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DIFFERENCES OF WEIGHTED COMPOSITION OPERATORS FROM H^∞ TO BLOCH SPACE

Takuya Hosokawa and Shûichi Ohno

Dedicated to Professor Kei Ji Izuchi on the occasion of his 65th birthday

Abstract. We consider the boundedness and compactness of the differences of two weighted composition operators acting from the Banach space of bounded analytic functions on the open unit disk to the Bloch space. In the sequel, we will present some explicit examples to bring our reseach into forcus.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. We denote by $\mathcal{H}(\mathbb{D})$ the set of analytic functions on $\mathbb{D}$ and by $\mathcal{S}(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. We here consider a problem concerning the operators induced by multiplying an analytic function and by the composition with an analytic self-map of $\mathbb{D}$. More precisely, for a function $u \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$, we define a *weighted composition operator* uC_φ by

$$uC_\varphi f = u \cdot (f \circ \varphi) \quad \text{for } f \in \mathcal{H}(\mathbb{D}).$$

It is clear that uC_φ is linear, but the boundedness and compactness are not obvious on any analytic function space. This operator can be regarded as a generalization of a multiplication operator and a composition operator and appears in the study of dynamical systems. Furthermore the isometries on many analytic function spaces are of the canonical forms of weighted composition operators.

Let $H^\infty = H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on $\mathbb{D}$. Then H^∞ is a Banach algebra with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)|; z \in \mathbb{D}\}.$$

Recall that the Bloch space $\mathcal{B}$ consists of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|\!|f|\!\| = \sup\{(1-|z|^2)|f'(z)|; z \in \mathbb{D}\} < \infty.$$

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Then $\|\!|\cdot|\!\|$ defines a complete semi-norm on $\mathcal{B}$. So $\mathcal{B}$ is a Banach space under the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|\!|f|\!\|$. Note that $H^\infty \subset \mathcal{B}$ and that $\|\!|f|\!\| \leq \|f\|_\infty$ for $f \in H^\infty$. See the books [2], [14] and [16] for a thorough treatment on such classical settings.

Many authors have investigated (weighted) composition operators on various analytic function spaces in these decades. One of recent main subjects is the study of the differences of two weighted composition operators, which was posed originally by Shapiro and Sundberg [15] to consider the topological structure in the space of composition operators on the Hilbert Hardy space. MacCluer, Zhao and the second author [9] studied the topological structure in the case of H^∞ and obtained that for $\varphi, \psi \in \mathcal{S}(\mathbb{D})$, the compactness of $C_\varphi - C_\psi : H^\infty \to H^\infty$ is equivalent to the compactness of $C_\varphi - C_\psi$ acting from $\mathcal{B}$ to H^∞ and also that C_φ and C_ψ are in the same path component of the space of composition operators on H^∞ if and only if $C_\varphi - C_\psi : \mathcal{B} \to H^\infty$ is bounded. So it has been noticed that the topological structure problem on H^∞ is closely related to the boundedness of the differences between H^∞ and $\mathcal{B}$. Then the authors have studied the topological structure of the set of composition operators on Bloch spaces and the compact differences ([5, 6]) and the first author [3] characterized boundedness and compactness of differences of two weighted composition operators on Bloch spaces (also see [4]). Furthermore the authors [7] considered the differences of two weighted composition operators acting from $\mathcal{B}$ to H^∞. There are a lot of researches on the differences of (weighted) composition operators between two analytic function spaces (see [1, 8, 11, 12], for example).

In this article we study the differences of two weighted composition operators acting from H^∞ to $\mathcal{B}$. In the next section we will prepare results to need in the sequel. In Section 3, we characterize the boundedness of the differences of two weighted composition operators acting from H^∞ to $\mathcal{B}$ and give examples showing the difference between the boundedness of $uC_\varphi - vC_\psi : H^\infty \to \mathcal{B}$ and of $uC_\varphi - vC_\psi : \mathcal{B} \to \mathcal{B}$. In Section 4, we will try to characterize the compactness of the differences and present an explicit example.

Throughout the paper, C will stand for positive constants whose values may change from one occurrence to another.

2. Prerequisites

For each $w \in \mathbb{D}$, let α_w be the Möbius transformation of $\mathbb{D}$ defined by

$$\alpha_w(z) = \frac{w - z}{1 - \overline{w}z}. \tag{2.1}$$

For z, w in $\mathbb{D}$, the pseudo-hyperbolic distance $\rho(z, w)$ between z and w is given by

$$\rho(z, w) = |\alpha_w(z)|.$$

We remark that

$$\frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{w}z|^2} = 1 - \rho(z, w)^2.$$

We use the following lemma.

