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A NOTE ON WEIGHTED NORM INEQUALITIES FOR FRACTIONAL MAXIMAL OPERATORS WITH NON-DOUBLING MEASURES

Weihong Wang, Chaoqiang Tan and Zengjian Lou*

Abstract. Let μ be a non-negative Borel measure on $\mathbb{R}^d$ which only satisfies some growth condition, we study two-weight norm inequalities for fractional maximal functions associated to such μ. A necessary and sufficient condition for the maximal operator to be bounded from $L^p(v)$ into weak $L^q(u)$ $(1 \le p \le q < \infty)$ is given. Furthermore, by using certain Orlicz norm, a strong type inequality is obtained.

1. Introduction

Let μ be a non-negative "n-dimensional" Borel measure on $\mathbb{R}^d$ which only satisfies the following growth condition: there exists $n \in (0, d]$ such that

$$\mu(Q) \le \ell(Q)^n \tag{1.1}$$

for any cube $Q \subset \mathbb{R}^d$, where $\ell(Q)$ stands for the side length of Q. Throughout this paper, by a cube $Q \subset \mathbb{R}^d$, we mean a closed cube whose sides are parallel to the coordinate axes and we shall always denote the side length as above. For $\lambda > 0$ and any cube Q, λQ is a cube concentric as Q and with $\ell(\lambda Q) = \lambda \ell(Q)$. Moreover, $Q(x, r)$ will be the cube centered at x with side length r.

The classical theory of harmonic analysis for maximal functions and singular integrals on $(\mathbb{R}^d, \mu)$ has been developed under the assumption that the underlying measure μ satisfies the doubling property, i.e., there exists a constant $C > 0$ such that for $x \in \mathbb{R}^d$ and $r > 0$, $\mu(B(x, 2r)) \le C\mu(B(x, r))$, where $B(x, r)$ stands for the open ball centered at x with radius r (see [1, 2, 4, 7, 12, 15]). However, it seems that this

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*Corresponding author.

doubling condition can be removed and many classical results of Calderón-Zygmund theory have been proved to continue to hold (see [5, 6, 8-11, 17-19). The motivation for developing the analysis on non-homogeneous spaces and some examples of non-doubling measures can be found in [20]. For a complete account of this topic the reader is referred to [3] (Chapter 5, pp. 137-147).

For $0 \le \alpha < 1$, define the non-centered fractional maximal function

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{\mu(5Q)^{1-\alpha}} \int_Q |f(y)|\, d\mu(y), \tag{1.2}$$

where the supremum is taken over all cubes that contain x. The purpose of the paper is to consider two-weight norm inequalities for the maximal function M_α. We shall investigate that for which pairs of weights, M_α satisfies a weak or a strong type inequality. A weight w will be a nonnegative and locally integrable function. For any measurable set E, we shall write $w(E) = \int_E w\, d\mu$ and $L^p(w) = L^p(w d\mu)$ for $0 < p < \infty$. If $1 \le p \le \infty$, as usual, p' will be the exponent conjugate to p, that is, the one satisfying $p' = p/(p-1)$.

Garcîa-Cuerva and Martell in [5] introduced the following radical fractional maximal functions: for $0 \le \alpha < n$,

$$\mathcal{M}_\alpha f(x) = \sup_{Q \ni x} \frac{1}{\ell(Q)^{n-\alpha}} \int_Q |f(y)|\, d\mu(y), \tag{1.3}$$

where the measure μ just satisfies the growth condition (1.1). As we can see in many papers (e.g. [13] and [16]), it is more natural to define the fractional maximal operators as showing in (1.2), which can be seen from the following example in the case of $\mathbb{R}$.

Example 1. Given a non-doubling measure $d\mu = \chi_{[0,1]} dx$, i.e. $\mu(Q) = \int_Q \chi_{[0,1]}\, dx$, for $Q \subset \mathbb{R}$. Obviously, the measure μ satisfies the growth condition (1.1), for $0 < n \le 1$. In fact, let $\ell(Q) = r \ge 0$. When $r \ge 1$, then $\mu(Q) \le 1 \le r^n$. When $0 < r < 1$, then $\mu(Q) \le r \le r^n$. So, for any $n \in (0, 1]$, the measure μ satisfies the growth condition. As a conclusion, there are infinite maximal functions for a same measure μ according to the definition of $\mathcal{M}_\alpha$ defined in (1.3). This fact causes difficulties in studying properties of the measure space. However, the maximal function M_α defined in (1.2) is unique in some sense.

