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BOUNDEDNESS FOR SEMILINEAR DUFFING EQUATIONS AT RESONANCE

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Abstract. In this paper, we prove the boundedness of all solutions for the equation $x'' + n^2x + \phi(x) + g''(x)q(t) = 0$, where $n \in \mathbb{N}$, $q(t) = q(t + 2\pi)$, $\phi(x)$ and $g(x)$ are bounded.

1. Introduction and Main Result

In this paper, we study the boundedness of solutions for the following semilinear Duffing-type equations:

$$x'' + n^2x + \psi(x,t) = 0, \quad \psi(x,t+2\pi) = \psi(x,t), \tag{1.1}$$

where $n \in \mathbb{N}$ and $\psi(x,t)$ is bounded.

It is well known that the linear equation

$$x'' + n^2x = \sin nt, \qquad n \in \mathbb{N}$$

has no bounded solutions. Due to the phenomenon of linear resonance, the boundedness of solutions for semilinear equation at resonance (1.1) is very delicate and interesting.

In 1999 Ortega [15] proved a variant of Moser's small twist theorem, by which he obtained the Lagrangian stability for the equation

$$x'' + n^2x + h_L(x) = p(t), \quad p(t) \in \mathcal{C}^5(\mathbb{R}/2\pi\mathbb{Z}),$$

where $h_L(x)$ is of the form

$$h_L(x) = \begin{cases} L, & \text{if} \quad x \geq 1, \\ Lx, & \text{if} \quad -1 \leq x \leq 1, \\ -L, & \text{if} \quad x \leq -1, \end{cases}$$

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and $p(t)$ satisfies

$$\frac{1}{2\pi}\left|\int_0^{2\pi} p(t)e^{-int}dt\right| < \frac{2L}{\pi}.$$

Then Liu [8] studied the equation

$$x'' + n^2x + \phi(x) = p(t), \quad p(t) \in \mathcal{C}^7(\mathbb{R}/2\pi\mathbb{Z}), \tag{1.2}$$

where $\phi(x) \in \mathcal{C}^6(\mathbb{R})$ and the limits

$$\phi(\pm\infty) = \lim_{x\to\pm\infty} \phi(x) \text{ exist and are finite} \tag{1.3}$$

and

$$\lim_{|x|\to+\infty} x^6\phi^{(6)}(x) = 0. \tag{1.4}$$

He showed that each solution of (1.2) is bounded if

$$\left|\int_0^{2\pi} p(t)e^{-int}dt\right| < 2(\phi(+\infty) - \phi(-\infty)). \tag{1.5}$$

If in addition $\phi(x)$ is monotone, then Lazer-Leach's result [5] implies that the condition (1.5) is also necessary for the Lagrangian stability of (1.2). Without monotone assumption on $\phi(x)$, Alonso and Ortega [1] showed that if

$$\left|\int_0^{2\pi} p(t)e^{-int}dt\right| > 2(\phi(+\infty) - \phi(-\infty)),$$

then all solutions of equation (1.2) with initial points sufficiently far from the origin are unbounded.

For more results on (1.1), one can see [9, 10, 11, 12] and references therein.

It is noted to point out that all the results stated above require the condition $\psi(x,t)$ satisfies the following growth condition:

$$\lim_{|x|\to+\infty} x^m D_x^m \psi(x,t) = 0 \tag{1.6}$$

for some finite m.

The reason that people impose the above growth condition on $\psi(x,t)$ is related to the fact that the application of KAM theorem requires the estimates of several derivatives of the solutions with respect to initial conditions, which is tedious and difficult.

In this paper, we try to study the boundedness of solutions of (1.1) without the condition (1.6). More precisely, we will consider the following equation:

$$x'' + n^2x + \phi(x) + g''(x)q(t) = 0, \tag{1.7}$$

where $n \in \mathbb{N},\ q(t) \in \mathcal{C}^{13}(\mathbb{R}/2\pi\mathbb{Z})$ and $\phi(x) \in \mathcal{C}^{12}(\mathbb{R})$ satisfies (1.3) and

$$\lim_{|x| \to +\infty} x^{12}\phi^{(12)}(x) = 0, \tag{1.8}$$

while $g(x)$ satisfies

$$|g^{(k)}(x)| \le C, \quad 0 \le k \le 14 \tag{1.9}$$

for some $C > 0$.

Obviously, in our situation $\psi(x,t) = \phi(x) + g''(x)q(t)$, which does not necessarily, which satisfy (1.6). For example, if $\phi(x) = \arctan x$ and $g(x) = \cos x$, then $\psi(x,t) = \arctan x - \cos x \cdot q(t)$.

The main result of this paper is as follows:

Theorem 1. *Let $n \in \mathbb{N}$, $q(t) \in \mathcal{C}^{13}(\mathbb{R}/2\pi\mathbb{Z})$. Suppose $\phi(x)$ satisfies (1.3), (1.8) and $\phi(+\infty) \neq \phi(-\infty)$, $g(x)$ satisfies (1.9). Then all solutions of (1.7) are bounded. That is, each solution $x = x(t)$ of (1.7) exists on $(-\infty, +\infty)$ and $\sup_{t\in\mathbb{R}}(|x(t)| + |x'(t)|) < +\infty$.*

Remark 1.1. Recently, Jiao, Piao and Wang [4] studied the equation

$$x'' + \omega^2 x + \phi(x) = G_x(x,t) + p(t), \tag{1.10}$$

where $\omega \in \mathbb{R}^+ \backslash \mathbb{Q}$ satisfies the Diophantine condition and ϕ, p are similar to in (1.2). They proved that if $|D_x^i D_t^j G(x,t)| \le C$, then all solutions of (1.10) are bounded. This result together with Theorem 1 implies that the growth condition is not necessary for the Lagrangian stability of Duffing equations. In our future work, we will extend Theorem 1 to a more general situation which includes (1.2) as a special case.

The study of the boundedness problem for Duffing equations was in early 1960's by Littlewood [7]. In 1976, by Moser's small twist theorem [14], Morris [13] first obtained the boundedness result for the equation

$$x'' + x^3 = p(t)$$

with $p(t + 2\pi) = p(t)$ piecewise continuous.

Since then Moser's small twist theorem has become the most important tool in the study of Littlewood's problem. The main idea is as follows.

By means of transformation theory the original system outside of a large disc $D = \{(x,x') \in R^2 : x^2 + x'^2 \le r^2\}$ in (x,x')-plane is transformed into a perturbation of an integrable Hamiltonian system. The Poincaré map of the transformed system is closed to a so-called twist map in $R^2 \backslash D$. Then Moser's twist theorem guarantees the existence of arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin in the (x, x')-plane. Every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space $(x, x', t) \in R^2 \times R$, which confines the solutions in the interior and leads to a bound of these solutions.

The rest of this paper is organized as follows. In Section 2, we deal with some technical lemmas. In Section 3, we will make some more canonical transformations such that the Poincaré map of the new system is close to the twist map. The proof of Theorem 1 will be given in the last section.

Throughout this paper, we denote by $c < 1$ and $C > 1$, respectively, two universal positive constants not concerning their quantities.

2. Some Canonical Transformations

In this section, we will state some technical lemmas which will be used later. Throughout this paper, without loss of generality we assume that $\phi(+\infty) > \phi(-\infty)$.

Let $y = -n^{-1}x'$. Then (1.7) is equivalent to the following equations:

$$x' = -ny, \quad y' = nx + n^{-1}\phi(x) + n^{-1}g''(x)q(t),$$

which is a planar non-autonomous Hamiltonian system

$$x' = -\frac{\partial H}{\partial y}(x, y, t), \qquad y' = \frac{\partial H}{\partial x}(x, y, t) \tag{2.1}$$

with Hamiltonian

$$H(x, y, t) = \frac{1}{2}n(x^2 + y^2) + \frac{1}{n}\Phi(x) + \frac{1}{n}g'(x)q(t), \tag{2.2}$$

where $\Phi(x) = \int_0^x \phi(x)dx$.

Under the transformation

$$x = r^{\frac{1}{2}} \cos n\theta, \quad y = r^{\frac{1}{2}} \sin n\theta$$

with $(r, \theta) \in \mathbb{R}^+ \times \mathbb{R}/\frac{2\pi\mathbb{Z}}{n}$, the system (2.1) is transformed into another system

$$r' = -\frac{\partial h}{\partial \theta}, \quad \theta' = \frac{\partial h}{\partial r}, \tag{2.3}$$

where

$$h(r, \theta, t) = r + f_1(r, \theta) + f_2(r, \theta, t), \tag{2.4}$$

and $f_1 = \frac{2}{n^2}\Phi(r^{\frac{1}{2}} \cos n\theta)$, $f_2 = \frac{2}{n^2}g'(r^{\frac{1}{2}} \cos n\theta) \cdot q(t)$.

As n is a positive integer, the function $h(r, \theta, t)$ is 2π-periodic in θ.

For any function $f(\cdot,\theta)$, we denote by $[f](\cdot)$ the average value of $f(\cdot,\theta)$ over S^1, that is,

$$[f](\cdot) := \frac{1}{2\pi}\int_0^{2\pi} f(\cdot,\theta)d\theta.$$

From (1.3), (1.8) and the rule of L'Hospital, it follows that

$$|x^k\phi^{(k)}(x)| \le C, \qquad for \ \ 0 \le k \le 12 \tag{2.5}$$

and

$$\lim_{|x|\to+\infty} x^k\phi^{(k)}(x) = 0, \qquad for \ \ 0 \le k \le 12. \tag{2.6}$$

Then similar to [8], the following two lemmas hold:

Lemma 2.1. *It holds that:*

$$\begin{aligned} &|f_1(r,\theta)| \le C\cdot r^{\frac{1}{2}}, \quad |[f_1](r)| \le C\cdot r^{\frac{1}{2}}, \\ &\left|\frac{\partial f_1}{\partial\theta}(r,\theta)\right| \le C\cdot r^{\frac{1}{2}}, \left|\frac{\partial^2 f_1}{\partial\theta^2}(r,\theta)\right| \le C\cdot r. \end{aligned} \tag{2.7}$$

Moreover, for $1 \le k \le 12$,

$$\begin{aligned} &\left|\frac{\partial^k f_1}{\partial r^k}(r,\theta)\right| \le C\cdot r^{-k+\frac{1}{2}}, \quad \left|\frac{d^k[f_1]}{dr^k}(r)\right| \le C\cdot r^{-k+\frac{1}{2}}, \\ &\left|\frac{\partial^{k+1} f_1}{\partial r^k\partial\theta}(r,\theta)\right| \le C\cdot r^{-k+\frac{1}{2}}. \end{aligned} \tag{2.8}$$