Lemma 2.1. *For any $z, w \in \mathbb{D}$,*

$$\left|1 - \frac{(1-|z|^2)(1-|w|^2)}{(1-\overline{w}z)^2}\right| \leq 3\,\rho(z,w).$$

It is known that for $z, w \in \mathbb{D}$ and $f \in H^\infty$

$$\sup_{\|f\|_\infty \leq 1} \bigl|f(z) - f(w)\bigr| = \frac{2 - 2\sqrt{1-\rho(z,w)^2}}{\rho(z,w)} \leq C\rho(z,w) \tag{2.2}$$

where C is a positive constant. We define a Bloch-type induced distance $\flat$ on $\mathbb{D}$:

$$\flat(z,w) = \sup_{\|\!|f|\!\| \leq 1} \bigl|(1-|z|^2)f'(z) - (1-|w|^2)f'(w)\bigr|. \tag{2.3}$$

In [5], $\flat(z,w)$ is estimated in the following form:

Proposition 2.2. *There exists a positive constant C such that*

$$\rho(z,w)^2 \leq \flat(z,w) \leq C\,\rho(z,w)$$

for any $z, w \in \mathbb{D}$.

Moreover we use the following notation: $\rho(z) = \rho(\varphi(z), \psi(z))$.

We introduce the hyperbolic derivative $\varphi^\#$ of $\varphi \in \mathcal{S}(\mathbb{D})$ which is closely related to the action of the composition operators on the Bloch space ([10]):

$$\varphi^\#(z) = \frac{1-|z|^2}{1-|\varphi(z)|^2}\varphi'(z).$$

The Schwarz-Pick lemma implies that $\|\varphi^\#\|_\infty \leq 1$.

The second author [13] studied the weighted composition operators from H^∞ to $\mathcal{B}$.

Theorem 2.3. *Let $u \in \mathcal{H}(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the followings hold.*

(i) *uC_φ is bounded from H^∞ to $\mathcal{B}$ if and only if $u \in \mathcal{B}$ and $\|u\,\varphi^\#\|_\infty < \infty$.*

(ii) *Suppose uC_φ is bounded from H^∞ to $\mathcal{B}$. Then uC_φ is compact from H^∞ to $\mathcal{B}$ if and only if $(1-|z|^2)\,|u'(z)| \to 0$ and $|u(z)\,\varphi^\#(z)| \to 0$ whenever $|\varphi(z)| \to 1$.*

3. Boundedness to the Bloch Space

In this section, we will consider the boundedness of differences of two weighted composition operators acting from H^∞ to $\mathcal{B}$. To characterize the boundedness, we will need the following assumption

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)|\,\rho(z) < \infty \quad \text{and} \quad \sup_{z\in\mathbb{D}}(1-|z|^2)|v'(z)|\,\rho(z) < \infty. \tag{A}$$

Remark that if $u, v \in \mathcal{B}$, then (A) holds.

Theorem 3.1. *Let u, v be in $\mathcal{H}(\mathbb{D})$ and φ, ψ be in $S(\mathbb{D})$. Assume that condition* (A) *holds. Then $uC_\varphi - vC_\psi$ is bounded from H^∞ to $\mathcal{B}$ if and only if the following conditions hold:*

(i) $u - v \in \mathcal{B}$.

(ii) $\sup_{z\in\mathbb{D}} \big|u(z)\,\varphi^{\#}(z) - v(z)\,\psi^{\#}(z)\big| < \infty$.

(iii) $\sup_{z\in\mathbb{D}} \big|v(z)\,\psi^{\#}(z)\big|\,\rho(z) < \infty$.

It is possible to replace φ with ψ and u with v in condition (iii).

Proof. We suppose that conditions (i), (ii), and (iii) hold. Let f be in H^∞. Then we have the following fundamental and important equalities in this text.

$$\begin{aligned}
&(1-|z|^2)((uC_\varphi - vC_\psi)f)'(z)\\
&= (1-|z|^2)\big(u'(z)\,f(\varphi(z)) - v'(z)\,f(\psi(z))\big)\\
&\quad + (1-|z|^2)\big(u(z)\,\varphi'(z)\,f'(\varphi(z)) - v(z)\,\psi'(z)\,f'(\psi(z))\big)\\
&= (1-|z|^2)\big(u'(z) - v'(z)\big)f(\varphi(z))\\
&\quad + (1-|z|^2)\,v'(z)\big(f(\varphi(z)) - f(\psi(z))\big)\\
&\quad + \big(u(z)\,\varphi^{\#}(z) - v(z)\,\psi^{\#}(z)\big)(1-|\varphi(z)|^2)f'(\varphi(z))\\
&\quad + v(z)\,\psi^{\#}(z)\big((1-|\varphi(z)|^2)f'(\varphi(z)) - (1-|\psi(z)|^2)f'(\psi(z))\big).
\end{aligned}$$

Using (2.2), condition (A) and Proposition 2.2, we obtain

$$\|(uC_\varphi - vC_\psi)f\| \le C\|f\|_\infty$$

and $uC_\varphi - vC_\psi$ is bounded from H^∞ to $\mathcal{B}$.