Definition 1.1. Let $1 \le p \le q < \infty$ and $0 \le \alpha < 1$. We shall say that the pair of weights $(u, v) \in A^\alpha_{p,q}$, if for every cube Q

$$(i) \quad \frac{1}{\mu(5Q)^{1-\alpha}} \left(\int_Q u(x)\, d\mu(x) \right)^{\frac{1}{q}} \left(\int_Q v(x)^{1-p'}\, d\mu(x) \right)^{\frac{1}{p'}} \le C, \quad \text{when } 1 < p < \infty.$$

$$(ii) \quad \frac{1}{\mu(5Q)^{1-\alpha}} \left(\int_Q u(x)\, d\mu(x) \right)^{\frac{1}{q}} \le C v(x), \text{ for } \mu\text{-almost every } x \in Q, \text{ when } p = 1.$$

Here and afterward, C denotes a constant independent of functions, whose value may differ from line to line.

Remark 1.1. In Definition 1.1, we are implicitly assuming that $u,\, v^{1-p'} \in L^1_{loc}(\mu)$ and so $u < \infty, \quad v > 0 \quad \mu$-almost everywhere.

The main results of the paper can be stated as follows. Theorem 1.1 concerns with the problem of finding pairs of weights such that the maximal operator M_α satisfies a weak type inequality; Theorem 1.2 characterizes those pairs of weights for which M_α satisfies a strong type inequality, which can be achieved by certain Orlize norm localized in cubes. Their proofs are given respectively in Sections 2 and 3.

Theorem 1.1. *Let* $1 \le p \le q < \infty$, $0 \le \alpha < 1$ *and* $0 < \lambda < \infty$, u *and* v *are two weights. Then the maximal operator* $f \mapsto M_\alpha f$ *is of weak-type* $(L^p(v),\, L^q(u))$, *i.e.,*

$$u(\{x \in \mathbb{R}^d :\, M_\alpha f(x) > \lambda\}) \le \frac{C}{\lambda^q}\left(\int_{\mathbb{R}^d} |f(x)|^p v(x)\, d\mu(x)\right)^{\frac{q}{p}} \tag{1.4}$$

if and only if the pair of weights $(u,\, v) \in A^{\,\alpha}_{p,\,q}$.

Theorem 1.2. *Let* $1 < p < q < \infty$ *and* $0 \le \alpha < 1$. *Let* $(u,\, v)$ *be a pair of weights such that for every cube* Q

$$\ell(Q)^{n(1-\frac{1}{p})}\mu(Q)^{\alpha-1}u(3Q)^{\frac{1}{q}}\|v^{-\frac{1}{p}}\|_{\Phi,\,Q} \le C, \tag{1.5}$$

where Φ *is a Young function whose complementary function* $\bar{\Phi} \in B_p$. *Then*

$$\left(\int_{\mathbb{R}^d} (M_\alpha f(x))^q\, u(x)\, d\mu(x)\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^d} |f(x)|^p v(x)\, d\mu(x)\right)^{\frac{1}{p}} \tag{1.6}$$

for $f \in L^p(v)$ *which is bounded with compact support.*

The definitions of Young function Φ, norm $\|\cdot\|_{\Phi,\,Q}$, and B_p condition are given in Section 3.

2. Proof of Theorem 1.1

To prove Theorem 1.1 we need the following lemma which provides an equivalence for the pair of weights in $A^\alpha_{p,q}$.

Lemma 2.1. *Let* $1 \le p \le q < \infty$ *and* $0 \le \alpha < 1$. *The pair of weights* $(u,\, v) \in A^\alpha_{p,q}$ *if and only if for every cube* Q *and every* $f \ge 0$,

$$\left(\frac{1}{\mu(5Q)^{1-\alpha}}\int_Q f(x)\, d\mu(x)\right)^q u(Q) \le C\left(\int_Q f(x)^p v(x)\, d\mu(x)\right)^{\frac{q}{p}}. \tag{2.1}$$

