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H-SEMI-INVARIANT SUBMERSIONS

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Abstract. In this paper, we introduce the notions of the almost h-semi-invariant submersion and the h-semi-invariant submersion which may be the extended version of the notion of the semi-invariant submersion [18]. Using them, we obtain some properties. Finally, we give some examples for them.

1. Introduction

Given a $C^{\infty}-$submersion F from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N), there are several kinds of submersions according to the conditions on it: e.g. Riemannian submersion ([9], [15]), slant submersion ([6], [17]), almost Hermitian submersion [19], contact-complex submersion [10], quaternionic submersion [11], almost h-slant submersion and h-slant submersion [16], semi-invariant submersion [18], etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([5], [20]), Kaluza-Klein theory ([4], [12]), Supergravity and superstring theories ([13], [14]), etc. And the quaternionic Kähler manifolds have applications in physics as the target spaces for nonlinear $\sigma-$models with supersymmetry [7]. For more information about Riemannian submersions, there is a book which covers recent results on this topic [8]. The paper is organized as follows. In section 2 we recall some notions needed for this paper. In section 3 we give the definitions of the almost h-semi-invariant submersion and the h-semi-invariant submersion and obtain some interesting properties about them. In section 4 we construct some examples for the almost h-semi-invariant submersions and the h-semi-invariant submersions.

2. Preliminaries

Let (M, E, g) be an almost quaternionic Hermitian manifold, where M is a $4n$-dimensional differentiable manifold, g is a Riemannian metric on M, and E is a rank

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3 subbundle of End(TM) such that for any point $p \in M$ with its some neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$
$$g(J_\alpha X, J_\alpha Y) = g(X, Y)$$

for all vector fields X, Y on M, where the indices are taken from $\{1, 2, 3\}$ modulo 3. The above basis $\{J_1, J_2, J_3\}$ is said to be a *quaternionic Hermitian basis*. We call (M, E, g) a *quaternionic Kähler manifold* if there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1, 2, 3\}$

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}$$

for any vector field X on M, where the indices are taken from $\{1, 2, 3\}$ modulo 3. If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on M, then (M, E, g) is said to be *hyperkähler*. Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on M and g a *hyperkähler metric*. Let (M, g_M) and (N, g_N) be Riemannian manifolds and $F : M \mapsto N$ a $C^\infty-$submersion. The map F is said to be *Riemannian submersion* if the differential F_* preserves the lengths of horizontal vectors [11]. Let (M, g_M, J) be an almost Hermitian manifold. A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *slant submersion* if the angle $\theta(X)$ between JX and the space $\ker(F_*)_p$ is constant for any nonzero $X \in T_pM$ and $p \in M$ [17]. We call $\theta(X)$ a *slant angle*. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is said to be an *almost h-slant submersion* if given a point $p \in M$ with its some neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$ [16]. We call such a basis $\{I, J, K\}$ an *almost h-slant basis*. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-slant submersion* if given a point $p \in M$ with its some neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$, and $\theta_I(X) = \theta_J(X) = \theta_K(X)$ [16]. We call such a basis $\{I, J, K\}$ a *h-slant basis* and the angle θ *h-slant angle*. Let (M, g_M, J) be an almost Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *semi-invariant submersion* if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1, \ J(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$ [17].

Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds. A map $F : M \mapsto N$ is called a $(E_M, E_N)-$*holomorphic map* if given a point $p \in M$, for any $J \in (E_M)_p$ there exists $J' \in (E_N)_{F(p)}$ such that

$$F_* \circ J = J' \circ F_*.$$

A Riemannian submersion $F : M \mapsto N$ which is a $(E_M, E_N)-$holomorphic map is called a *quaternionic submersion*. Moreover, if (M, E_M, g_M) is a quaternionic Kähler manifold (or a hyperkähler manifold), then we say that F is a *quaternionic Kähler submersion* (or a *hyperkähler submersion*) [11].

Let (M, g_M) and (N, g_N) be Riemannian manifolds and $F : (M, g_M) \mapsto (N, g_N)$ a smooth map. The second fundamental form of F is given by

$$(\nabla F_*)(X, Y) := \nabla^F_X F_* Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),$$

where ∇^F is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N [1]. Recall that F is said to be *harmonic* if $trace(\nabla F_*) = 0$ and F is called a *totally geodesic* map if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ [1].