Next we assume that $uC_\varphi - vC_\psi$ is bounded from H^∞ to $\mathcal{B}$. Then we get $u - v = (uC_\varphi - vC_\psi)1 \in \mathcal{B}$. Let N be the operator norm of $uC_\varphi - vC_\psi$ from H^∞ to $\mathcal{B}$ and α_λ be the Möbius transformation of $\mathbb{D}$ in the form of (2.1). For nonnegative integers k, m and $\lambda, \mu \in \mathbb{D}$, we have that $(\alpha_\lambda^k\,\alpha_\mu^m)' = k\,\alpha_\lambda'\alpha_\lambda^{k-1}\,\alpha_\mu^m + m\,\alpha_\lambda^k\,\alpha_\mu'\,\alpha_\mu^{m-1}$ and $\|\alpha_\lambda^k\,\alpha_\mu^m\|_\infty = 1$.

Fix $w \in \mathbb{D}$ and put $k = 1$, $m = 2$, $\lambda = \varphi(w)$, and $\mu = \psi(w)$. Since $\alpha_\lambda(\lambda) = 0$, we have that

$$\begin{aligned}
N &\ge \|\,(uC_\varphi - vC_\psi)\,(\alpha_{\varphi(w)}\,\alpha^2_{\psi(w)})\|\\
&\ge (1-|w|^2)\,\Big|u(w)\,\varphi'(w)\,\alpha'_{\varphi(w)}(\varphi(w))\,\alpha^2_{\psi(w)}(\varphi(w))\Big|\\
&= \big|u(w)\,\varphi^{\#}(w)\big|\,\rho(w)^2.
\end{aligned} \tag{3.1}$$

Considering $\|(uC_\varphi - vC_\psi)\alpha^2_{\psi(w)}\|$, we obtain that

$$
\begin{aligned}
N \geq & \left|(1-|w|^2)u'(w)\,\alpha^2_{\psi(w)}(\varphi(w))\right.\\
& \left. -2u(w)\,\varphi^{\#}(w)\,\alpha_{\psi(w)}(\varphi(w))\,\frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\psi(w)}\varphi(w))^2}\right|\\
\geq & \ 2|u(w)\,\varphi^{\#}(w)|\,\rho(w)(1-\rho(w)^2)-(1-|w|^2)|u'(w)|\,\rho(w)^2.
\end{aligned}
$$

By condition (A), we have that

$$
|u(w)\,\varphi^{\#}(w)|\,\rho(w)(1-\rho(w)^2) \leq C. \tag{3.2}
$$

By (3.1) and (3.2), we have that

$$
|u(w)\,\varphi^{\#}(w)|\,\rho(w) \leq C.
$$

Similarly we get

$$
|v(w)\,\psi^{\#}(w)|\,\rho(w) \leq C. \tag{3.3}
$$

Since $w \in \mathbb{D}$ is arbitrary, we get (iii).

Finally, take $\alpha_{\varphi(w)}$ as a test function. Then we obtain that

$$
\begin{aligned}
N \geq & \left| -u(w)\,\varphi^{\#}(w)-(1-|w|^2)v'(w)\,\alpha_{\varphi(w)}(\psi(w))\right.\\
& \left. +v(w)\,\psi^{\#}(w)\,\frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\psi(w)}\varphi(w))^2}\right|\\
\geq & \left|u(w)\,\varphi^{\#}(w)-v(w)\,\psi^{\#}(w)\right|-(1-|w|^2)|v'(w)|\,\rho(w)\\
& -|v(w)\,\psi^{\#}(w)|\left|1-\frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\psi(w)}\varphi(w))^2}\right|.
\end{aligned}
$$

Using condition (A), Lemma 2.1, and (3.3), we obtain

$$
\left|u(w)\,\varphi^{\#}(w)-v(w)\,\psi^{\#}(w)\right| < C.
$$

Thus we get (ii) and the proof is finished. ∎

Letting $u, v \in H^\infty$, we have the following fact.

Corollary 3.2. *For $u, v \in H^\infty$ and $\varphi, \psi \in \mathcal{S}(\mathbb{D})$, $uC_\varphi - vC_\psi$ is always bounded from H^∞ to $\mathcal{B}$.*

We give examples for Theorem 3.1.