Proof. When $p = 1$, for $(u, v) \in A^{\alpha}_{1,q}$,

$$\begin{aligned}
&\left(\frac{1}{\mu(5Q)^{1-\alpha}} \int_Q f(x)\, d\mu(x)\right)^q u(Q) \\
&= \left(\int_Q f(x) \frac{1}{\mu(5Q)^{1-\alpha}} \left(\int_Q u(y)\, d\mu(y)\right)^{\frac{1}{q}} d\mu(x)\right)^q \\
&\le C \left(\int_Q f(x) v(x)\, d\mu(x)\right)^q.
\end{aligned}$$

When $1 < p < \infty$, by Hölder's inequality, we obtain

$$\begin{aligned}
&\left(\frac{1}{\mu(5Q)^{1-\alpha}} \int_Q f(x)\, d\mu(x)\right)^q u(Q) \\
&\le \frac{1}{\mu(5Q)^{(1-\alpha)q}} \left(\int_Q f(x)^p v(x)\, d\mu(x)\right)^{\frac{q}{p}} \left(\int_Q v(x)^{1-p'}\, d\mu(x)\right)^{\frac{q}{p'}} \int_Q u(x)\, d\mu(x) \\
&= \left(\int_Q f(x)^p v(x)\, d\mu(x)\right)^{\frac{q}{p}} \left(\frac{1}{\mu(5Q)^{1-\alpha}} \left(\int_Q u(y)\, d\mu(y)\right)^{\frac{1}{q}} \left(\int_Q v(x)^{1-p'} d\mu(x)\right)^{\frac{1}{p'}}\right)^q \\
&\le C \left(\int_Q f(x)^p v(x)\, d\mu(x)\right)^{\frac{q}{p}}.
\end{aligned}$$

To prove the converse, for any $S \subset Q$, apply (2.1) to $f\chi_S$,

$$\left(\frac{1}{\mu(5Q)^{1-\alpha}} \int_S f(x)\, d\mu(x)\right)^q u(Q) \le C \left(\int_S f(x)^p v(x)\, d\mu(x)\right)^{\frac{q}{p}}. \tag{2.2}$$

Take $f \equiv 1$, (2.2) gives

$$\left(\frac{\mu(S)}{\mu(5Q)^{1-\alpha}}\right)^q u(Q) \le C v(S)^{\frac{q}{p}}. \tag{2.3}$$

From the inequality above it follows that $u \in L^1_{loc}(\mu)$ (unless $v = \infty$ μ-almost everywhere) and that $v > 0$ μ-almost everywhere (unless $u = 0$ μ-almost everywhere). Now, we are going to show that $(u, v) \in A^{\alpha}_{p,q}$.

For $p = 1$, note that (2.3) can be written as

$$\frac{1}{\mu(5Q)^{1-\alpha}} \left(\int_Q u(x)\, d\mu(x)\right)^{\frac{1}{q}} \le C \frac{v(S)}{\mu(S)}, \quad \text{for any } S \subset Q \text{ with } \mu(S) > 0.$$

Fix Q and consider

$$a > \operatorname*{ess\,inf}_Q v = \inf\{t > 0 : \mu(S_t) > 0\},$$

where $S_t = \{x \in Q : \quad v(x) < t\} \subset Q$. Then $\mu(S_a) > 0$, and

$$\begin{aligned}\frac{1}{\mu(5Q)^{1-\alpha}} \left(\int_Q u(x)\, d\mu(x) \right)^{\frac{1}{q}} &\leq \frac{C}{\mu(S_a)} \int_{S_a} v(x)\, d\mu(x) \\ &\leq Ca.\end{aligned}$$

Let $a \to \operatorname{ess\,inf}_Q v$, we get

$$\frac{1}{\mu(5Q)^{1-\alpha}} \left(\int_Q u(x)\, d\mu(x) \right)^{\frac{1}{q}} \leq C \quad \operatorname*{ess\,inf}_Q v \leq Cv(x) \qquad \text{for } \mu\text{-a.e. } x \in Q.$$

That is $(u,\, v) \in A^{\alpha}_{1,\, q}$.