3. H-Semi-invariant Submersions

Definition 3.1. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-semi-invariant submersion* if given a point $p \in M$ with its some neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1 \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ R(\mathcal{D}_1) = \mathcal{D}_1, \ R(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

We call such a basis $\{I, J, K\}$ a *h-semi-invariant basis*.

Definition 3.2. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *almost h-semi-invariant submersion* if given a point $p \in M$ with its some neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R, \ R(\mathcal{D}_2^R) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$.

We call such a basis $\{I, J, K\}$ an *almost h-semi-invariant basis*.

Remark 3.1. Let F be a h-semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-invariant basis. Then the fibers of the map F are quaternionic CR-submanifolds [3]. More generally, it is also true when $F : (M, E, g_M) \mapsto (N, g_N)$ is a h-semi-invariant submersion with some additional conditions.

Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an almost h-semi-invariant submersion with an almost h-semi-invariant basis $\{I, J, K\}$. We denote the orthogonal complement of $R\mathcal{D}_2^R$ in $(\ker F_*)^\perp$ by μ^R for $R \in \{I, J, K\}$.

Then for $X \in \Gamma(\ker F_*)$, we have

$$RX = \phi_R X + \omega_R X,$$

where $\phi_R X \in \Gamma(\mathcal{D}_1^R)$ and $\omega_R X \in \Gamma(R\mathcal{D}_2^R)$ for $R \in \{I, J, K\}$.

For $Z \in \Gamma((\ker F_*)^\perp)$, we get

$$RZ = B_R Z + C_R Z,$$

where $B_R Z \in \Gamma(\mathcal{D}_2^R)$ and $C_R Z \in \Gamma(\mu^R)$ for $R \in \{I, J, K\}$.

Note that we denote the projection morphisms on the distributions $\ker F_*$ and $(\ker F_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively. Define the tensor $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$$
$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$$

for vector fields E, F on M, where ∇ is the Levi-Civita connection of g_M. Define

$$(\nabla_X \phi_R)Y := \widehat{\nabla}_X \phi_R Y - \phi_R \widehat{\nabla}_X Y$$

and

$$(\nabla_X \omega_R)Y := \mathcal{H}\nabla_X \omega_R Y - \omega_R \widehat{\nabla}_X Y$$

for $X, Y \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$, where $\widehat{\nabla}_X Y := \mathcal{V}\nabla_X Y$.

We look at the integrability of the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$. Using the results of ([3], [2]), we easily have

Lemma 3.1. *Let F be a h-semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-invariant basis. Then*

(i) *the distribution $\mathcal{D}_2$ is always integrable.*

(ii) *the following conditions are equivalent :*

- (a) *the distribution $\mathcal{D}_1$ is integrable.*
- (b) $g_M(\mathcal{T}_X IY - \mathcal{T}_Y IX, IZ) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_1)$ *and* $Z \in \Gamma(\mathcal{D}_2)$.

(c) $g_M(\mathcal{T}_X JY - \mathcal{T}_Y JX, JZ) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_1)$ *and* $Z \in \Gamma(\mathcal{D}_2)$.

(d) $g_M(\mathcal{T}_X KY - \mathcal{T}_Y KX, KZ) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_1)$ *and* $Z \in \Gamma(\mathcal{D}_2)$.

Using Theorem 5.1 of [2, p.63], we get

Proposition 3.1. *Let* F *be a h-semi-invariant submersion from a hyperkähler manifold* (M, I, J, K, g_M) *onto a Riemannian manifold* (N, g_N) *such that* (I, J, K) *is a h-semi-invariant basis. Then the following conditions are equivalent :*

(a) *the fibers of* F *are locally product Riemannian manifolds.*

(b) $(\nabla_X \phi_I)Y = 0$ *for* $X, Y \in \Gamma(\ker F_*)$.

(c) $(\nabla_X \phi_J)Y = 0$ *for* $X, Y \in \Gamma(\ker F_*)$.

(d) $(\nabla_X \phi_K)Y = 0$ *for* $X, Y \in \Gamma(\ker F_*)$.