Lemma 2.2. *The following conclusion holds true:*

$$|f_2(r,\theta,t)| \le C, \quad \left|\frac{\partial f_2}{\partial\theta}(r,\theta,t)\right| \le C\cdot r^{\frac{1}{2}}, \quad \left|\frac{\partial^2 f_2}{\partial\theta^2}(r,\theta,t)\right| \le C\cdot r. \tag{2.9}$$

Moreover, for $1 \le k+l \le 12$, *we have*

$$\left|\frac{\partial^{k+l} f_2}{\partial r^k\partial t^l}(r,\theta,t)\right| \le C\cdot r^{-\frac{k}{2}}, \quad \left|\frac{\partial^{k+l+1} f_2}{\partial r^k\partial t^l\partial\theta}(r,\theta,t)\right| \le C\cdot r^{-\frac{k}{2}+\frac{1}{2}}. \tag{2.10}$$

Again from [8], we have that

Lemma 2.3. *The following conclusions hold:*

$$\lim_{r\to+\infty}\sqrt{r}[f_1]'(r) = \frac{1}{\pi n^2}\{\phi(+\infty)-\phi(-\infty)\}, \tag{2.11}$$

$$\lim_{r\to+\infty}\frac{[f_1](r)}{\sqrt{r}} = \frac{2}{\pi n^2}\{\phi(+\infty)-\phi(-\infty)\}, \tag{2.12}$$

$$\lim_{r\to+\infty} r^{\frac{3}{2}}[f_1]''(r) = -\frac{1}{2\pi n^2}\{\phi(+\infty)-\phi(-\infty)\}. \tag{2.13}$$

From Lemma 2.3, it follows that for $r \gg 1$

(2.14) $c \cdot r^{\frac{1}{2}} < [f_1](r) < C \cdot r^{\frac{1}{2}}, \quad c \cdot r^{-\frac{1}{2}} < [f_1]'(r) < C \cdot r^{-\frac{1}{2}}, \quad c \cdot r^{-\frac{3}{2}} < |[f_1]''(r)|.$

2.1. Exchange of the role of time and angle variables

Observe that

$$r d\theta - h dt = -(h dt - r d\theta).$$

Thus if one can solve (2.4) for r such that $r = r(h, t, \theta)$, then

$$\frac{dh}{d\theta} = -\frac{\partial r}{\partial t}(h, t, \theta), \quad \frac{dt}{d\theta} = \frac{\partial r}{\partial h}(h, t, \theta), \tag{2.15}$$

i.e., (2.15) is a Hamiltonian system with Hamiltonian $r = r(h, t, \theta)$ and now the action, angle and time variables are $h, t,$ and θ, respectively. This trick has been used in [6] and [8].

From Eq.(2.4) and Lemma 2.1, 2.2, one can easily see that

$$\lim_{r \to +\infty} \frac{h}{r} = 1 > 0$$

and for $r \gg 1$

$$\frac{\partial h}{\partial r} = 1 + \frac{\partial}{\partial r} f_1(r, \theta) + \frac{\partial}{\partial r} f_2(r, \theta, t) > 0.$$

Thus by the implicit function theorem, we have that there exists a function $R = R(h, t, \theta)$ such that

$$r(h, t, \theta) = h - R(h, t, \theta). \tag{2.16}$$

Moreover, for $h \gg 1$,

$$|R(h, t, \theta)| \leq h/2$$

and $R(h, t, \theta)$ is $\mathcal{C}^{13}$ in h, t and θ.

From (2.4), it holds that

$$R = f_1(h - R, \theta) + f_2(h - R, \theta, t). \tag{2.17}$$

The following three lemmas are similar to the ones in [8] and we give the proof of them in the appendix for the convenience of readers.

Lemma 2.4. *Assume $R(h, t, \theta)$ is defined by (2.17). Then for $h \gg 1$ it holds that*

$$|R| \leq C \cdot h^{\frac{1}{2}}, \quad |\frac{\partial R}{\partial \theta}| \leq C \cdot h^{\frac{1}{2}}, \quad |\frac{\partial^2 R}{\partial \theta^2}| \leq C \cdot h, \quad |\frac{\partial^{k+l} R}{\partial h^k \partial t^l}| \leq C \cdot h^{-\frac{k}{2}}, \tag{2.18}$$

for $1 \leq k + l \leq 13$.

In (2.17), let

$$R = f_1(h,\theta) + R_1(h,t,\theta). \tag{2.19}$$

Then

$$\begin{aligned} &R_1(h,t,\theta)\\ &= -\frac{\partial}{\partial r} f_1(h,\theta)\cdot R + \int_0^1\int_0^1 \frac{\partial^2}{\partial r^2} f_1(h-s\mu R,\theta)\mu R^2 ds d\mu + f_2(h-R,\theta,t). \end{aligned} \tag{2.20}$$

The following lemma gives an estimate of $R_1(h,t,\theta)$.

Lemma 2.5. *It holds that*

$$\left|\frac{\partial^{k+l} R_1}{\partial h^k \partial t^l}\right| \le C\cdot h^{-\frac{k}{2}}, \quad 0\le k+l\le 11. \tag{2.21}$$

From Lemma 2.4 and 2.5, we have the following estimate on R:

Lemma 2.6. *It holds that*

$$\left|\frac{\partial^{k+l+1} R}{\partial h^k \partial t^l \partial\theta}\right| \le C\cdot h^{-\frac{k}{2}+\frac{1}{2}}, \quad 1\le k+l\le 11. \tag{2.22}$$

The following technical lemma will be used to refine the estimates on $[R_1](h,t)$.

Lemma 2.7. *Let $h \gg 1$. Assume that:*

(*i*) $g(x)\in \mathcal{C}^1(\mathbb{R})$ *and* $R, \mathcal{R} \in \mathcal{C}^2(\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}/2\pi\mathbb{Z})$*;*

(*ii*) $|g(x)|\le C, \quad |g'(x)\le C$;

(*iii*) $|R(h,t,\theta)|\le C\cdot h^{\frac{1}{2}}, \quad |\frac{\partial R}{\partial\theta}|\le C\cdot h^{\frac{1}{2}}, \quad |\frac{\partial^2 R}{\partial\theta^2}|\le C\cdot h$;

(*iv*) $|\mathcal{R}|, |\frac{\partial \mathcal{R}}{\partial\theta}| \le C\cdot h^{-a}$ *with* $a\ge 0$.

Then for any constant $\delta\in(0,\frac{1}{3})$*, it holds that*

$$\left|\int_0^{2\pi} \mathcal{R}(h,t,\theta) g'((h-R(h,t,\theta))^{\frac{1}{2}}\cos n\theta)d\theta\right| \le C\cdot h^{-(\frac{\delta}{2}+a)}. \tag{2.23}$$

Proof. Let $[0,2\pi] = I_1\bigcup I_2\cdots\bigcup I_n$, $I_i = \left[\frac{(i-1)2\pi}{n}, \frac{i2\pi}{n}\right] = I_i^1\bigcup I_i^2$, where

$$\begin{aligned} I_i^1 = &\Big[\frac{(i-1)2\pi}{n}, \frac{(i-1)2\pi + h^{-\frac{\delta}{2}}}{n}\Big]\\ &\bigcup\Big[\frac{(i-1)2\pi+\pi-h^{-\frac{\delta}{2}}}{n}, \frac{(i-1)2\pi+\pi+h^{-\frac{\delta}{2}}}{n}\Big]\\ &\bigcup\Big[\frac{(i-1)2\pi+2\pi-h^{-\frac{\delta}{2}}}{n}, \frac{(i-1)2\pi+2\pi}{n}\Big], \end{aligned}$$

$$I_i^2 = \Big[\frac{(i-1)2\pi + h^{-\frac{\delta}{2}}}{n}, \frac{(i-1)2\pi + \pi - h^{-\frac{\delta}{2}}}{n}\Big] \bigcup \Big[\frac{(i-1)2\pi + \pi + h^{-\frac{\delta}{2}}}{n}, \frac{(i-1)2\pi + 2\pi - h^{-\frac{\delta}{2}}}{n}\Big].$$

Then

$$\begin{aligned} & \int_0^{2\pi} \mathcal{R}(h,t,\theta) g'((h-R)^{\frac{1}{2}} \cos n\theta) d\theta \\ = & \sum_{i=1}^{n} \Big(\int_{I_i^1} \mathcal{R}(h,t,\theta) g'((h-R)^{\frac{1}{2}} \cos n\theta) d\theta + \int_{I_i^2} \mathcal{R}(h,t,\theta) g'((h-R)^{\frac{1}{2}} \cos n\theta) d\theta \Big). \end{aligned}$$

Obviously, $|I_i^1| \leq C \cdot h^{-\frac{\delta}{2}}$, where $|\cdot|$ denotes the Lebesgue measure. Then from condition (ii) and (iv), it is easy to see that

$$\left| \sum_{i=1}^{n} \int_{I_i^1} \mathcal{R}(h,t,\theta) g'((h-R)^{\frac{1}{2}} \cos n\theta) d\theta \right| \leq C \cdot h^{-\frac{\delta}{2} - a}.$$

To estimate the integral on I_i^2, we first estimate the integral on the interval

$$I_i^{21} = \Big[\frac{(i-1)2\pi + h^{-\frac{\delta}{2}}}{n}, \quad \frac{(i-1)2\pi + \pi - h^{-\frac{\delta}{2}}}{n}\Big].$$

By direct computation, we have

$$D_\theta((h-R)^{\frac{1}{2}} \cos n\theta) = -\frac{1}{2}(h-R)^{-\frac{1}{2}} \cdot \frac{\partial R}{\partial \theta} \cdot \cos n\theta - n(h-R)^{\frac{1}{2}} \sin n\theta.$$

From condition (iii), it holds that $|n(h-R)^{\frac{1}{2}} \sin n\theta| \geq c \cdot h^{\frac{1-\delta}{2}}$ and $|\frac{1}{2}(h-R)^{-\frac{1}{2}} \frac{\partial R}{\partial \theta} \cos n\theta| \leq C$ for $\theta \in I_i^{21}$, which implies

$$|D_\theta((h-R)^{\frac{1}{2}} \cos n\theta)| \geq c \cdot h^{\frac{1-\delta}{2}}. \tag{2.24}$$