Example 3.3. Let

$$\varphi(z) = \frac{z+1}{2}$$

and $\psi(z) = \varphi(z) + t(1-z)^3$ where t is real and $|t|$ is so small that $\psi(\mathbb{D}) \subset \mathbb{D}$. We fix a polynomial p and put

$$u(z) = \log \frac{1}{1-z} + p(z) \quad \text{and} \quad v(z) = \log \frac{1}{1-z} - p(z).$$

Then neither uC_φ nor vC_ψ is bounded from H^∞ to $\mathcal{B}$, but $uC_\varphi - vC_\psi$ is bounded from H^∞ to $\mathcal{B}$.

Proof. We remark that $\overline{\mathbb{D}} \cap \overline{\varphi(\mathbb{D})} = \overline{\mathbb{D}} \cap \overline{\psi(\mathbb{D})} = \{1\}$. Let x be a real number. It is easy to check

$$\lim_{x\to 1} |\varphi^{\#}(x)| = 1$$

and

$$\rho(z) \le C|1-z| \to 0$$

as $z \to 1$. Since $|\varphi^{\#}(z) - \psi^{\#}(z)| \le C\rho(z)$ (see [12]), we have $\psi^{\#}(z) \to 1$ as $z \to 1$. Hence we conclude that $|u(z)\varphi^{\#}(z)| \to \infty$ and $|v(z)\psi^{\#}(z)| \to \infty$, and then neither uC_φ nor vC_ψ is bounded from H^∞ to $\mathcal{B}$.

Since $u, v \in \mathcal{B}$, condition (A) and (i) hold. Moreover, we have that

$$\begin{aligned} &|u(z)\,\varphi^{\#}(z) - v(z)\,\psi^{\#}(z)| \\ &\le \left|\log \frac{1}{1-z}\right| |\varphi^{\#}(z) - \psi^{\#}(z)| + \left|p(z)\big(\varphi^{\#}(z) + \psi^{\#}(z)\big)\right| \\ &\le C \left|\log \frac{1}{1-z}\right| |1-z| + 2\|p\|_\infty < \infty. \end{aligned}$$

Then we conclude $\|u\varphi^{\#} - v\psi^{\#}\|_\infty < \infty$. We get (ii). Similary we can check $u(z)\,\varphi^{\#}(z)\,\rho(z)$ is bounded on $\mathbb{D}$. Hence we have $uC_\varphi - vC_\psi$ is bounded from H^∞ to $\mathcal{B}$. ■

But the example above yields the boundedness of $uC_\varphi - vC_\psi : \mathcal{B} \to \mathcal{B}$. We here give an example showing the difference between the boundedness of $uC_\varphi - vC_\psi : H^\infty \to \mathcal{B}$ and of $uC_\varphi - vC_\psi : \mathcal{B} \to \mathcal{B}$.

Example 3.4. Let φ and ψ be the same maps as in Example 3.3. We denote a singular inner function by

$$S(z) = \exp \frac{1+z}{1-z},$$

and put $u(z) = \log \dfrac{1}{1-z} + S(z)$ and $v(z) = \log \dfrac{1}{1-z} - S(z)$. Neither uC_φ nor vC_ψ is bounded from H^∞ to $\mathcal{B}$. And $uC_\varphi - vC_\psi$ is not bounded from $\mathcal{B}$ to $\mathcal{B}$ but is bounded from H^∞ to $\mathcal{B}$.

Proof. We can easily check that neither uC_φ nor vC_ψ is bounded from H^∞ to $\mathcal{B}$.

Since $u, v \in \mathcal{B}$, condition (A) and (i) hold. Moreover, we have that

$$\begin{aligned} &|u(z)\,\varphi^{\#}(z) - v(z)\,\psi^{\#}(z)| \\ \le &\left|\log \frac{1}{1-z}\right| |\varphi^{\#}(z) - \psi^{\#}(z)| + \left|S(z)\big(\varphi^{\#}(z) + \psi^{\#}(z)\big)\right| \\ \le &\, C\left|\log \frac{1}{1-z}\right| |1-z| + 2 < \infty. \end{aligned}$$

Then we conclude $\|u\varphi^{\#} - v\psi^{\#}\|_\infty < \infty$. We get (ii). Similary we can check $u(z)\,\varphi^{\#}(z)\,\rho(z)$ is bounded on $\mathbb{D}$. Hence we have $uC_\varphi - vC_\psi$ is bounded from H^∞ to $\mathcal{B}$.