Theorem 3.1. *Let* F *be an almost h-semi-invariant submersion from a hyperkähler manifold* (M, I, J, K, g_M) *onto a Riemannian manifold* (N, g_N) *such that* (I, J, K) *is an almost h-semi-invariant basis. Then the following conditions are equivalent :*

(a) F *is a totally geodesic map.*

(b)

$$\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y, \quad \widehat{\nabla}_X B_I Z + \mathcal{T}_X C_I Z \in \Gamma(\mathcal{D}_1^I),$$
$$\mathcal{H}\nabla_X \omega_I Y + \mathcal{T}_X \phi_I Y, \quad \mathcal{T}_X B_I Z + \mathcal{H}\nabla_X C_I Z \in \Gamma(I\mathcal{D}_2^I)$$

for $X, Y \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^\perp)$.

(c)

$$\widehat{\nabla}_X \phi_J Y + \mathcal{T}_X \omega_J Y, \quad \widehat{\nabla}_X B_J Z + \mathcal{T}_X C_J Z \in \Gamma(\mathcal{D}_1^J),$$
$$\mathcal{H}\nabla_X \omega_J Y + \mathcal{T}_X \phi_J Y, \quad \mathcal{T}_X B_J Z + \mathcal{H}\nabla_X C_J Z \in \Gamma(J\mathcal{D}_2^J)$$

for $X, Y \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^\perp)$.

(d)

$$\widehat{\nabla}_X \phi_K Y + \mathcal{T}_X \omega_K Y, \quad \widehat{\nabla}_X B_K Z + \mathcal{T}_X C_K Z \in \Gamma(\mathcal{D}_1^K),$$
$$\mathcal{H}\nabla_X \omega_K Y + \mathcal{T}_X \phi_K Y, \quad \mathcal{T}_X B_K Z + \mathcal{H}\nabla_X C_K Z \in \Gamma(K\mathcal{D}_2^K)$$

for $X, Y \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^\perp)$.

Proof. Given a complex structure $R \in \{I, J, K\}$, for $X, Y \in \Gamma(\ker F_*)$ and $Z, Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$ we have

$$(\nabla F_*)(Z_1, Z_2) = 0,$$

since F is a Riemannian submersion.

Furthermore, using the properties $\nabla R = 0$ and $R^2 = -id$, we obtain

$$\begin{aligned}(\nabla F_*)(X,Y) &= -F_*(\nabla_X Y) = F_*(R\nabla_X RY)\\ &= F_*(R\nabla_X \phi_R Y + R\nabla_X \omega_R Y)\\ &= F_*(R(\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \phi_R Y + \mathcal{H}\nabla_X \omega_R Y + \mathcal{T}_X \omega_R Y))\\ &= F_*(\phi_R \widehat{\nabla}_X \phi_R Y + \omega_R \widehat{\nabla}_X \phi_R Y + B_R \mathcal{T}_X \phi_R Y + C_R \mathcal{T}_X \phi_R Y\\ &+ B_R \mathcal{H}\nabla_X \omega_R Y + C_R \mathcal{H}\nabla_X \omega_R Y + \phi_R \mathcal{T}_X \omega_R Y + \omega_R \mathcal{T}_X \omega_R Y).\end{aligned}$$

Thus,

$$(\nabla F_*)(X,Y) = 0 \Leftrightarrow \omega_R(\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \omega_R Y) = 0, C_R(\mathcal{T}_X \phi_R Y + \mathcal{H}\nabla_X \omega_R Y) = 0.$$

Similarly,

$$(\nabla F_*)(X,Z) = 0 \Leftrightarrow \omega_R(\widehat{\nabla}_X B_R Z + \mathcal{T}_X C_R Z) = 0, C_R(\mathcal{T}_X B_R Z + \mathcal{H}\nabla_X C_R Z) = 0.$$

Hence, we get

$$a) \Leftrightarrow b),\ a) \Leftrightarrow c),\ a) \Leftrightarrow d).$$

Therefore, we obtain the result. ■

Proposition 3.2. *Let F be an almost h-semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-invariant basis. Then the following conditions are equivalent :*