For $1 < p < \infty$, take $f(x) = f(x)^p v(x)$, that is $f(x) = v(x)^{1-p'}$. Fix Q and define

$$S_j = \{x \in Q : \quad v(x) > \frac{1}{j}\}, \qquad j = 1,\, 2, \cdots.$$

Then, f is bounded in every S_j and $\int_{S_j} f\, d\mu < \infty$ (fix j and Q). Applying (2.2) for $S = S_j$ and $f(x) = v(x)^{1-p'}$ gives

$$\left(\frac{1}{\mu(5Q)^{1-\alpha}} \int_{S_j} v(x)^{1-p'}\, d\mu(x) \right)^q u(Q) \leq C \left(\int_{S_j} v(x)^{1-p'}\, d\mu(x) \right)^{\frac{q}{p}}.$$

So

$$\frac{1}{\mu(5Q)^{1-\alpha}} \left(\int_Q u(x)\, d\mu(x) \right)^{\frac{1}{q}} \left(\int_{S_j} v(x)^{1-p'}\, d\mu(x) \right)^{\frac{1}{p'}} \leq C.$$

Since $S_1 \subset S_2 \subset \cdots \subset S_j \cdots$, and $\cup_j S_j = \{x \in Q : \quad v(x) > 0\} = Q$. Let $j \to \infty$, we get

$$\frac{1}{\mu(5Q)^{1-\alpha}} \left(\int_Q u(x)\, d\mu(x) \right)^{\frac{1}{q}} \left(\int_Q v(x)^{1-p'}\, d\mu(x) \right)^{\frac{1}{p'}} \leq C.$$

That is $(u,\, v) \in A^{\alpha}_{p,\, q}$. The lemma is proved. ∎

Proof of Theorem 1.1. Suppose that (1.4) holds. For $f \geq 0$ and a cube Q, Take λ with

$$0 < \lambda < m_{\alpha,\, Q}(f) =: \frac{1}{\mu(5Q)^{1-\alpha}} \int_Q f(x)\, d\mu(x).$$

Note that

$$m_{\alpha,\, Q}(f) \leq M_\alpha(f\chi_Q)(x), \quad x \in Q,$$

we have $Q \subset \{x \in \mathbb{R}^d : M_\alpha(f\chi_Q)(x) > \lambda\}$. Using (1.4),

$$u(Q) \leq u\Big(\{x \in \mathbb{R}^d : M_\alpha(f\chi_Q)(x) > \lambda\}\Big) \leq \frac{C}{\lambda^q}\left(\int_Q f(x)^p v(x)\, d\mu(x)\right)^{\frac{q}{p}}.$$

That is,

$$\lambda^q u(Q) \leq C\left(\int_Q f(x)^p v(x)\, d\mu(x)\right)^{\frac{q}{p}} \quad \text{for} \quad 0 < \lambda < m_{\alpha,Q}(f).$$

Let $\lambda \to m_{\alpha,Q}(f)$, then

$$(m_{\alpha,Q}(f))^q\, u(Q) \leq C\left(\int_Q f(x)^p v(x)\, d\mu(x)\right)^{\frac{q}{p}}.$$

Hence $(u,\, v) \in A^\alpha_{p,\,q}$ by Lemma 2.1.

In proving the converse, we still invoke Lemma 2.1. For fixed $\lambda > 0$ and $A > 0$ large enough, set

$$E^A_\lambda = \{x \in \mathbb{R}^d : M_\alpha f(x) > \lambda,\, |x| \leq A\}.$$

Then, for any $x \in E^A_\lambda$, there is a cube Q_x containing x such that

$$\frac{1}{\mu(5Q_x)^{1-\alpha}} \int_{Q_x} |f(y)|\, d\mu(y) > \lambda. \tag{2.4}$$

By Besicovitch covering lemma, there exists a countable collection of quasi-disjoint cubes $\{Q_j\}_j = \{Q(x_j,\, r_j)\}_j$ with $x_j \in E^A_\lambda$ and $r_j = r_{x_j}$, such that

$$E^A_\lambda \subset \bigcup_j Q_j\,, \qquad \chi_{Q_j}(x) \leq B(d),$$

where $B(d) > 1$ is usually called the Besicovitch constant. Recall the equivalence of (2.1) and $(u,\, v) \in A^\alpha_{p,q}$ along with (2.4), we have