But we get that

$$\begin{aligned} &(1-|z|^2)|u'(z) - v'(z)| \log \frac{1}{1-|\varphi(z)|^2} \\ = &\,(1-|z|^2)|S'(z)| \log \frac{1}{1-|\varphi(z)|^2} \end{aligned}$$

is not bounded as $z \to 1$. So $uC_\varphi - vC_\psi$ is not bounded from $\mathcal{B}$ to $\mathcal{B}$ by [3, Theorem 3.6]. ∎

4. Compactness to the Bloch Space

In this section, we will consider the compactness of differences of two weighted composition operators acting from H^∞ to $\mathcal{B}$. It is easy to prove the next lemma which is a generalization of Proposition 3.11 of [2].

Lemma 4.1. *Let u, v be in $\mathcal{H}(\mathbb{D})$ and φ, ψ be in $S(\mathbb{D})$. Suppose that $uC_\varphi - vC_\psi$ is bounded from H^∞ to $\mathcal{B}$. Then the following are equivalent:*

- (i) *$uC_\varphi - vC_\psi$ is compact from H^∞ to $\mathcal{B}$.*
- (ii) *$\|(uC_\varphi - vC_\psi)f_n\|_{\mathcal{B}} \to 0$ for any bounded sequence $\{f_n\}$ in H^∞ that converges to 0 uniformly on every compact subset of $\mathbb{D}$.*
- (iii) *$|||(uC_\varphi - vC_\psi)f_n||| \to 0$ for any sequence $\{f_n\}$ as in* (ii).

We define the following notation to discuss the behaviors of u and φ near the boundary of $\mathbb{D}$.

Definition 4.2. For $u \in \mathcal{H}(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, let Δ be the set of all convergent sequence $\{z_n\}$ in $\mathbb{D}$. Put

$$\Lambda_{u,\varphi} = \Big\{\{z_n\} \in \Delta : |\varphi(z_n)| \to 1, (1-|z_n|^2)|u'(z_n)| \not\to 0\Big\}$$

and

$$\Gamma^{\#}_{u,\varphi} = \Big\{ \{z_n\} \in \Delta : |\varphi(z_n)| \to 1, |u(z_n)\varphi^{\#}(z_n)| \not\to 0 \Big\}.$$

Then $uC_\varphi : H^\infty \to \mathcal{B}$ is compact if and only if $\Gamma^{\#}_{u,\varphi} = \Lambda_{u,\varphi} = \emptyset$.

To characterize the compactness of $uC_\varphi - vC_\psi : H^\infty \to \mathcal{B}$, we need the assumption:

(B) $\quad (1-|z|^2)|u'(z)|\,\rho(z) \to 0$ and $(1-|z|^2)|v'(z)|\,\rho(z) \to 0$ if $\rho(z) \to 0$.

Theorem 4.3. *Let $u, v \in \mathcal{H}(\mathbb{D})$ and $\varphi, \psi \in S(\mathbb{D})$. Assume that conditions* (A) *and* (B) *hold. Suppose that $uC_\varphi - vC_\psi$ is bounded from H^∞ to $\mathcal{B}$, but neither uC_φ nor vC_ψ is compact. Then $uC_\varphi - vC_\psi$ is compact from H^∞ to $\mathcal{B}$ if and only if the followings hold:*

(i) $\Lambda_{u,\varphi} = \Lambda_{v,\psi}$.

(ii) $\lim\limits_{n\to\infty} (1-|z_n|^2)|u'(z_n) - v'(z_n)| = 0$ *for any* $\{z_n\} \in \Lambda_{u,\varphi}$.

(iii) $\lim\limits_{n\to\infty} (1-|z_n|^2)|u'(z_n)|\,\rho(z_n) = 0$ *for any* $\{z_n\} \in \Lambda_{u,\varphi}$.

(iv) $\Gamma^{\#}_{u,\varphi} = \Gamma^{\#}_{v,\psi}$.

(v) $\lim\limits_{n\to\infty} |u(z_n)\,\varphi^{\#}(z_n) - v(z_n)\,\psi^{\#}(z_n)| = 0$ *for any* $\{z_n\} \in \Gamma^{\#}_{u,\varphi}$.

(vi) $\lim\limits_{n\to\infty} |u(z_n)\,\varphi^{\#}(z_n)|\,\rho(z_n) = 0$ *for any* $\{z_n\} \in \Gamma^{\#}_{u,\varphi}$.

It is possible to replace u with v in condition (iii)*, and also u with v, and φ with ψ in condition* (vi).