Similarly, we have

$$\begin{aligned} & D_\theta^2((h-R)^{\frac{1}{2}} \cos n\theta) \\ = & -\frac{1}{4}(h-R)^{-\frac{3}{2}} (\frac{\partial R}{\partial \theta})^2 \cos n\theta - (h-R)^{-\frac{1}{2}} \left[\frac{1}{2} \frac{\partial^2 R}{\partial \theta^2} \cos n\theta - n \frac{\partial R}{\partial \theta} \sin n\theta \right] \\ & -n^2 (h-R)^{\frac{1}{2}} \cos n\theta, \end{aligned}$$

which together with condition (iii) implies

$$|D_\theta^2((h-R)^{\frac{1}{2}} \cos n\theta)| \leq C \cdot h^{\frac{1}{2}}. \tag{2.25}$$

Let $\theta_1 = \frac{(i-1)2\pi + h^{-\frac{\delta}{2}}}{n}$, $\theta_2 = \frac{(i-1)2\pi+\pi-h^{-\frac{\delta}{2}}}{n}$. By integration by parts, we have that

$$\int_{I_i^{21}} \mathcal{R}(h,t,\theta)g'((h-R)^{\frac{1}{2}}\cos n\theta)d\theta = \int_{I_i^{21}} \frac{\mathcal{R}(h,t,\theta)dg((h-R)^{\frac{1}{2}}\cos n\theta)}{D_\theta((h-R)^{\frac{1}{2}}\cos n\theta)}$$
$$= \frac{\mathcal{R}(h,t,\theta)g((h-R)^{\frac{1}{2}}\cos n\theta)}{D_\theta((h-R)^{\frac{1}{2}}\cos n\theta)}\Big|_{\theta=\theta_1}^{\theta=\theta_2} - \int_{I_i^{21}} \frac{g((h-R)^{\frac{1}{2}}\cos n\theta)\frac{\partial \mathcal{R}}{\partial\theta}}{D_\theta((h-R)^{\frac{1}{2}}\cos n\theta)}d\theta$$
$$+ \int_{I_i^{21}} \frac{\mathcal{R}(h,t,\theta)g((h-R)^{\frac{1}{2}}\cos n\theta)D_\theta^2((h-R)^{\frac{1}{2}}\cos n\theta)}{(D_\theta((h-R)^{\frac{1}{2}}\cos n\theta))^2}d\theta.$$

Then conditions (i), (iv) and (2.24) imply

$$|\mathcal{R}(h,t,\theta_i)g((h-R)^{\frac{1}{2}}\cos n\theta_i)\cdot(D_\theta((h-R)^{\frac{1}{2}}\cos n\theta_i))^{-1}| \le C\cdot h^{-a-\frac{1}{2}+\frac{\delta}{2}}, \quad i=1,2$$
$$|g((h-R)^{\frac{1}{2}}\cos n\theta)\frac{\partial \mathcal{R}}{\partial\theta}\cdot(D_\theta((h-R)^{\frac{1}{2}}\cos n\theta))^{-1}| \le C\cdot h^{-a-\frac{1}{2}+\frac{\delta}{2}}, \quad \theta\in I_i^{21}.$$

Thus conditions (i), (iv) and (2.25) imply that for $\theta \in I_i^{21}$, it holds that

$$|\mathcal{R}(h,t,\theta)((h-R)^{\frac{1}{2}}\cos n\theta)D_\theta^2((h-R)^{\frac{1}{2}}\cos n\theta)\cdot(D_\theta((h-R)^{\frac{1}{2}}\cos n\theta))^{-2}|$$
$$\le C\cdot h^{-a-\frac{1}{2}+\delta}.$$

Similarly, we have the same estimate for the other parts of I_i^{22}. Hence from the fact $0<\delta<\frac{1}{3}$, we obtain (2.23). This completes the proof of this lemma. ■

For $F(h,t,\theta) = g'((h-R)^{\frac{1}{2}}\cos n\theta)v(t,\theta)$, we have

$$[F](h,t) = \frac{1}{2\pi}\int_0^{2\pi} g'((h-R)^{\frac{1}{2}}\cos n\theta)v(t,\theta)d\theta.$$

The following estimates hold true for $[F]$:

Corollary 1. *Assume that*

(i) $g(x)$ *satisfies (1.9);*

(ii) $|\frac{\partial^l v(t,\theta)}{\partial t^l}|$, $|\frac{\partial^{l+1}v(t,\theta)}{\partial t^l\partial\theta}| \le C$, $l\le 10$;

(iii) $|R|$, $|\frac{\partial R}{\partial\theta}| \le C\cdot h^{\frac{1}{2}}$, $|\frac{\partial^2 R}{\partial\theta^2}| \le C\cdot h$;

(iv) $|\frac{\partial^{k+l}R}{\partial h^k\partial t^l}| \le C\cdot h^{-\frac{k}{2}}$, $|\frac{\partial^{k+l+1}R}{\partial h^k\partial t^l\partial\theta}| \le C\cdot h^{-\frac{k}{2}+\frac{1}{2}}$, $1\le k+l\le 10$.

Then for any constant $\delta\in(0,\frac{1}{3})$, it holds that

(2.26) $$\left|\frac{\partial^{k+l}}{\partial h^k\partial t^l}[F](h,t)\right| \le C\cdot h^{-\frac{\delta}{2}-\frac{k}{2}}, \quad 0\le k+l\le 10.$$

Proof.

(a) When $k+l=0$, (2.26) is deduced from (2.23), where we set $\mathcal{R}=v(t,\theta)$.

(b) When $k+l>0$, one can easily prove that $\frac{\partial^{k+l}[F]}{\partial h^k \partial t^l}(h,t)$ is the sum of some finite terms with the following form:

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{2}{n^2}\frac{\partial^{k+l_1}}{\partial h^k \partial t^{l_1}}(g'((h-R)^{\frac{1}{2}}\cos n\theta))\cdot\frac{\partial^{l_2}v(t,\theta)}{\partial t^{l_2}}d\theta,$$

where $l_1\geq 0,\quad l_2\geq 0,\quad l_1+l_2=l$.

(b1) when $k=0,\ l_1=0,\ l_2=l>0$. Then (2.26) is deduced from (2.23), where we set $\mathcal{R}=\frac{\partial^l v(t,\theta)}{\partial t^l}$.

(b2) when $k+l_1\geq 1$, it can be seen that $\frac{\partial^{k+l_1}}{\partial h^k \partial t^{l_1}}g'((h-R)^{\frac{1}{2}}\cos n\theta)$ is the sum of the term like

$$g^{(m+1)}(u\cdot\cos n\theta)\frac{\partial^{k_1+l_1^1}u}{\partial h^{k_1}\partial t^{l_1^1}}\cdot\frac{\partial^{k_2+l_2^1}u}{\partial h^{k_2}\partial t^{l_2^1}}\cdots\frac{\partial^{k_m+l_m^1}u}{\partial h^{k_m}\partial t^{l_m^1}}\cdot(\cos n\theta)^m,$$

where $u=(h-R)^{\frac{1}{2}},\quad k_1+\cdots+k_m=k,\quad l_1^1+\cdots+l_m^1=l_1,\quad k_i+l_i^1\geq 1$. Notice that for $i+j\geq 1$, conditions (iii), (iv) imply the following conclusions

$$\left|\frac{\partial^{i+j}u}{\partial h^i\partial t^j}\right|\leq C\cdot h^{-\frac{i}{2}},\qquad \left|\frac{\partial^{i+j+1}u}{\partial h^i\partial t^j\partial\theta}\right|\leq C\cdot h^{-\frac{i}{2}}.$$

Consequently we have that

$$\left|\frac{\partial^{k_1+l_1^1}u}{\partial h^{k_1}\partial t^{l_1^1}}\cdot\frac{\partial^{k_2+l_2^1}u}{\partial h^{k_2}\partial t^{l_2^1}}\cdots\frac{\partial^{k_m+l_m^1}u}{\partial h^{k_m}\partial t^{l_m^1}}\right|\leq C\cdot h^{-\frac{k}{2}},\tag{2.27}$$

$$\left|\frac{\partial}{\partial\theta}\left\{\frac{\partial^{k_1+l_1^1}u}{\partial h^{k_1}\partial t^{l_1^1}}\cdot\frac{\partial^{k_2+l_2^1}u}{\partial h^{k_2}\partial t^{l_2^1}}\cdots\frac{\partial^{k_m+l_m^1}u}{\partial h^{k_m}\partial t^{l_m^1}}\right\}\right|\leq C\cdot h^{-\frac{k}{2}}.\tag{2.28}$$

So (2.26) follows from condition (ii), (2.23), (2.27), (2.28) and Lemma 2.7 with $\mathcal{R}=\frac{\partial^{k_1+l_1^1}u}{\partial h^{k_1}\partial t^{l_1^1}}\cdot\frac{\partial^{k_2+l_2^1}u}{\partial h^{k_2}\partial t^{l_2^1}}\cdots\frac{\partial^{k_m+l_m^1}u}{\partial h^{k_m}\partial t^{l_m^1}}\cdot(\cos n\theta)^m\cdot\frac{\partial^{l_2}v(t,\theta)}{\partial t^{l_2}}$. Thus the proof of the corollary ends. ■

From (2.19), we obtain that the Hamiltonian $r(h,t,\theta)$ in (2.16) is of the form:

$$r=h-f_1(h,\theta)-R_1(h,t,\theta),\tag{2.29}$$

where $f_1(h,\theta)\ \ R_1(h,t,\theta)$ satisfy (2.8) and (2.21), respectively.

Now the system (2.15) can be written in the form

$$\frac{dh}{d\theta}=\frac{\partial R_1}{\partial t},\quad \frac{dt}{d\theta}=1-\frac{\partial f_1}{\partial h}(h,\theta)-\frac{\partial R_1}{\partial h}.\tag{2.30}$$

3. More Canonical Transformations

In this section, we will make some more canonical transformations such that the Poincaré map of the new system is close to a twist map.