- (a) *the distribution* $\ker F_*$ *defines a totally geodesic foliation.*
- (b) $\mathcal{T}_{X_1}\phi_I X_2 + \mathcal{H}\nabla_{X_1}\omega_I X_2 \in \Gamma(I\mathcal{D}_2^I)$, $\widehat{\nabla}_{X_1}\phi_I X_2 + \mathcal{T}_{X_1}\omega_I X_2 \in \Gamma(\mathcal{D}_1^I)$ *for* $X_1, X_2 \in \Gamma(\ker F_*)$.
- (c) $\mathcal{T}_{X_1}\phi_J X_2 + \mathcal{H}\nabla_{X_1}\omega_J X_2 \in \Gamma(J\mathcal{D}_2^J)$, $\widehat{\nabla}_{X_1}\phi_J X_2 + \mathcal{T}_{X_1}\omega_J X_2 \in \Gamma(\mathcal{D}_1^J)$ *for* $X_1, X_2 \in \Gamma(\ker F_*)$.
- (d) $\mathcal{T}_{X_1}\phi_K X_2 + \mathcal{H}\nabla_{X_1}\omega_K X_2 \in \Gamma(K\mathcal{D}_2^K)$, $\widehat{\nabla}_{X_1}\phi_K X_2 + \mathcal{T}_{X_1}\omega_K X_2 \in \Gamma(\mathcal{D}_1^K)$ *for* $X_1, X_2 \in \Gamma(\ker F_*)$.

Proof. For $X, Y \in \Gamma(\ker F_*)$,

$$\begin{aligned}\nabla_X Y &= -I\nabla_X IY = -I(\nabla_X \phi_I Y + \nabla_X \omega_I Y)\\ &= -I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \phi_I Y + \mathcal{T}_X \omega_I Y + \mathcal{H}\nabla_X \omega_I Y)\\ &= -(\phi_I \widehat{\nabla}_X \phi_I Y + \omega_I \widehat{\nabla}_X \phi_I Y + B_I \mathcal{T}_X \phi_I Y + C_I \mathcal{T}_X \phi_I Y\\ &+ \phi_I \mathcal{T}_X \omega_I Y + \omega_I \mathcal{T}_X \omega_I Y + B_I \mathcal{H}\nabla_X \omega_I Y + C_I \mathcal{H}\nabla_X \omega_I Y).\end{aligned}$$

Thus,

$$\begin{aligned}
&\nabla_X Y \in \Gamma(\ker F_*) \\
&\Leftrightarrow \omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0 \\
&\Leftrightarrow \widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y \in \Gamma(\mathcal{D}_1),\ \mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y \in \Gamma(I\mathcal{D}_2).
\end{aligned}$$

Hence,

$$a) \Leftrightarrow b).$$

Similarly, we get

$$a) \Leftrightarrow c) \text{ and } a) \Leftrightarrow d).$$

Therefore, we obtain the result. ∎

Similarly, we have

Proposition 3.3. *Let F be an almost h-semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-invariant basis. Then the following conditions are equivalent :*

- (a) *the distribution* $(\ker F_*)^\perp$ *defines a totally geodesic foliation.*
- (b) $\mathcal{A}_{Z_1} B_I Z_2 + \mathcal{H}\nabla_{Z_1} C_I Z_2 \in \Gamma(\mu^I)$, $\mathcal{A}_{Z_1} C_I Z_2 + \mathcal{V}\nabla_{Z_1} Z_2 \in \Gamma(\mathcal{D}_2^I)$ *for* $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.
- (c) $\mathcal{A}_{Z_1} B_J Z_2 + \mathcal{H}\nabla_{Z_1} C_J Z_2 \in \Gamma(\mu^J)$, $\mathcal{A}_{Z_1} C_J Z_2 + \mathcal{V}\nabla_{Z_1} Z_2 \in \Gamma(\mathcal{D}_2^J)$ *for* $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.
- (d) $\mathcal{A}_{Z_1} B_K Z_2 + \mathcal{H}\nabla_{Z_1} C_K Z_2 \in \Gamma(\mu^K)$, $\mathcal{A}_{Z_1} C_K Z_2 + \mathcal{V}\nabla_{Z_1} Z_2 \in \Gamma(\mathcal{D}_2^K)$ *for* $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.