Proof. We assume that conditions (i) - (vi) hold. Let $\{f_n\}$ be a bounded sequence in H^∞ that converges to 0 uniformly on every compact subset of $\mathbb{D}$. To prove that $\|(uC_\varphi - vC_\psi)f_n\| \to 0$, suppose not. Then there exists a positive constant δ such that $\|(uC_\varphi - vC_\psi)f_n\| > \delta$ for any n. We can choose $z_n \in \mathbb{D}$ which satisfies that

$$(1-|z_n|^2)\big|(u\,f_n\circ\varphi - v\,f_n\circ\psi)'(z_n)\big| > \delta \tag{4.1}$$

for each n. Then it follows that $|z_n| \to 1$. By conditions (i) - (iii),

$$(1-|z_n|^2)\big|u'(z_n)\,f_n(\varphi(z_n)) - v'(z_n)\,f_n(\psi(z_n))\big| \to 0.$$

Furthermore, by conditions (iv) - (vi),

$$\begin{aligned}&\big|u(z_n)\,\varphi^{\#}(z_n)\,(1-|\varphi(z_n)|^2)\,f_n'(\varphi(z_n))\\&\quad -v(z_n)\,\psi^{\#}(z_n)\,(1-|\psi(z_n)|^2)\,f_n'(\psi(z_n))\big| \to 0\end{aligned}$$

as $|z_n| \to 1$. Here we have that

$$\begin{aligned}
&(1-|z_n|^2)\big|(u\, f_n \circ \varphi - v\, f_n \circ \psi)'(z_n)\big| \\
&\le (1-|z_n|^2)\big|u'(z_n)\, f_n(\varphi(z_n)) - v'(z_n)\, f_n(\psi(z_n))\big| \\
&\quad + \big|u(z_n)\, \varphi^{\#}(z_n)\, (1-|\varphi(z_n)|^2)\, f_n'(\varphi(z_n)) \\
&\quad - v(z_n)\, \psi^{\#}(z_n)\, (1-|\psi(z_n)|^2)\, f_n'(\psi(z_n))\big| \\
&\to 0
\end{aligned}$$

as $|z_n| \to 1$. This contradicts (4.1).

Next we assume the compactness of $uC_\varphi - vC_\psi$ from H^∞ to $\mathcal{B}$. Suppose that $\Gamma^{\#}_{u,\varphi} \neq \emptyset$. For a sequence $\{z_n\} \in \Gamma^{\#}_{u,\varphi}$, put

$$f_n(z) = \big(\alpha_{\varphi(z_n)}(z)^2 - \varphi(z_n)\, \alpha_{\varphi(z_n)}(z)\big)\, \alpha_{\psi(z_n)}(z).$$

Then $\|f_n\|_\infty \le 2$ and $\{f_n\}$ converges to 0 uniformly on every compact subset of $\mathbb{D}$ as $|\varphi(z_n)| \to 1$. Here we have

$$\begin{aligned}
\|(uC_\varphi - vC_\psi) f_n\| &\ge (1-|z_n|^2)\big|(u\, f_n \circ \varphi - v\, f_n \circ \psi)'(z_n)\big| \\
&= |u(z_n)\, \varphi^{\#}(z_n)\, \varphi(z_n)|\, \rho(z_n)^2.
\end{aligned}$$

So, by Lemma 4.1, we get

$$\lim_{n\to\infty} |u(z_n)\, \varphi^{\#}(z_n)\varphi(z_n)|\, \rho(z_n)^2 = 0. \tag{4.2}$$

Since $|\varphi(z_n)| \to 1$ and $|u(z_n)\, \varphi^{\#}(z_n)| \not\to 0$, we get $\rho(z_n) \to 0$.

Put

$$g_n(z) = (\alpha_{\varphi(z_n)}(z) - \varphi(z_n))\, \alpha_{\psi(z_n)}(z)^2$$

and

$$\sigma(z) = \frac{\psi(z) - \varphi(z)}{1 - \overline{\psi(z)}\varphi(z)}.$$

Then $g_n \in H^\infty$, $\|g_n\|_\infty \le 2$ and $\{g_n\}$ also converges to 0 uniformly on every compact subset of $\mathbb{D}$. Then we obtain that