Lemma 3.1. *There exists a canonical transformation Ψ_1 of the form:*

$$\Psi_1 : \quad h = \varrho, \quad t = \varphi + T(\varrho, \theta)$$

with $T(\varrho,\theta) = T(\varrho, \theta + 2\pi)$ such that the transformed system of (2.30) is of the form

$$\frac{d\varrho}{d\theta} = -\frac{\partial r_1}{\partial \varphi}(\varrho, \varphi, \theta), \quad \frac{d\varphi}{d\theta} = \frac{\partial r_1}{\partial \varrho}(\varrho, \varphi, \theta), \tag{3.1}$$

where

$$r_1(\varrho, \varphi, \theta) = \varrho - [f_1](\varrho) - R_2(\varrho, \varphi, \theta).$$

Moreover, the new perturbation R_2 satisfies

$$\left| \frac{\partial^{k+l} R_2}{\partial \varrho^k \partial \varphi^l}(\varrho, \varphi, \theta) \right| \le C \cdot \varrho^{-\frac{k}{2}}, \tag{3.2}$$

for $k + l \le 10$.

proof. We construct the canonical transformation by means of a generating function:

$$\Psi_1 : \quad \varrho = h + \frac{\partial S_1}{\partial \varphi}(h, \varphi, \theta), \quad t = \varphi + \frac{\partial S_1}{\partial h}(h, \varphi, \theta).$$

We choose

$$S_1 = \int_0^\theta (f_1(h, s) - [f_1](h)) ds.$$

Let $T(h, \theta) = \frac{\partial S_1}{\partial h}$, then the transformation Ψ_1 is of the form

$$h = \varrho, \quad t = \varphi + T(\varrho, \theta).$$

Define

$$R_2(\varrho, \varphi, \theta) = R_1(\varrho, \varphi, \theta) + \int_0^1 \frac{\partial R_1}{\partial t}(\varrho, \varphi + \mu T(\varrho, \theta), \theta) T(\varrho, \theta) d\mu. \tag{3.3}$$

Then the transformed Hamiltonian function is of the form

$$r_1(\varrho, \varphi, \theta) = \varrho - [f_1](\varrho) - R_2(\varrho, \varphi, \theta).$$

By the definition of T and (2.8), we can obtain

$$\left|\frac{\partial^k T}{\partial \varrho^k}(\varrho,\theta)\right| \leq C\cdot \varrho^{-k-\frac{1}{2}}. \tag{3.4}$$

Combining (3.4) with (2.21), we have

$$\left|\frac{\partial^{k+l}}{\partial \varrho^k \partial \varphi^l}\int_0^1 \frac{\partial R_1}{\partial t}(\varrho,\varphi+\mu T(\varrho,\theta),\theta)T(\varrho,\theta)d\mu\right| \leq C\cdot \varrho^{-\frac{k}{2}-\frac{1}{2}}, \tag{3.5}$$

which together with (2.21) yields (3.2). So the proof of the lemma ends. ∎

With the idea in [16], we construct a canonical transformation as follows:

Lemma 3.2. *Under the following transformation* Ψ_2*:*

$$\Psi_2: \quad \vartheta=\theta, \quad \rho=\varrho, \quad \tau=\varphi-\theta,$$

(*3.1*) *is transformed into the following system*

$$\frac{d\rho}{d\vartheta}=-\frac{\partial r_2}{\partial \tau}(\rho,\tau,\vartheta), \quad \frac{d\tau}{d\vartheta}=\frac{\partial r_2}{\partial \rho}(\rho,\tau,\vartheta), \tag{3.6}$$

where

$$r_2(\rho,\tau,\vartheta)=-[f_1](\rho)-R_3(\rho,\tau,\vartheta).$$

Moreover the new perturbation R_3 *satisfies*

$$\left|\frac{\partial^{k+l} R_3}{\partial \rho^k \partial \tau^l}(\rho,\tau,\vartheta)\right| \leq C\cdot \rho^{-\frac{k}{2}}, \quad \left|\frac{\partial^{k+l}[R_3]}{\partial \rho^k \partial \tau^l}(\rho,\tau)\right| \leq C\cdot(\rho^{-k}+\rho^{-\frac{k}{2}-\frac{\delta}{2}}) \tag{3.7}$$

for $k+l\leq 10$.

Proof. Under the transformation Ψ_2, the transformed system is of the form

$$\frac{d\rho}{d\vartheta}=-\frac{\partial r_2}{\partial \tau}(\rho,\tau,\vartheta), \quad \frac{d\tau}{d\vartheta}=\frac{\partial r_2}{\partial \rho}(\rho,\tau,\vartheta),$$

where $r_2(\rho,\tau,\vartheta)=-[f_1](\rho)-R_3(\rho,\tau,\vartheta)$ and R_3 is of the following form

$$R_3(\rho,\tau,\vartheta)=R_2(\rho,\tau+\vartheta,\vartheta), \tag{3.8}$$

which implies $\frac{\partial^{k+l}}{\partial\rho^k\partial\tau^l}R_3(\rho,\tau,\vartheta)=\frac{\partial^{k+l}}{\partial\rho^k\partial\tau^l}R_2(\rho,\tau,\vartheta)$.
From (3.3) and (3.8), it follows that

$$R_3(\rho,\tau,\vartheta)=R_1(\rho,\tau+\vartheta,\vartheta)+\int_0^1 \frac{\partial R_1}{\partial t}(\rho,\tau+\vartheta+\mu T(\rho,\vartheta),\vartheta)T(\rho,\vartheta)d\mu.$$

(2.20) gives

$$R_1(\rho,\tau+\vartheta,\vartheta) = -\frac{\partial f_1}{\partial r}(\rho,\vartheta)R(\rho,\tau+\vartheta,\vartheta)+f_2(\rho-R(\rho,\tau+\vartheta,\vartheta),\vartheta,\tau+\vartheta) + \int_0^1\int_0^1 \frac{\partial^2 f_1}{\partial r^2}(\rho-s\mu R(\rho,\tau+\vartheta,\vartheta),\vartheta)\mu R^2(\rho,\tau+\vartheta,\vartheta)dsd\mu.$$

With Lemma 2.1, 2.5 and (3.5), we have

$$\left|\frac{\partial^{k+l}}{\partial\rho^k\partial\tau^l}\left(\frac{\partial f_1}{\partial r}(\rho,\vartheta)R(\rho,\tau+\vartheta,\vartheta)\right)\right| \le C\cdot(\rho^{-k}+\rho^{-\frac{k}{2}-\frac{1}{2}}),$$

$$\left|\frac{\partial^{k+l}}{\partial\rho^k\partial\tau^l}\int_0^1\int_0^1\frac{\partial^2 f_1}{\partial r^2}(\rho-s\mu R(\rho,\tau+\vartheta,\vartheta),\vartheta)\mu R^2(\rho,\tau+\vartheta,\vartheta)dsd\mu\right| \le C\cdot\rho^{-\frac{k}{2}-\frac{1}{2}},$$

$$\left|\frac{\partial^{k+l}}{\partial\rho^k\partial\tau^l}\int_0^1\frac{\partial R_1}{\partial t}(\rho,\tau+\vartheta+\mu T(\rho,\vartheta),\vartheta)T(\rho,\vartheta)d\mu\right| \le C\cdot\rho^{-\frac{k}{2}-\frac{1}{2}}.$$

Let $\tilde{R}(\rho,\tau,\vartheta)=R(\rho,\tau+\vartheta,\vartheta)$. From Lemma 2.4, 2.6, it is not difficulty to see that

$$|\tilde{R}|\le C\cdot\rho^{\frac{1}{2}}, \qquad |\frac{\partial}{\partial\vartheta}\tilde{R}|\le C\cdot\rho^{\frac{1}{2}}, \qquad |\frac{\partial^2}{\partial\vartheta^2}\tilde{R}|\le C\cdot\rho,$$
$$|\frac{\partial^{k+l}}{\partial\rho^k\partial\tau^l}\tilde{R}|\le C\cdot\rho^{-\frac{k}{2}}, \qquad |\frac{\partial^{k+l+1}}{\partial\rho^k\partial\tau^l\partial\vartheta}\tilde{R}|\le C\cdot\rho^{-\frac{k}{2}+\frac{1}{2}}. \tag{3.9}$$

To finish the proof of (3.7), it suffices to see that

$$[\tilde{f}_2](\rho,\tau)=\frac{1}{2\pi}\int_0^{2\pi}\frac{2}{n^2}g'((\rho-\tilde{R})^{\frac{1}{2}}\cos n\vartheta)q(\tau+\vartheta)d\vartheta$$

satisfies

$$\left|\frac{\partial^{k+l}[\tilde{f}_2]}{\partial\rho^k\partial\tau^l}(\rho,\tau)\right|\le C\cdot\rho^{-\frac{\delta}{2}-\frac{k}{2}}, \quad k+l\le 10, \tag{3.10}$$

which can be obtained from Corollary 1 and (3.9). Thus we complete the proof. ∎

Lemma 3.3. *Consider the system with Hamiltonian*

$$r(\rho,\tau,\theta)=-[f_1](\rho)+g(\rho,\tau)+R(\rho,\tau,\theta), \tag{3.11}$$

where $[f_1](\rho)$ satisfies (2.7), (2.8) and (2.14), while $g(\rho,\tau), R(\rho,\tau,\theta)$ satisfy respectively

$$\left|\frac{\partial^{k+l}g}{\partial\rho^k\partial\tau^l}(\rho,\tau)\right|\le C\cdot(\rho^{-k}+\rho^{-\frac{k}{2}-\frac{\delta}{2}}), \quad \left|\frac{\partial^{k+l}}{\partial\rho^k\partial\tau^l}R(\rho,\tau,\theta)\right|\le C\cdot\rho^{-\frac{k}{2}-\varepsilon}, \tag{3.12}$$

for $k+l\le n$ with $\delta,\ \varepsilon\ge 0$. Then there exists a canonical transformation Ψ of the form:

$$\Psi: \quad \rho = I + u(I,\psi,\theta), \quad \tau = \psi + v(I,\psi,\theta)$$

such that the Hamiltonian of the transformed system of (3.11) is

$$\hat{r}(I,\psi,\theta) = -[f_1](I) + \hat{g}(I,\psi) + \hat{R}(I,\psi,\theta),$$

where $\hat{g}, \hat{R}$ *satisfy*

$$\text{(3.13)} \quad \Big|\frac{\partial^{k+l}\hat{g}}{\partial I^k \partial \psi^l}(I,\psi)\Big| \le C\cdot(I^{-k} + I^{-\frac{k}{2}-\frac{\delta}{2}}), \quad \Big|\frac{\partial^{k+l}\hat{R}}{\partial I^k \partial \psi^l}(I,\psi,\theta)\Big| \le C\cdot I^{-\frac{k}{2}-\frac{1}{2}-\varepsilon},$$

for $k+l \le n-1$.