Using Proposition 3.1 and Proposition 3.3, we obtain

Theorem 3.2. *Let F be a h-semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-invariant basis. Then the following conditions are equivalent :*

- (a) M *is locally a product Riemannian manifold* $M_{\mathcal{D}_1} \times M_{\mathcal{D}_2} \times M_{(\ker F_*)^\perp}$, *where* $M_{\mathcal{D}_1}$, $M_{\mathcal{D}_2}$, *and* $M_{(\ker F_*)^\perp}$ *are integral manifolds of the distributions* $\mathcal{D}_1$, $\mathcal{D}_2$, *and* $(\ker F_*)^\perp$, *respectively* .
- (b) $(\nabla \phi_I) = 0$ *on* $\ker F_*$ *and* $\mathcal{A}_{Z_1} B_I Z_2 + \mathcal{H}\nabla_{Z_1} C_I Z_2 \in \Gamma(\mu)$, $\mathcal{A}_{Z_1} C_I Z_2 + \mathcal{V}\nabla_{Z_1} Z_2 \in \Gamma(\mathcal{D}_2)$ *for* $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.
- (c) $(\nabla \phi_J) = 0$ *on* $\ker F_*$ *and* $\mathcal{A}_{Z_1} B_J Z_2 + \mathcal{H}\nabla_{Z_1} C_J Z_2 \in \Gamma(\mu)$, $\mathcal{A}_{Z_1} C_J Z_2 + \mathcal{V}\nabla_{Z_1} Z_2 \in \Gamma(\mathcal{D}_2)$ *for* $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.
- (d) $(\nabla \phi_K) = 0$ *on* $\ker F_*$ *and* $\mathcal{A}_{Z_1} B_K Z_2 + \mathcal{H}\nabla_{Z_1} C_K Z_2 \in \Gamma(\mu)$, $\mathcal{A}_{Z_1} C_K Z_2 + \mathcal{V}\nabla_{Z_1} Z_2 \in \Gamma(\mathcal{D}_2)$ *for* $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.

Using Proposition 3.2 and Proposition 3.3, we have

Theorem 3.3. *Let F be a h-semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-invariant basis. Then the following conditions are equivalent :*

(a) *M is locally a product Riemannian manifold $M_{\ker F_*} \times M_{(\ker F_*)^\perp}$, where $M_{\ker F_*}$ and $M_{(\ker F_*)^\perp}$ are integral manifolds of the distributions $\ker F_*$ and $(\ker F_*)^\perp$, respectively .*

(b) *$\mathcal{T}_{X_1}\phi_I X_2 + \mathcal{H}\nabla_{X_1}\omega_I X_2 \in \Gamma(I\mathcal{D}_2)$, $\widehat{\nabla}_{X_1}\phi_I X_2 + \mathcal{T}_{X_1}\omega_I X_2 \in \Gamma(\mathcal{D}_1)$ and $\mathcal{A}_{Z_1}B_I Z_2 + \mathcal{H}\nabla_{Z_1}C_I Z_2 \in \Gamma(\mu)$, $\mathcal{A}_{Z_1}C_I Z_2 + \mathcal{V}\nabla_{Z_1}Z_2 \in \Gamma(\mathcal{D}_2)$ for $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.*

(c) *$\mathcal{T}_{X_1}\phi_J X_2 + \mathcal{H}\nabla_{X_1}\omega_J X_2 \in \Gamma(J\mathcal{D}_2)$, $\widehat{\nabla}_{X_1}\phi_J X_2 + \mathcal{T}_{X_1}\omega_J X_2 \in \Gamma(\mathcal{D}_1)$ and $\mathcal{A}_{Z_1}B_J Z_2 + \mathcal{H}\nabla_{Z_1}C_J Z_2 \in \Gamma(\mu)$, $\mathcal{A}_{Z_1}C_J Z_2 + \mathcal{V}\nabla_{Z_1}Z_2 \in \Gamma(\mathcal{D}_2)$ for $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.*

(d) *$\mathcal{T}_{X_1}\phi_K X_2 + \mathcal{H}\nabla_{X_1}\omega_K X_2 \in \Gamma(K\mathcal{D}_2)$, $\widehat{\nabla}_{X_1}\phi_K X_2 + \mathcal{T}_{X_1}\omega_K X_2 \in \Gamma(\mathcal{D}_1)$ and $\mathcal{A}_{Z_1}B_K Z_2 + \mathcal{H}\nabla_{Z_1}C_K Z_2 \in \Gamma(\mu)$, $\mathcal{A}_{Z_1}C_K Z_2 + \mathcal{V}\nabla_{Z_1}Z_2 \in \Gamma(\mathcal{D}_2)$ for $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.*

Let F be a semi-invariant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \;\; J(\mathcal{D}_1) = \mathcal{D}_1, \;\; J(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

We choose a local orthonormal frame $\{v_1, \cdots, v_l\}$ of $\mathcal{D}_2$ and a local orthonormal frame $\{e_1, \cdots, e_{2k}\}$ of $\mathcal{D}_1$ such that $e_{2i} = Je_{2i-1}$ for $1 \le i \le k$.

