\alpha(x,y)| \lesssim d(x,y)^{-a}V(x,y)^{-1+\alpha}. \tag{4.1}$$

Note that the smoothness condition (4.1) implies that condition (1.5) holds for $\tilde{K}_\alpha$ with $\varepsilon = 1$. Thus by theorems 1.3 and 1.4, we have

Theorem 4.2. *Let $0 < \alpha < \epsilon_1/n$. For all p, q with $n/(1+n) \leq p \leq 1$ and $1/p - 1/q = \alpha$, $\tilde{T}_\alpha$ is bounded from $H^{p,\infty}(M)$ to $L^{q,\infty}(M)$. If $\tilde{K}_\alpha$ further satisfies the cancelation condition* (3.12)*, then $\tilde{T}_\alpha$ is also bounded from $H^{p,\infty}(M)$ to $H^{q,\infty}(M)$.*

Remark 4.3. The (L^p, L^q) boundedness of $\tilde{T}_\alpha$ can be obtained from a more general result in [7], where the kernel of the fractional integral operator only assumed to satisfies some weak size condition. The $(H^1(\mathcal{X}), L^{\frac{1}{1-\alpha}}(\mathcal{X}))$ and $(H^{1,\infty}(\mathcal{X}), L^{\frac{1}{1-\alpha},\infty}(\mathcal{X}))$ boundedness of $\tilde{T}_\alpha$ were given in [6]. When $p < 1$, the conclusion of Theorem 4.2 is new.

ACKNOWLEDGMENTS

The authors wish to thank the referee for his/her valuable comments, suggestions and corrections, which improve the exposition of the paper.

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