Proof. We construct the canonical transformation by means of a generating function:

$$\Psi: \quad \rho = I + \frac{\partial S}{\partial \tau}(I,\tau,\theta), \quad \psi = \tau + \frac{\partial S}{\partial I}(I,\tau,\theta).$$

Then the transformed Hamiltonian function $\hat{r}$ is of the form

$$\begin{aligned}
\hat{r} &= -[f_1](I + \frac{\partial S}{\partial \tau}) + g(I + \frac{\partial S}{\partial \tau}, \tau) + R(I + \frac{\partial S}{\partial \tau}, \tau, \theta) + \frac{\partial S}{\partial \theta} \\
&= -[f_1](I) + g(I,\tau) + R(I,\tau,\theta) + \frac{\partial S}{\partial \theta} - \int_0^1 [f_1]'(I + \mu\frac{\partial S}{\partial \tau})\frac{\partial S}{\partial \tau} d\mu \\
&\quad + \int_0^1 \frac{\partial g}{\partial \rho}(I + \mu\frac{\partial S}{\partial \tau}, \tau)\frac{\partial S}{\partial \tau} d\mu + \int_0^1 \frac{\partial R}{\partial \rho}(I + \mu\frac{\partial S}{\partial \tau}, \tau, \theta)\frac{\partial S}{\partial \tau} d\mu.
\end{aligned}$$

Define

$$\text{(3.14)} \qquad [R](I,\tau) = \frac{1}{2\pi}\int_0^{2\pi} R(I,\tau,\theta)d\theta.$$

Now we choose $S(I,\tau,\theta)$

$$S(I,\tau,\theta) = -\int_0^{\theta} (R(I,\tau,s) - [R](I,\tau))ds.$$

It is obvious that $S(I,\tau,\theta)$ is 2π-periodic in τ, θ and

$$\text{(3.15)} \qquad \Big|\frac{\partial^{k+l} S}{\partial I^k \partial \tau^l}(I,\tau,\theta)\Big| \le C\cdot I^{-\frac{k}{2}-\varepsilon}.$$

Since $\tau = \psi - \frac{\partial S}{\partial I}$, by Taylor's formula, we can write

$$g(I,\tau) = g(I,\psi) - \int_0^1 \frac{\partial g}{\partial \tau}(I, \psi - \mu\frac{\partial S}{\partial I})\frac{\partial S}{\partial I} d\mu,$$

$$[R](I,\tau) = [R](I,\psi) - \int_0^1 \frac{\partial [R]}{\partial \tau}(I, \psi - \mu \frac{\partial S}{\partial I}) \frac{\partial S}{\partial I} d\mu.$$

Let $\hat{g}(I,\psi) = g(I,\psi) + [R](I,\psi)$ and

$$\begin{aligned}\hat{R}(I,\psi,\theta) = & -\int_0^1 \frac{\partial g}{\partial \tau}(I,\psi - \mu\frac{\partial S}{\partial I})\frac{\partial S}{\partial I} d\mu - \int_0^1 \frac{\partial [R]}{\partial \tau}(I,\psi - \mu\frac{\partial S}{\partial I})\frac{\partial S}{\partial I} d\mu \\ & - \int_0^1 [f_1]'(I + \mu\frac{\partial S}{\partial \tau})\frac{\partial S}{\partial \tau} d\mu + \int_0^1 \frac{\partial g}{\partial \rho}(I + \mu\frac{\partial S}{\partial \tau}, \tau)\frac{\partial S}{\partial \tau} d\mu \\ & + \int_0^1 \frac{\partial R}{\partial \rho}(I + \mu \frac{\partial S}{\partial \tau}, \tau, \theta)\frac{\partial S}{\partial \tau} d\mu.\end{aligned}$$

Then the transformed Hamiltonian is of the form

$$\hat{r}(I,\psi,\theta) = -[f_1](I) + \hat{g}(I,\psi) + \hat{R}(I,\psi,\theta).$$

We can solve the two equations in Ψ for ρ and τ due to (3.15) and write $\rho = \rho(I,\psi,\theta)$ and $\tau = \tau(I,\psi,\theta)$. Moreover we have that

$$(3.16)\qquad \begin{aligned} & \left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l}\tau\right| \le C \cdot I^{-\frac{k}{2}}, && 1 \le k+l \le n, \\ & \left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l}\rho\right| \le C, \quad k+l=1; \left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l}\rho\right| \le C \cdot I^{-\frac{k}{2}}, && 2 \le k+l \le n. \end{aligned}$$

For the convenience of readers, we give the proof of (3.16) in Appendix.

Thus (3.13) follows from (2.8) and (3.16), (3.12). This ends the proof of this lemma. ∎

By repeated use of Lemma 3.3, we obtain from Lemma 3.2 that

Corollary 2. *There exists a canonical transformation Ψ_3 of the form:*

$$\Psi_3 : \quad \rho = I + u_3(I,\psi,\vartheta), \quad \tau = \psi + v_3(I,\psi,\vartheta)$$

such that the system (3.6) is transformed into the one with Hamiltonian

$$(3.17)\qquad r_3(I,\psi,\vartheta) = -[f_1](I) - g(I,\psi) - R_4(I,\psi,\vartheta)$$

where $[f_1](I)$ satisfies Lemma 2.3 and (2.14), while $g(I,\psi)$ and R_4 satisfy

$$(3.18)\qquad \left|\frac{\partial^{k+l} g}{\partial I^k \partial \psi^l}(I,\psi)\right| \le C \cdot (I^{-k} + I^{-\frac{k}{2} - \frac{\delta}{2}}),$$

$$(3.19)\qquad \left|\frac{\partial^{k+l} R_4}{\partial I^k \partial \psi^l}(I,\psi,\vartheta)\right| \le C \cdot I^{-\frac{k}{2} - \frac{5}{2}}$$

for $k + l \le 5$, respectively.

Introducing a new time variable by $\gamma = -\vartheta$, let $\mathcal{R}_4(I,\psi,\gamma) = R_4(I,\psi,-\gamma)$, then the system with Hamiltonian (3.17) can be rewritten as

$$\frac{dI}{d\gamma} = -\frac{\partial H_3}{\partial \psi}(I,\psi,\gamma), \quad \frac{d\psi}{d\gamma} = \frac{\partial H_3}{\partial I}(I,\psi,\gamma), \tag{3.20}$$

where

$$H_3(I,\psi,\gamma) = -r_3(I,\psi,-\gamma) = [f_1](I) + g(I,\psi) + \mathcal{R}_4(I,\psi,\gamma)$$

and $g(I,\psi)$, $\mathcal{R}_4$ satisfy respectively

$$\Big|\frac{\partial^{k+l} g}{\partial I^k \partial \psi^l}(I,\psi)\Big| \le C\cdot(I^{-k} + I^{-\frac{k}{2}-\frac{\delta}{2}}), \quad k+l \le 5, \tag{3.21}$$

$$\Big|\frac{\partial^{k+l}\mathcal{R}_4}{\partial I^k \partial \psi^l}(I,\psi,\gamma)\Big| \le C\cdot I^{-\frac{k}{2}-\frac{5}{2}}, \quad k+l \le 5. \tag{3.22}$$

Next we will define a canonical transformation such that the leading term in Hamiltonian is independent of angle variable.

Lemma 3.4. *For any* $I_0 > 1$, *there exist* $c\cdot I_0 \le A_1, A_2, B_1, B_2 \le C\cdot I_0$ *satisfying* $c\cdot I_0^{\frac{\delta}{4}} \le A_2 - A_1,\ B_2 - B_1 \le C\cdot I_0^{\frac{\delta}{4}}$ *and a canonical transformation* $\Psi_4 : [B_1,B_2]\times\mathbb{S}^1 \mapsto [A_1,A_2]\times\mathbb{S}^1,\ (\zeta,\eta) \mapsto (I,\psi) = \Psi_4(\zeta,\eta)$ *such that the system (3.20) is transformed into the one with Hamiltonian*

$$H_4(\zeta,\eta,\gamma) = G(\zeta) + R_5(\zeta,\eta,\gamma). \tag{3.23}$$

Moreover, $G(\zeta)$ *satisfies*

$$\begin{aligned} &c\cdot I < \zeta < C\cdot I, && c\cdot\zeta^{-\frac{1}{2}} < G'(\zeta) < C\cdot\zeta^{-\frac{1}{2}}, \\ &c\cdot\zeta^{-\frac{3}{2}} < |G''(\zeta)|, && |G^{(k)}(\zeta)| < C\cdot\zeta^{-\frac{k}{2}} \quad k = 1,2,3,4,5, \end{aligned} \tag{3.24}$$

and $R_5(\zeta,\eta,\gamma)$ *satisfies*

$$\Big|\frac{\partial^{k+l}}{\partial\zeta^k\partial\eta^l}R_5(\zeta,\eta,\gamma)\Big| \le C\cdot\zeta^{-\frac{k}{2}-\frac{5}{2}}, \quad for \quad k+l \le 5. \tag{3.25}$$

Proof. In the following, we will follow the method of [2] (also see [6]) to eliminate angle variable ψ from the leading term $[f_1](I) + g(I,\psi)$ in H_3. We define a canonical transformation Ψ_4 by means of a generating function $S_4(\psi,\zeta)$:

$$\Psi_4 : \quad \frac{\partial S_4}{\partial\psi}(\psi,\zeta) = I, \quad \frac{\partial S_4}{\partial\zeta}(\psi,\zeta) = \eta \tag{3.26}$$

such that the Hamiltonian of the transformed system would be η-independent in its leading part, that is,

$$[f_1](I) + g(I,\psi) \equiv G(\zeta). \tag{3.27}$$

As an additional condition, we suppose the map preserves the periodicity: if $\Psi_4 : (I,\psi) \longrightarrow (\zeta,\eta)$, then $\Psi_4 : (I,\psi+2\pi) \longrightarrow (\zeta,\eta+2\pi)$. These two conditions can define $G(\zeta)$ (up to a constant).
In fact, owing to (2.14), (3.21), we can solve (3.27) for I such that $I = I(G,\psi)$ by Implicit function theorem.

Define

$$\bar{I}(G) = \frac{1}{2\pi}\int_0^{2\pi} I(G,\psi)d\psi.$$

From (3.26), $S_4(\psi,\zeta)$ can be taken as

$$S_4(\psi,\zeta) = \int_0^{\psi} I(G,s)ds. \tag{3.28}$$

The periodicity condition translates into

$$\frac{\partial S_4}{\partial \zeta}(2\pi,\zeta) - \frac{\partial S_4}{\partial \zeta}(0,\zeta) = \int_0^{2\pi} \frac{\partial I}{\partial G}(G,s)ds \cdot G'(\zeta) = 2\pi,$$

which holds if we choose $G = \bar{I}^{-1}(\zeta)$.