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi) g_n\| \\
&\ge \Bigg|\varphi(z_n)(1-|z_n|^2)\, u'(z_n)\, \sigma(z_n)^2 \\
&\quad - u(z_n)\, \varphi^{\#}(z_n) \left(\sigma(z_n)^2 - 2\varphi(z_n)\, \sigma(z_n)\, \frac{(1-|\varphi(z_n)|^2)(1-|\psi(z_n)|^2)}{(1-\overline{\psi(z_n)}\varphi(z_n))^2}\right)\Bigg| \\
&\ge 2|u(z_n)\, \varphi^{\#}(z_n)\, \varphi(z_n)|\, \rho(z_n)(1-\rho(z_n)^2) \\
&\quad - (1-|z_n|^2)|u'(z_n)\, \varphi(z_n)|\, \rho(z_n)^2 - |u(z_n)\, \varphi^{\#}(z_n)\,|\, \rho(z_n)^2 \\
&= |u(z_n)\, \varphi^{\#}(z_n)\, \varphi(z_n)|\, (2\rho(z_n) - \rho(z_n)^2 - 2\rho(z_n)^3) \\
&\quad - (1-|z_n|^2)|u'(z_n)\, \varphi(z_n)|\, \rho(z_n)^2.
\end{aligned}$$

Using conditions (B) and (4.2), and noticing that $|\varphi(z_n)| \to 1$, we have that

$$|u(z_n)\,\varphi^{\#}(z_n)|\,\rho(z_n) \to 0.$$

We get condition (vi).

Since $\rho(z_n) \to 0$ for $\{z_n\} \in \Gamma^{\#}_{u,\varphi}$, we have $|\psi(z_n)| \to 1$. We can consider $(\alpha_{\psi(z_n)}(z)^2 - \psi(z_n)\,\alpha_{\psi(z_n)}(z))\,\alpha_{\varphi(z_n)}(z)$ and $(\alpha_{\psi(z_n)}(z) - \psi(z_n))\,\alpha_{\varphi(z_n)}(z)^2$ as test functions converging to 0 uniformly on compact subsets of $\mathbb{D}$. Hence we conclude

$$|v(z_n)\,\psi^{\#}(z_n)|\rho(z_n) \to 0.$$

Next consider

$$h_n(z) = (\alpha_{\varphi(z_n)}(z) - \varphi(z_n))\,\alpha_{\psi(z_n)}(z)$$

as test functions converging to 0 uniformly on compact subsets of $\mathbb{D}$. Then we get that

$$\begin{aligned}\Bigg| & -\varphi(z_n)(1-|z_n|^2)\,u'(z_n)\,\sigma(z_n) - u(z_n)\,\varphi^{\#}(z_n)\,\sigma(z_n) \\ & + u(z_n)\,\varphi^{\#}(z_n)\,\varphi(z_n)\,\frac{(1-|\varphi(z_n)|^2)(1-|\psi(z_n)|^2)}{(1-\overline{\psi(z_n)}\varphi(z_n))^2} \\ & + v(z_n)\,\psi^{\#}(z_n)\left(\frac{\varphi(z_n)-\psi(z_n)}{1-\overline{\varphi(z_n)}\psi(z_n)} - \varphi(z_n)\right)\Bigg|\end{aligned}$$

tends to 0.

By Lemma 2.1, we get

$$\Big|u(z_n)\,\varphi^{\#}(z_n) - v(z_n)\,\psi^{\#}(z_n)\Big| \to 0.$$

We get condition (v).

For $\{z_n\} \in \Gamma^{\#}_{u,\varphi}$, condition (v) implies $|v(z_n)\,\psi^{\#}(z_n)| \not\to 0$. Thus $\Gamma^{\#}_{u,\varphi} \subset \Gamma^{\#}_{v,\psi}$. The converse inclusion can be shown by the same way. Hence we get $\Gamma^{\#}_{u,\varphi} = \Gamma^{\#}_{v,\psi}$ for the case $\Gamma^{\#}_{u,\varphi} \neq \emptyset$. By the fact above, we can check $\Gamma^{\#}_{v,\psi} = \emptyset$ if and only if $\Gamma^{\#}_{u,\varphi} = \emptyset$. Here we get (iv).

Condition (A) and the boundedness of $uC_\varphi - vC_\psi$ imply conditions (i) - (iii) in Theorem 3.1. Then, by conditions (i) - (iii) in Theorem 3.1 and conditions (iv) - (vi), we have

$$\begin{aligned}\sup_{z\in\mathbb{D}} \big| & u(z)\,\varphi^{\#}(z)\,(1-|\varphi(z)|^2)\,f_n'(\varphi(z)) \\ & - v(z)\,\psi^{\#}(z)\,(1-|\psi(z)|^2)\,f_n'(\psi(z))\big| \to 0\end{aligned}$$

for any bounded sequence $\{f_n\}$ in H^∞ that converges to 0 uniformly on every compact subset of $\mathbb{D}$. By the triangle inequality, we get