Since $F_*(\nabla_{Je_{2i-1}}Je_{2i-1}) = -F_*(\nabla_{e_{2i-1}}e_{2i-1})$, $1 \le i \le k$, we easily have

$$trace(\nabla F_*) = 0 \Leftrightarrow \sum_{j=1}^{l} F_*(\nabla_{v_j}v_j) = 0.$$

Thus, we get

Theorem 3.4. *Let F be a semi-invariant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then F is a harmonic map if and only if $trace(\nabla F_*) = 0$ on $\mathcal{D}_2$.*

Corollary 3.1. *Let F be a semi-invariant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) such that $\ker F_* = \mathcal{D}_1$. Then F is a harmonic map.*

Theorem 3.5. *Let F be an almost h-semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-invariant basis. Then the following conditions are equivalent :*
(a) *F is a harmonic map.*
(b) $trace(\nabla F_*) = 0$ *on* $\mathcal{D}_2^I$.
(c) $trace(\nabla F_*) = 0$ *on* $\mathcal{D}_2^J$.
(d) $trace(\nabla F_*) = 0$ *on* $\mathcal{D}_2^K$.

Proof. By Theorem 3.4, we have

$$(a) \Leftrightarrow (b), \ (a) \Leftrightarrow c), \ (a) \Leftrightarrow (d).$$

Therefore, we obtain the result. ■

Corollary 3.2. *Let F be an almost h-semi-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-invariant basis such that* $\ker F_* = \mathcal{D}_1^R$ *for some* $R \in \{I, J, K\}$. *Then F is a harmonic map.*

Let $F : (M, g_M) \mapsto (N, g_N)$ be a Riemannian submersion. The map F is called a Riemannian submersion *with totally umbilical fibers* if

$$\mathcal{T}_X Y = g_M(X, Y) H \quad \text{for } X, Y \in \Gamma(\ker F_*),$$

where H is the mean curvature vector field of the fiber.

Lemma 3.2. *Let F be a h-semi-invariant submersion with totally umbilical fibers from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-invariant basis. Then*

$$H \in \Gamma(R\mathcal{D}_2) \quad \textit{for } R \in \{I, J, K\}.$$

Proof. Given a complex structure $R \in \{I, J, K\}$, for $X, Y \in \Gamma(\mathcal{D}_1)$ and $W \in \Gamma(\mu)$ we have

$$\begin{aligned} \mathcal{T}_X RY + \widehat{\nabla}_X RY &= \nabla_X RY = R\nabla_X Y \\ &= B_R \mathcal{T}_X Y + C_R \mathcal{T}_X Y + \phi_R \widehat{\nabla}_X Y + \omega_R \widehat{\nabla}_X Y \end{aligned}$$

so that

$$g_M(\mathcal{T}_X RY, W) = g_M(C_R \mathcal{T}_X Y, W).$$

By the assumption, with some computations we obtain

$$g_M(X, RY) g_M(H, W) = -g_M(X, Y) g_M(H, RW).$$

Interchanging the role of X and Y, we get

$$g_M(Y, RX)g_M(H, W) = -g_M(Y, X)g_M(H, RW)$$

so that combining the above two equations, we have

$$g_M(X, Y)g_M(H, RW) = 0$$

which means $H \in \Gamma(R\mathcal{D}_2)$, since $R\mu = \mu$.