We will look for an interval $[A_1, A_2]$ such that $\frac{\partial^2 G}{\partial I^2}(I,\psi)$($\psi$ is a parameter) is nonzero on it. Consider the interval $[I_0, 2I_0]$ with $I_0 > 1$. From (2.14) and (3.21), it follows that the set $\{\frac{\partial G}{\partial I}(I,\psi)\big| I \in [\frac{5}{4}I_0, \frac{7}{4}I_0]\}$ covers some interval with length longer than $c \cdot I_0^{-\frac{1}{2}}$. In fact, we have

$$\begin{aligned}
\left|\frac{\partial G}{\partial I}\Big|_{\frac{5}{4}I_0}^{\frac{7}{4}I_0}\right| &> \Big|\int_{\frac{5}{4}I_0}^{\frac{7}{4}I_0} [f_1]''(I)dI\Big| - \left|\frac{\partial}{\partial I} g(I,\psi)\Big|_{\frac{5}{4}I_0}^{\frac{7}{4}I_0}\right| \\
&> \Big|\int_{\frac{5}{4}I_0}^{\frac{7}{4}I_0} [f_1]''(I)dI\Big| - C \cdot I_0^{-\frac{1}{2}-\frac{\delta}{2}} \\
&= \int_{\frac{5}{4}I_0}^{\frac{7}{4}I_0} |[f_1]''(I)|dI - C \cdot I_0^{-\frac{1}{2}-\frac{\delta}{2}} > c \cdot I_0^{-\frac{1}{2}}.
\end{aligned}$$

Thus by Mean Value theorem, there exists some point $A_1 \in [\frac{5}{4}I_0, \frac{7}{4}I_0]$ such that $|\frac{\partial^2}{\partial I^2}G(A_1,\psi)| \geq c \cdot I_0^{-\frac{3}{2}}$. On the other hand, (2.14) and (3.21) imply $|\frac{\partial^3}{\partial I^3}G(I,\psi)| \leq C \cdot I^{-\frac{3}{2}-\frac{\delta}{2}}$. Let $A_2 = A_1 + I_0^{\frac{\delta}{4}}$. Consequently, for each $I \in [A_1, A_2]$, we have

$$\begin{aligned}\left|\frac{\partial^2}{\partial I^2}G(I,\psi)\right| &= |\frac{\partial^2 G}{\partial I^2}(A_1,\psi)+\frac{\partial^2}{\partial I^2}G(I,\psi)-\frac{\partial^2 G}{\partial I^2}(A_1,\psi)| \\ &\geq |\frac{\partial^2 G}{\partial I^2}(A_1,\psi)|-|\frac{\partial^2 G}{\partial I^2}(I,\psi)-\frac{\partial^2 G}{\partial I^2}(A_1,\psi)| \\ &\geq c\cdot I_0^{-\frac{3}{2}}-|\frac{\partial^3 G}{\partial I^3}(A_1+\mu(I-A_1),\psi)(I-A_1)| \\ &\geq c\cdot I_0^{-\frac{3}{2}}-C\cdot I_0^{-\frac{3}{2}-\frac{\delta}{2}}\cdot I_0^{\frac{\delta}{4}}\geq \frac{1}{2}c\cdot I_0^{-\frac{3}{2}}\geq \frac{1}{2}c\cdot I^{-\frac{3}{2}}.\end{aligned} \tag{3.29}$$

On $[A_1, A_2]$, (2.14), (3.21) and (3.29) yield that $G(I,\psi) = [f_1](I) + g(I,\psi)$ satisfies

$$\begin{aligned} c\cdot I^{-k+\frac{1}{2}} < \frac{\partial^k G}{\partial I^k} &< C\cdot I^{-k+\frac{1}{2}}, \qquad k=0,1; \quad c\cdot I^{-\frac{3}{2}} < |\tfrac{\partial^2 G}{\partial I^2}|; \\ |\frac{\partial^{k+l}}{\partial I^k\partial\psi^l}G(I,\psi)| &\leq C\cdot I^{-\frac{k}{2}}, \qquad 0<k+l\leq 5. \end{aligned} \tag{3.30}$$

Moreover, we can easily prove the following estimates:

$$\begin{aligned} c\cdot I^{\frac{1}{2}} < |\frac{\partial I}{\partial G}(G,\psi)|; \qquad & c < |\frac{\partial^2 I}{\partial G^2}(G,\psi)|; \\ |\frac{\partial^{k+l}}{\partial G^k\partial\psi^l}I(G,\psi)| \leq C\cdot I^{\frac{1}{2}}, \qquad & 0<k+l\leq 5. \end{aligned} \tag{3.31}$$

(3.31) yields that

$$\begin{aligned} & c\cdot I < \bar I < C\cdot I; \qquad c\cdot I^{\frac{1}{2}} < \bar I'(G) < C\cdot I^{\frac{1}{2}}; \quad c < |\bar I''(G)|; \\ & |\bar I^{(k)}(G)| < C\cdot I^{\frac{1}{2}} \qquad k=1,2,3,4,5. \end{aligned} \tag{3.32}$$

Let

$$B_1 = \bar I(G(A_1,\psi)), \qquad B_2 = \bar I(G(A_2,\psi)). \tag{3.33}$$

(3.30) and (3.32) together with the definition of A_2 yield that $[B_1, B_2] \subset [c\cdot I_0, C\cdot I_0]$ is of length between $c\cdot I_0^{\frac{\delta}{4}}$ and $C\cdot I_0^{\frac{\delta}{4}}$.

Next, we prove that $G(\zeta)$ satisfies (3.24) on $[B_1, B_2]$. By the definition of $G = \bar I^{-1}(\zeta)$ and (3.32), we have

$$c\cdot I < c\cdot G^2 < \zeta = \bar I(G) < C\cdot G^2 < C\cdot I.$$

Differentiating on both sides of $\zeta = \bar I(G)$ with respect to ζ, one has

$$\begin{aligned} & G'(\zeta)\cdot \bar I'(G) && = 1 \\ & G''(\zeta)\bar I'(G) + \bar I''(G)\cdot (G'(\zeta))^2 && = 0 \\ & G^{(K_0+1)}(\zeta)\bar I'(G) + \sum \bar I^{(m)}(G)\cdot G^{(k_1)}(\zeta)\cdots G^{(k_m)}(\zeta) && = 0, \end{aligned} \tag{3.34}$$

where $1 < m \le k, \quad k_i > 0, \quad k_1 + \cdots + k_m = K_0 + 1$. Then (3.24) follows from (3.32) and (3.34) by induction.
The remain part is devoted to the proof of (3.25). Due to

$$\frac{\partial^2 S_4}{\partial \psi \partial \zeta} = \frac{\partial I}{\partial G}(G, \psi) \cdot G'(\zeta) > cI^{\frac{1}{2}} \cdot c\zeta^{-\frac{1}{2}} > 0,$$

we can solve $\frac{\partial S_4}{\partial \zeta}(\psi, \zeta) = \eta$ for $\psi = \psi(\zeta, \eta)$ and thus I has the following expression

$$I = \frac{\partial S_4}{\partial \psi}(\psi, \zeta) = \frac{\partial S_4}{\partial \psi}(\psi(\zeta, \eta), \zeta) = I(\zeta, \eta).$$

Moreover, these two functions satisfy, for $1 \le k + l \le 5$,

$$|\frac{\partial^{k+l} I}{\partial \zeta^k \partial \eta^l}(\zeta, \eta)| \le C \cdot \zeta^{-\frac{k}{2}+\frac{1}{2}}, \qquad |\frac{\partial^{k+l} \psi}{\partial \zeta^k \partial \eta^l}(\zeta, \eta)| \le C \cdot \zeta^{-\frac{k}{2}}. \tag{3.35}$$

In the following, we first express (ζ, η) in terms of (I, ψ) and estimate their derivatives.
From $G = \bar{I}^{-1}(\zeta)$ and (3.27), we have the expression for ζ:

$$\zeta = \bar{I}(G) = \bar{I}([f_1](I) + g(I, \psi)) \triangleq M(I, \psi). \tag{3.36}$$

Combining (3.26) with (3.28), we have the expression for η:

$$\begin{aligned} \eta &= \frac{\partial S_4}{\partial \zeta}(\psi, \zeta) \\ &= \int_0^{\psi} \frac{\partial}{\partial G} I(G, s)\Big|_{G=[f_1](I)+g(I,\psi)} ds \cdot G'(\zeta)\Big|_{\zeta=M(I,\psi)} \triangleq N(I, \psi). \end{aligned} \tag{3.37}$$

Note that $M(I, \psi), N(I, \psi)$ obey the estimate

$$\left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} M(I, \psi)\right| \le C \cdot I^{-\frac{k}{2}+\frac{1}{2}}, \qquad \left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} N(I, \psi)\right| \le C \cdot I^{-\frac{k}{2}}. \tag{3.38}$$

In fact, by the expression of $M(I, \psi)$, it follows that

$$\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} M(I, \psi) = \sum \bar{I}^{(m)}(G) \frac{\partial^{k_1+l_1}}{\partial I^{k_1} \partial \psi^{l_1}} G(I, \psi) \cdots \frac{\partial^{k_m+l_m}}{\partial I^{k_m} \partial \psi^{l_m}} G(I, \psi),$$

where $0 < m \le k + l, \quad k_i + l_i > 0, \quad k_1 + \cdots + k_m = k, \quad l_1 + \cdots + l_m = l$.
By (3.30) and (3.32), we have

$$\left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} M(I, \psi)\right| \le C \cdot I^{\frac{1}{2}} \cdot I^{-\frac{1}{2}(k_1+\cdots+k_m)} \le C \cdot I^{-\frac{k}{2}+\frac{1}{2}}.$$

To estimate $\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} N(I,\psi)$, let

$$J_1(I,\psi) = \int_0^{\psi} \frac{\partial}{\partial G} I(G,s)\Big|_{G=[f_1](I)+g(I,\psi)} ds, \qquad J_2(I,\psi) = G'(\zeta)\Big|_{\zeta=M(I,\psi)}.$$

we have

$$\left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} J_1(I,\psi)\right| \le C \cdot I^{-\frac{k}{2}+\frac{1}{2}}, \qquad \left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} J_2(I,\psi)\right| \le C \cdot I^{-\frac{k}{2}-\frac{1}{2}}. \tag{3.39}$$

In fact, by direct computation, we have

$$\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} J_1(I,\psi) = \int_0^{\psi} \frac{\partial^{k+l}}{\partial I^k \partial \psi^l} \frac{\partial}{\partial G} I(G,s)\Big|_{G=[f_1](I)+g(I,\psi)} ds + \frac{\partial^{k+l-1}}{\partial I^k \partial \psi^{l-1}} \frac{\partial}{\partial G} I(G,\psi).$$

Thus one can easily prove the first inequality of (3.39) from (3.30) and (3.31). The second inequality of (3.39) is a consequence of (3.24) and (3.38).
Since $J_1(I,\psi) \cdot J_2(I,\psi) = N(I,\psi)$, it follows that

$$\begin{aligned} \left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} N(I,\psi)\right| &\le \sum_{k_1+k_2=k,\ l_1+l_2=l} \left|\frac{\partial^{k_1+l_1}}{\partial I_1^{k} \partial \psi^{l_1}} J_1(I,\psi) \cdot \frac{\partial^{k_2+l_2}}{\partial I_2^{k} \partial \psi^{l_2}} J_2(I,\psi)\right| \\ &\le C \cdot I^{-\frac{k_1}{2}+\frac{1}{2}} \cdot I^{-\frac{k_2}{2}-\frac{1}{2}} \le C \cdot I^{-\frac{k}{2}}. \end{aligned}$$

From the fact that $(M(I,\psi), N(I,\psi))$ is the inverse of $(I(\zeta,\eta), \psi(\zeta,\eta))$, (3.35) can be easily obtained from (3.38).
Obviously

$$R_5(\zeta,\eta,\gamma) = \mathcal{R}_4(I(\zeta,\eta), \psi(\zeta,\eta), \gamma).$$

So (3.25) follows from (3.22) and (3.35). Thus we end the proof of this lemma. ■

4. Proof of Theorem 1

In this section, we first give the expression of the Poincaré map of the Hamiltonian system with the Hamiltonian (3.23). Then we will prove Theorem 1 via Moser's twist theorem.