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)\big|u'(z)\,f_n(\varphi(z))-v'(z)\,f_n(\psi(z))\big|\\
&\le \|(uC_\varphi - vC_\psi)f_n\|\\
&\quad+\sup_{z\in\mathbb{D}}\big|u(z)\,\varphi^{\#}(z)\,(1-|\varphi(z)|^2)\,f_n'(\varphi(z))\\
&\quad-v(z)\,\psi^{\#}(z)\,(1-|\psi(z)|^2)\,f_n'(\psi(z))\big|\\
&\to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Assume that $\Lambda_{u,\varphi}\neq\emptyset$ and take a sequence $\{z_n\}\in\Lambda_{u,\varphi}$. We remark that $|u(z_n)\,\varphi^{\#}(z_n)|\,\rho(z_n)\to 0$ if $|\varphi(z_n)|\to 1$. Here we will use the test function $h_n(z)=(\alpha_{\varphi(z_n)}(z)-\varphi(z_n))\,\alpha_{\psi(z_n)}(z)$ again. Then we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)\big|u'(z)\,h_n(\varphi(z))-v'(z)\,h_n(\psi(z))\big|\\
&\ge (1-|z_n|^2)|u'(z_n)\,\varphi(z_n)|\,\rho(z_n).
\end{aligned}$$

We get condition (iii). Hence we have $\rho(z_n)\to 0$ for $\{z_n\}\in\Lambda_{u,\varphi}$, and also $|\psi(z_n)|\to 1$. Taking $\alpha_{\varphi(z_n)}(z)\,\alpha_{\psi(z_n)}(z)-\varphi(z_n)\,\psi(z_n)$ as a test function,

$$(1-|z_n|^2)|u'(z_n)-v'(z_n)|\,|\varphi(z_n)\psi(z_n)|\to 0.$$

We obtain condition (ii) and $\Lambda_{u,\varphi}=\Lambda_{v,\psi}$ for the case $\Lambda_{u,\varphi}\neq\emptyset$. By the fact above, we can check $\Lambda_{v,\psi}=\emptyset$ if and only if $\Lambda_{u,\varphi}=\emptyset$. Here we get condition (i). ∎

Letting $u,v\in H^\infty$, we can easily show the following.

Corollary 4.4. *For $u,v\in H^\infty$ and $\varphi,\psi\in\mathcal{S}(\mathbb{D})$, $uC_\varphi - vC_\psi : H^\infty\to\mathcal{B}$ is compact if and only if $uC_\varphi - vC_\psi:\mathcal{B}\to\mathcal{B}$ is compact.*

The latter compactness was characterized in [3]. Finally we give an example for Theorem 4.3.

Example 4.5. Let φ and ψ be the same maps as in Example 3.3. Put $u(z)=\log\frac{1}{1-z}+(1-z)$ and $v(z)=\log\dfrac{1}{1-z}-(1-z)$. Then neither uC_φ nor vC_ψ is compact from H^∞ to $\mathcal{B}$, but $uC_\varphi - vC_\psi$ is compact from H^∞ to $\mathcal{B}$.

Proof. Since $u,v\in B$, condition ($\mathcal{B}$) holds. It is easy to check that $\Lambda_{u,\varphi}=\Lambda_{v,\psi}\neq\emptyset$ and $\Gamma^{\#}_{u,\varphi}=\Gamma^{\#}_{v,\psi}\neq\emptyset$. Hence conditions (i) and (iv) hold and neither uC_φ nor vC_ψ is compact from H^∞ to $\mathcal{B}$. Since $u(z)-v(z)=2(1-z)$, condition (ii) holds. If $z \to 1$, then $\rho(z) \to 0$. Then condition (B) implies condition (iii). Moreover, we have that

$$\begin{aligned}
&|u(z_n)\,\varphi^{\#}(z_n) - v(z_n)\,\psi^{\#}(z_n)| \\
&\le 2|1-z_n||\varphi^{\#}(z_n)| + |v(z_n)||\varphi^{\#}(z_n) - \psi^{\#}(z_n)| \\
&\le 2|1-z_n| + C\left|\log\frac{1}{1-z_n} - (1-z_n)\right||1-z_n| \\
&\to 0
\end{aligned}$$

for $\{z_n\} \in \Gamma^{\#}_{u,\varphi}$. We obtain condition (v). It is easy to show that condition (vi) holds. Hence we conclude that $uC_\varphi - vC_\psi$ is compact from H^∞ to $\mathcal{B}$. ∎

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Takuya Hosokawa
College of Engineering
Ibaraki University
Hitachi, Ibaraki 316-8511
Japan
E-mail: hoso-t@mx.ibaraki.ac.jp

Shûichi Ohno
Nippon Institute of Technology
Miyashiro, Minami-Saitama 345-8501
Japan
E-mail: ohno@nit.ac.jp