Therefore, we obtain the result. ■

Theorem 3.6. *Let F be a h-semi-invariant submersion with totally umbilical fibers from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-invariant basis. Then the fibers are totally geodesic.*

Proof. By Lemma 3.2, we have

$$H \in \Gamma(R\mathcal{D}_2) \quad \text{for } R \in \{I, J, K\}$$

so that

$$< IH, JH, KH > \subset \mathcal{D}_2.$$

By Theorem 4.3 of [18], we obtain the result. ■

Remark 3.2. Let F be a semi-invariant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then there are distributions $\mathcal{D}_1 \subset \ker F_*$ and $\mu \subset (\ker F_*)^\perp$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2,\ J(\mathcal{D}_1) = \mathcal{D}_1,\ J(\mathcal{D}_2) \subset (\ker F_*)^\perp, (\ker F_*)^\perp = J(\mathcal{D}_2) \oplus \mu,$$

where $\mathcal{D}_2$ and μ are the orthogonal complements of $\mathcal{D}_1$ and $J(\mathcal{D}_2)$ in $\ker F_*$ and $(\ker F_*)^\perp$, respectively. As we know, the holomorphic sectional curvatures determine the Riemannian curvature tensor in a Kähler manifold.

Given a plane P invariant by J in T_pM, $p \in M$, there is an orthonormal basis $\{X, JX\}$ of P. Denote by $K(P)$, $K_*(P)$, and $\widehat{K}(P)$ the sectional curvatures of the plane P in M, N, and the fiber $F^{-1}(F(p))$, respectively, where $K_*(P)$ denotes the sectional curvature of the plane $P_* =< F_*X, F_*JX >$ in N. Using Corollary 1 of [15, p.465], we obtain the following :

1. If $P \subset (\mathcal{D}_1)_p$, then with some computations we have

$$K(P) = \widehat{K}(P) + |\mathcal{T}_X X|^2 - |\mathcal{T}_X JX|^2 - g_M(\mathcal{T}_X X, J[JX, X]).$$

2. If $P \subset (\mathcal{D}_2 \oplus J\mathcal{D}_2)_p$ with $X \in (\mathcal{D}_2)_p$, then we get

$$K(P) = g_M((\nabla_{JX}\mathcal{T})_X X, JX) + |\mathcal{H}J\nabla_X X|^2 - |\mathcal{V}J\nabla_X X|^2.$$

3. If $P \subset (\mu)_p$, then we obtain

$$K(P) = K_*(P) - 3|\mathcal{V}J\nabla_X X|^2.$$

4. Examples

Example 4.1. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an almost h-slant submersion with its slant angles $\{\theta_I, \theta_J, \theta_K\} \subset \{0, \frac{\pi}{2}\}$ [16]. Then the map F is an almost h-semi-invariant submersion.

Example 4.2. Let (M, E, g) be an almost quaternionic Hermitian manifold. Let $\pi : TM \mapsto M$ be the natural projection. Then the map π is a h-semi-invariant submersion with $\mathcal{D}_1 = \ker F_*$ [11]. Furthermore, by Corollary 3.2, π is harmonic.

Example 4.3. Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds. Let $F : M \mapsto N$ be a quaternionic submersion. Then the map F is a h-semi-invariant submersion with $\mathcal{D}_1 = \ker F_*$ [11]. By Corollary 3.2, F is harmonic.

Example 4.4. Define a map $F : \mathbb{R}^4 \mapsto \mathbb{R}^3$ by

$$F(x_1, \cdots, x_4) = (x_1 \sin\alpha - x_3 \cos\alpha, x_4, x_2),$$

where α is constant. Then the map F is a h-semi-invariant submersion with $\mathcal{D}_2 = \ker F_*$.

Example 4.5. Let $F : \mathbb{R}^4 \mapsto \mathbb{R}^3$ be a Riemannian submersion. Then the map F is a h-semi-invariant submersion with $\mathcal{D}_2 = \ker F_*$.

We can check it as follows: Given coordinates (x_1, x_2, x_3, x_4) on $\mathbb{R}^4$, we can naturally choose the complex structures I, J, and K on $\mathbb{R}^4$ defined by

$$I\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_2}, I\left(\frac{\partial}{\partial x_2}\right) = -\frac{\partial}{\partial x_1}, I\left(\frac{\partial}{\partial x_3}\right) = \frac{\partial}{\partial x_4}, I\left(\frac{\partial}{\partial x_4}\right) = -\frac{\partial}{\partial x_3},$$
$$J\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_3}, J\left(\frac{\partial}{\partial x_2}\right) = -\frac{\partial}{\partial x_4}, J\left(\frac{\partial}{\partial x_3}\right) = -\frac{\partial}{\partial x_1}, J\left(\frac{\partial}{\partial x_4}\right) = \frac{\partial}{\partial x_2},$$
$$K\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_4}, K\left(\frac{\partial}{\partial x_2}\right) = \frac{\partial}{\partial x_3}, K\left(\frac{\partial}{\partial x_3}\right) = -\frac{\partial}{\partial x_2}, K\left(\frac{\partial}{\partial x_4}\right) = -\frac{\partial}{\partial x_1}.$$