Expression of the Poincaré map

From Lemma 3.4, the Hamiltonian system with the Hamiltonian (3.23) is of the form

$$\begin{cases} \dfrac{d\eta}{d\gamma} &= G'(\zeta) + \dfrac{\partial R_5}{\partial \zeta}(\zeta,\eta,\gamma), \\ \dfrac{d\zeta}{d\gamma} &= -\dfrac{\partial R_5}{\partial \eta}(\zeta,\eta,\gamma). \end{cases} \tag{4.1}$$

Moreover for $\zeta \in [B_1, B_2]$, G and R_5 satisfy (3.24) and (3.25), respectively.

Thus the Poincaré map of the equation (4.1) is of the form:

$$P: \begin{cases} \eta_1 &= \eta + \alpha(\zeta) + F_1(\zeta, \eta), \\ \zeta_1 &= \zeta + F_2(\zeta, \eta), \end{cases} \tag{4.2}$$

where, for $\zeta \in [B_1, B_2]$,

$$\begin{aligned} &c \cdot \zeta^{-\frac{1}{2}} < \alpha(\zeta) < C \cdot \zeta^{-\frac{1}{2}}, \qquad c \cdot \zeta^{-\frac{3}{2}} < |\alpha'(\zeta)|, \\ &|\alpha^{(k)}(\zeta)| < C \cdot \zeta^{-\frac{k}{2}-\frac{1}{2}}, \qquad\qquad k = 1, 2, 3, 4 \end{aligned} \tag{4.3}$$

and

$$\Big|\frac{\partial^{k+l} F_1}{\partial \zeta^k \partial \eta^l}(\zeta, \eta)\Big| \le C \cdot \zeta^{-3-\frac{k}{2}}, \quad \Big|\frac{\partial^{k+l} F_2}{\partial \zeta^k \partial \eta^l}(\zeta, \eta)\Big| \le C \cdot \zeta^{-\frac{5}{2}-\frac{k}{2}}, \; 0 \le k+l \le 4. \tag{4.4}$$

Next we make a scale transformation as follows:

$$\alpha(\zeta) - \alpha(B_1) = B_1^{-\frac{3}{2}} \nu, \qquad \nu \in [1,\ 2]. \tag{4.5}$$

We solve (4.5) for ζ to obtain $\zeta = \zeta(\nu)$. From the fact $B_2 - B_1 \ge c \cdot B_1^{\frac{\delta}{4}}$, we have that $\zeta([1, 2]) \subset [B_1, B_2]$.

Then the Poincaré map P is changed into the following one:

$$\tilde{P}: \begin{cases} \eta_1 &= \eta + \alpha(B_1) + B_1^{-\frac{3}{2}} \nu + \tilde{F}_1(\nu, \eta) \\ \nu_1 &= \nu + \tilde{F}_2(\nu, \eta), \end{cases} \tag{4.6}$$

where

$$\tilde{F}_1(\nu, \eta) = F_1(\zeta(\nu), \eta), \quad \tilde{F}_2(\nu, \eta) = B_1^{\frac{3}{2}} (\alpha(\zeta(\nu) + F_2(\zeta(\nu), \eta)) - \alpha(\zeta(\nu))). \tag{4.7}$$

From (4.3) and (4.5), we have that

$$|\zeta^{(i)}(\nu)| \le C, \quad 0 < i \le 4, \tag{4.8}$$

which together with (4.3), (4.4) and (4.7) implies

$$|\frac{\partial^{k+l} \tilde{F}_1}{\partial \nu^k \partial \eta^l}| \le C \cdot B_1^{-3}, \quad |\frac{\partial^{k+l} \tilde{F}_2}{\partial \nu^k \partial \eta^l}| \le C \cdot B_1^{-2}, \quad 0 \le k+l \le 4. \tag{4.9}$$

Since the map $\tilde{P}$ is time 1 map of the Hamiltonian system (3.23), it is area-preserving. Thus it possesses the intersection property in the annulus $[1, 2] \times S^1$, that is, if Γ is an embedded circle in $[1,2]\times S^1$ homotopic to a circle $\nu=$ constant then $\tilde{P}(\Gamma)\cap\Gamma\neq\emptyset$. The proof can be found in [3].

Proof of Theorem 1 via Moser's twist theorem

Until now, we have verified that the mapping $\tilde{P}$ satisfies all the conditions of Moser's small twist theorem [14]. Hence there is an invariant curve Γ of $\tilde{P}$ surrounding $\nu\equiv 1$ if $B_1\gg 1$. This means that there exist invariant curves of the Poincaré mapping of the system (3.23), which surround the origin $(x,y)=(0,0)$ and are arbitrarily far from the origin. Theorem 1 thus is proved.

5. Appendix

5.1. Proof of Lemma 2.4, 2.5 and 2.6

Proof of Lemma 2.4

(i) $k+l=0$. The proof for this case can be easily obtained from Lemma 2.1 and 2.2.

(ii) $k+l=1$. It is clear that for $h\gg 1, |\frac{\partial f_1}{\partial r}(h-R,\theta)+\frac{\partial f_2}{\partial r}(h-R,\theta,t)|\leq\frac{1}{2}$, Define

$$\Delta(h,t,\theta)=1+\frac{\partial f_1}{\partial r}(h-R,\theta)+\frac{\partial f_2}{\partial r}(h-R,\theta,t),\quad g_1=\Delta(h,t,\theta)-1,$$
$$g_2=\frac{\partial f_2}{\partial t}(h-R,\theta,t),\quad g_3=\frac{\partial}{\partial\theta}f_1(h-R,\theta)+\frac{\partial}{\partial\theta}f_2(h-R,\theta,t).$$

Then it follows that

$$\Delta\cdot\frac{\partial R}{\partial h}=g_1,\quad \Delta\cdot\frac{\partial R}{\partial t}=g_2,\quad \Delta\cdot\frac{\partial R}{\partial\theta}=g_3. \tag{5.1}$$

From Lemma 2.1 and 2.2, we have $|g_1|\leq C\cdot h^{-\frac{1}{2}}$, $|g_2|\leq C$, $|g_3|\leq C\cdot h^{\frac{1}{2}}$ which imply

$$|\frac{\partial R}{\partial h}|\leq C\cdot h^{-\frac{1}{2}},\quad |\frac{\partial R}{\partial t}|\leq C,\quad |\frac{\partial R}{\partial\theta}|\leq C\cdot h^{\frac{1}{2}}. \tag{5.2}$$

Thus the proof for this case is completed.

(iii) $k+l=2$. Lemma 2.1, 2.2 and (5.2) imply that

$$|\frac{\partial\Delta}{\partial t}|\leq C\cdot h^{-\frac{1}{2}},\quad |\frac{\partial\Delta}{\partial h}|\leq C\cdot h^{-1},\quad |\frac{\partial g_1}{\partial h}|\leq C\cdot h^{-1},\quad |\frac{\partial g_1}{\partial t}|\leq C\cdot h^{-\frac{1}{2}},$$
$$|\frac{\partial\Delta}{\partial\theta}|\leq C,\quad |\frac{\partial g_2}{\partial h}|\leq C\cdot h^{-\frac{1}{2}},\quad |\frac{\partial g_2}{\partial t}|\leq C,\quad |\frac{\partial g_3}{\partial h}|\leq C$$
$$|\frac{\partial g_3}{\partial t}|\leq C\cdot h^{\frac{1}{2}},\quad |\frac{\partial g_3}{\partial\theta}|\leq C\cdot h.$$

From the second equation of (5.1), we obtain

$$\Delta\frac{\partial^2 R}{\partial t^2}+\frac{\partial\Delta}{\partial t}\cdot\frac{\partial R}{\partial t}=\frac{\partial g_2}{\partial t},\qquad \Delta\frac{\partial^2 R}{\partial t\partial h}+\frac{\partial\Delta}{\partial h}\cdot\frac{\partial R}{\partial t}=\frac{\partial g_2}{\partial h}.$$

The above inequalities and equations imply that

$$|\frac{\partial^2 R}{\partial t^2}|\le C,\quad |\frac{\partial^2 R}{\partial h\partial t}|\le C\cdot h^{-\frac{1}{2}}.$$

From the first equation of (5.1), it follows that

$$\Delta\frac{\partial^2 R}{\partial h^2}+\frac{\partial\Delta}{\partial h}\cdot\frac{\partial R}{\partial h}=\frac{\partial g_1}{\partial h},$$

which implies $|\frac{\partial^2 R}{\partial h^2}|\le C\cdot h^{-1}$.
From the third equation of (5.1), we obtain

$$\Delta\frac{\partial^2 R}{\partial \theta^2}+\frac{\partial\Delta}{\partial \theta}\cdot\frac{\partial R}{\partial \theta}=\frac{\partial g_3}{\partial \theta},$$

which implies $|\frac{\partial^2 R}{\partial \theta^2}|\le C\cdot h$. Thus we complete the proof for this case.
By induction, suppose

$$|\frac{\partial^{k+l} R}{\partial h^k\partial t^l}|\le C\cdot h^{-\frac{k}{2}},\quad 1\le k+l\le m,$$

then it holds that

$$|\frac{\partial^{k+l}\Delta}{\partial h^k\partial t^l}|\le C\cdot h^{-\frac{1}{2}-\frac{k}{2}},\quad |\frac{\partial^{k+l}g_1}{\partial h^k\partial t^l}|\le C\cdot h^{-\frac{1}{2}-\frac{k}{2}}$$
$$|\frac{\partial^{k+l}g_2}{\partial h^k\partial t^l}|\le C\cdot h^{-\frac{k}{2}},\quad |\frac{\partial^{k+l}g_3}{\partial h^k\partial t^l}|\le C\cdot h^{-\frac{k}{2}+\frac{1}{2}}$$

for $\quad 1\le k+l\le m$. Consequently, we obtain

$$|\frac{\partial^{k+l} R}{\partial h^k\partial t^l}|\le C\cdot h^{-\frac{k}{2}}.$$

for $1\le k+l\le m+1$. So the proof is completed. ∎

Proof of Lemma 2.5

The lemma is easily followed from the following claim:

Claim

$$|\frac{\partial^{k+l}}{\partial h^k\partial t^l}\frac{\partial f_1}{\partial r}(h-\mu R,\theta)|\le C\cdot h^{-\frac{1}{2}-\frac{k}{2}},$$
(5.3)
$$|\frac{\partial^{k+l}}{\partial h^k\partial t^l}f_2(h-R,\theta,t)|\le C\cdot h^{-\frac{k}{2}}$$

for $0\le k+l\le 11$.

Proof. We only prove the first inequality of (5.3) and the proof for second one is similar.

(i) $k+l=0$. The proof for this case can be obtained directly from Lemma 2.1 and 2.4.

(ii) $k>0, l=0$. We have the following equality:

$$\frac{\partial^k}{\partial h^k}\frac{\partial f_1}{\partial r}(h-\mu R,\theta)=\sum \frac{\partial^{m+1}}{\partial r^{m+1}}f_1(u,\theta)\cdot\frac{\partial^{k_1}u}{\partial h^{k_1}}\cdots\frac{\partial^{k_m}u}{\partial h^{k_m}}$$

with $0<m\le k,\ k_1,\cdots,k_m>0,\ k_1+\cdots+k_m=k$ and $u=h-\mu R$. Assume there are $l(\le m)$ numbers in $\{k_1,\cdots,k_m\}$ which is equal to 1. Then we obtain

$$\Big|\frac{\partial^k}{\partial h^k}\frac{\partial f_1}{\partial r}(u,\theta)\Big|\le C\cdot h^{-m-\frac{1}{2}}\cdot h^{-\frac{k_1+\cdots+k_m-l}{2}}\le C\cdot h^{-\frac{k+1}{2}}.$$

(iii) $k=0, l>0$. By direct computation, we have

$$\frac{\partial^l}{\partial t^l}\frac{\partial f_1}{\partial r}(h-\mu R,\theta)=\sum \frac{\partial^{n+1}}{\partial r^{n+1}}f_1(u,\theta)\cdot\frac{\partial^{l_1}u}{\partial t^{l_1}}\cdots\frac{\partial^{l_n}u}{\partial t^{l_n}}$$

with $0<n\le l,\ l_1,\cdots,l_n>0,\ l_1+\cdots l_n=l$. It follows that

$$|\frac{\partial^l}{\partial t^l}\frac{\partial f_1}{\partial r}(u,\theta)|\le C\cdot h^{-n-\frac{1}{2}}\le C\cdot h^{-\frac{1}{2}}.$$

(iv) $k>0, l>0$. By direct computation, we have

$$\begin{aligned}&\frac{\partial^{k+l}}{\partial h^k\partial t^l}\frac{\partial f_1}{\partial r}(u,\theta)\\ &=\sum \frac{\partial^{m+n+1}}{\partial r^{m+n+1}}f_1(u,\theta)\cdot\frac{\partial^{k_1}u}{\partial h^{k_1}}\cdots\frac{\partial^{k_m}u}{\partial h^{k_m}}\cdot\frac{\partial^{k_{m+i}+l_1}u}{\partial h^{k_{m+1}}\partial t^{l_1}}\cdots\frac{\partial^{k_{m+n}}+l_n}{\partial h^{k_{m+n}}\partial t^{l_n}},\end{aligned}$$

where $u=h-\mu R$ and

$$0\le m\le k,\ 0\le n\le l,\ k_1,\cdots,k_m,\ l_1,\cdots,l_n>0,\quad k_{m+1},\cdots,k_{m+n}\ge 0,$$
$$k_1+\cdots k_m+k_{m+1}+\cdots+k_{m+n}=k,\quad l_1+\cdots+l_n=l.$$

Assume that there are $b(\le m)$ numbers in $\{k_1,\cdots,k_m\}$ which is equal to 1, Then

$$|\frac{\partial^{k+l}}{\partial h^k\partial t^l}\frac{\partial f_1}{\partial r}|\le C\cdot h^{-m-n-\frac{1}{2}}\cdot h^{-\frac{k_1+\cdots+k_m+k_{m+1}+\cdots+k_{m+n}-b}{2}}\le C\cdot h^{-\frac{k+1}{2}}.$$

This ends the proof of the claim. ∎

Proof of Lemma 2.6

We prove it by induction.

(i) When $k+l=1$. From the third equation of (5.1), we obtain

$$\Delta \cdot \frac{\partial^2 R}{\partial h \partial \theta} + \frac{\partial \Delta}{\partial h} \cdot \frac{\partial R}{\partial \theta} = \frac{\partial g_3}{\partial h}, \qquad \Delta \cdot \frac{\partial^2 R}{\partial t \partial \theta} + \frac{\partial \Delta}{\partial t} \cdot \frac{\partial R}{\partial \theta} = \frac{\partial g_3}{\partial t}$$

which implies $|\frac{\partial^2 R}{\partial h \partial \theta}| \leq C, \quad |\frac{\partial^2 R}{\partial t \partial \theta}| \leq C \cdot h^{\frac{1}{2}}$.

(ii) By induction, we assume that

$$|\frac{\partial^{k+l+1} R}{\partial h^k \partial t^l \partial \theta}| \leq C \cdot h^{-\frac{k}{2}+\frac{1}{2}}$$

for all $k+l \leq m$.

(iii) Then applying $\frac{\partial^{k+l}}{\partial h^k \partial t^l}$ to both sides of the third equation of (5.1), we have

$$\Delta \cdot \frac{\partial^{k+l+1}}{\partial h^k \partial t^l \partial \theta} R + \frac{\partial^{k_1+l_1} \Delta}{\partial h^{k_1} \partial t^{l_1}} \cdot \frac{\partial^{k_2+l_2}}{\partial h^{k_2} \partial t^{l_2}} \frac{\partial R}{\partial \theta} = \frac{\partial^{k+l}}{\partial h^k \partial t^l} g_3$$

where $k_1+k_2=k, \;\; l_1+l_2=l, \;\; k_1+l_1>0$. So by the hypothesis of induction,

$$\begin{aligned} \Big|\frac{\partial^{k+l+1}}{\partial h^k \partial t^l \partial \theta} R\Big| &\leq 2|\Delta| \cdot |\frac{\partial^{k+l+1}}{\partial h^k \partial t^l \partial \theta} R| \\ &\leq 2 \sum \Big|\frac{\partial^{k_1+l_1} \Delta}{\partial h^{k_1} \partial t^{l_1}}\Big| \cdot \Big|\frac{\partial^{k_2+l_2}}{\partial h^{k_2} \partial t^{l_2}} \frac{\partial R}{\partial \theta}\Big| + 2\Big|\frac{\partial^{k+l}}{\partial h^k \partial t^l} g_3\Big| \\ &\leq h^{-\frac{k_1}{2}-\frac{1}{2}} \cdot h^{-\frac{k_2}{2}+\frac{1}{2}} + C \cdot h^{-\frac{k}{2}+\frac{1}{2}} \\ &\leq C \cdot h^{-\frac{k}{2}+\frac{1}{2}}, \end{aligned}$$

where we have used the inequalities

$$\Big|\frac{\partial^{k_1+l_1} \Delta}{\partial h^{k_1} \partial t^{l_1}}\Big| \leq h^{-\frac{k_1}{2}-\frac{1}{2}}, \qquad \Big|\frac{\partial^{k+l}}{\partial h^k \partial t^l} g_3\Big| \leq C \cdot h^{-\frac{k}{2}+\frac{1}{2}}.$$

That is, if the conclusion is true for $k+l \leq m$, then it is also valid for $k+l \leq m+1$. Thus the proof is finished. ■

5.2. Proof of (3.16)

In fact when $k+l=1$, let

$$\Pi = 1 + \frac{\partial^2 S}{\partial I \partial \tau}(I, \tau, \theta), \qquad \Xi = -\frac{\partial^2 S}{\partial I^2}(I, \tau, \theta).$$

By direct computation we have

$$\Pi \cdot \frac{\partial \tau}{\partial \psi} = 1, \quad \Pi \cdot \frac{\partial \tau}{\partial I} = \Xi. \tag{5.4}$$

Then (3.15) gives $\Pi \geq \frac{1}{2}$, for $I \gg 1$ and

$$|\frac{\partial \tau}{\partial \psi}| \leq C, \quad |\frac{\partial \tau}{\partial I}| \leq C \cdot I^{-1}.$$

When $k+l=2$, from the result above we have

$$|\frac{\partial \Pi}{\partial I}| \leq C \cdot I^{-1}, \quad |\frac{\partial \Pi}{\partial \psi}| \leq C \cdot I^{-\frac{1}{2}}, \quad |\frac{\partial \Xi}{\partial I}| \leq C \cdot I^{-\frac{3}{2}}, \quad |\frac{\partial \Xi}{\partial \psi}| \leq C \cdot I^{-1}.$$

Thus from (5.4), it follows that

$$|\frac{\partial^2 \tau}{\partial I \partial \psi}| \leq CI^{-1}, \quad |\frac{\partial^2 \tau}{\partial \psi^2}| \leq CI^{-\frac{1}{2}}, \quad |\frac{\partial^2 \tau}{\partial I^2}| \leq CI^{-\frac{3}{2}}.$$

In general, if

$$\left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} \tau\right| \leq C \cdot I^{-\frac{k}{2}}, \quad 1 \leq k+l \leq q,$$

then

$$\left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} \Pi\right| \leq C \cdot I^{-\frac{k+1}{2}}, \qquad \left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} \Xi\right| \leq C \cdot I^{-\frac{k+2}{2}}, \quad 1 \leq k+l \leq q.$$

From (5.4) and the above estimates, we obtain

$$\left|\frac{\partial^{k+l}}{\partial I^k \partial \psi^l} \tau\right| \leq C \cdot I^{-\frac{k}{2}}, \quad 1 \leq k+l \leq q+1.$$

Thus (3.16) is proved by induction.

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