Since F is a Riemannian submersion, the dimension of the space $\ker(F_*)_p$ is equal to 1 for any $p \in \mathbb{R}^4$. Using the properties $< RX, X >= 0$ for $X \in T_p\mathbb{R}^4$ and $R \in \{I, J, K\}$, where $<, >$ denotes the Euclidean metric on $\mathbb{R}^4$, we obtain the result.

Example 4.6. Let (M, I, J, K, g_M) be a $4n-$dimensional hyperkähler manifold and (N, g_N) a $(4n-1)-$dimensional Riemannian manifold. Let $F : (M, I, J, K, g_M) \mapsto (N, g_N)$ be a Riemannian submersion. Then the map F is a h-semi-invariant submersion with $\mathcal{D}_2 = \ker F_*$.

Example 4.7. Let $(M_1, I_1, J_1, K_1, g_1)$ be a $4m-$dimensional hyperkähler manifold and $(M_2, I_2, J_2, K_2, g_2)$ a $4n-$dimensional hyperkähler manifold. Let (N_1, g_1') be a $(4m-1)$-dimensional Riemannian manifold and (N_2, g_2') a $(4n-1)$-dimensional Riemannian manifold. Let $F_i : (M_i, I_i, J_i, K_i, g_i) \mapsto (N_i, g_i')$ be a Riemannian submersion for $i \in \{1, 2\}$. Consider the product map $F = F_1 \times F_2 : M_1 \times M_2 \mapsto N_1 \times N_2$ given by

$$(F_1 \times F_2)(x, y) = (F_1(x), F_2(y)) \quad \text{for } x \in M_1 \text{ and } y \in M_2.$$

Then the map F is a h-semi-invariant submersion with $\mathcal{D}_2 = \ker F_*$.

Example 4.8. Define a map $F : \mathbb{R}^4 \mapsto \mathbb{R}^2$ by

$$F(x_1, \cdots, x_4) = (x_1 \cos\alpha - x_3 \sin\alpha, x_2 \sin\beta - x_4 \cos\beta),$$

where α and β are constant with $\alpha + \beta = \frac{\pi}{2}$. Then the map F is an almost h-semi-invariant submersion such that $\mathcal{D}_1^I = \mathcal{D}_2^J = \mathcal{D}_2^K = \ker F_*$. Furthermore, F is harmonic, by Corollary 3.2.

Example 4.9. Define a map $F : \mathbb{R}^4 \mapsto \mathbb{R}^2$ by

$$F(x_1, \cdots, x_4) = (x_1, x_2).$$

Then the map F is an almost h-semi-invariant submersion such that $I(\ker F_*) = \ker F_*$, $J(\ker F_*) = (\ker F_*)^\perp$, and $K(\ker F_*) = (\ker F_*)^\perp$. By Corollary 3.2, F is also harmonic.

Example 4.10. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^6$ by

$$F(x_1, \cdots, x_8) = (x_3, \cdots, x_8).$$

Then the map F is an almost h-semi-invariant submersion such that $I(\ker F_*) = \ker F_*$, $J(\ker F_*) \subset (\ker F_*)^\perp$, and $K(\ker F_*) \subset (\ker F_*)^\perp$. By Corollary 3.2, F is harmonic.

Example 4.11. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$ by

$$F(x_1, \cdots, x_8) = (x_1, x_2, x_5, x_7).$$

Then the map F is an almost h-semi-invariant submersion such that $\mathcal{D}_1^I = \mathcal{D}_2^J =< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} >$, $\mathcal{D}_2^I = \mathcal{D}_1^J =< \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8} >$, and $K(\ker F_*) = (\ker F_*)^\perp$.

Example 4.12. Define a map $F:\mathbb{R}^8 \mapsto \mathbb{R}^3$ by

$$F(x_1,\cdots,x_8)=(x_6,x_7,x_8).$$

Then the map F is a h-semi-invariant submersion such that $\mathcal{D}_1 =< \frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_4} >$ and $\mathcal{D}_2 =< \frac{\partial}{\partial x_5} >$.

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