$$\begin{aligned}
u(E^A_\lambda) &\leq \sum_j u(Q_j)\\
&\leq C \sum_j \left(\frac{1}{\mu(5Q_j)^{1-\alpha}} \int_{Q_j} |f(x)|\, d\mu(x)\right)^{-q} \left(\int_{Q_j} |f(x)|^p v(x)\, d\mu(x)\right)^{\frac{q}{p}}\\
&\leq \frac{C}{\lambda^q} \sum_j \left(\int_{Q_j} |f(x)|^p v(x)\, d\mu(x)\right)^{\frac{q}{p}}\\
&\leq \frac{C}{\lambda^q} \left(\sum_j \int_{Q_j} |f(x)|^p v(x)\, d\mu(x)\right)^{\frac{q}{p}}\\
&\leq \frac{C}{\lambda^q} \left(\int_{\mathbb{R}^d} |f(x)|^p v(x)\, d\mu(x)\right)^{\frac{q}{p}},
\end{aligned}$$

where the constant C is independent of A. Letting $A \to \infty$, the Monotone Convergence Theorem leads to the desired weak-type inequality (1.4). This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

We first recall some definitions and basic facts related to Orlicz spaces (see [14]). Let $\Phi : [0, \infty) \mapsto [0, \infty)$ be a Young function, i.e. a continuous, convex, increasing function with $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. The Orlicz space $L_\Phi(\mathbb{R}^d, \mu)$ consists of measurable functions f such that

$$\int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) < \infty, \qquad \text{for some } \lambda > 0.$$

The space $L_\Phi(\mathbb{R}^d, \mu)$ is a Banach space if it is endowed with the Luxemburg norm

$$\|f\|_\Phi = \inf\{\lambda > 0 : \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\}.$$

Each Young function Φ has associated to it a complementary Young function $\bar{\Phi}$ which satisfies

$$t \le \Phi^{-1}(t)\bar{\Phi}^{-1}(t) \le 2t, \qquad \text{for all } \ t > 0.$$

Let us define the following localized version of the Orlisz norm: for every cube $Q \subset \mathbb{R}^d$ with $\mu(Q) < \infty$

$$\|f\|_{\Phi,Q} = \inf\{\lambda > 0 : \frac{1}{\ell(Q)^n} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\}.$$

By the properties of Young function, it is easy to check that $\|\cdot\|_{\Phi,Q}$ provides a norm over $L_\Phi(Q, \mu)$: the space of all measurable functions on Q such that there exists $\lambda > 0$ for which

$$\frac{1}{\ell(Q)^n} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) < \infty.$$

From [14], the following generalization of Hölder inequality holds

$$\frac{1}{\ell(Q)^n} \int_Q |f(x)h(x)|\, d\mu(x) \le C\, \|f\|_{\Phi,Q} \|h\|_{\bar{\Phi},Q}.$$

For $1 < p < \infty$, it is said that a Young function Φ satisfies B_p condition ($\Phi \in B_P$), if

$$\int_c^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} < \infty, \quad \text{for some } \ c > 0.$$

For the proof of Theorem 1.2, we also need the following lemma.

Lemma 3.1. *Let* $0 \le \alpha < 1$ *and* $f \ge 0$ *be a locally integrable function. If*

$$\frac{1}{\mu(5Q)^{1-\alpha}} \int_Q f(y)\, d\mu(y) > t,$$

for some cube Q *and* $t > 0$. *Then there exists a dyadic cube* P *such that* $P \subset 5Q$, $Q \subset 3P$ *and*

$$\frac{1}{\mu(P)^{1-\alpha}} \int_P f(y)\, d\mu(y) > 2^{-d} t.$$

Proof. Take $s \in \mathbb{Z}$ such that $2^{s-1} \le \ell(Q) < 2^s$, there exist dyadic cubes $P_1, \cdots, P_j, \cdots, P_N$ ($1 \le N \le 2^d$) which intersect Q with the side length 2^s, and $P_j \subset 5Q$, $Q \subset 3P_j$, $j \in [1, N]$, and for at least one of them, say P, the following estimate holds

$$\int_P f(y)\, d\mu(y) > \frac{t\mu(5Q)^{1-\alpha}}{2^d}.$$

Otherwise,

$$\begin{aligned} \int_Q f(y)\, d\mu(y) &\le \sum_{j=1}^{N} \int_{P_j} f(y)\, d\mu(y) \\ &\le \sum_{j=1}^{N} \frac{t\mu(5Q)^{1-\alpha}}{2^d} \\ &\le t\mu(5Q)^{1-\alpha}, \end{aligned}$$

which contradicts the hypothesis. Note that $P \subset 5Q$, we get

$$\frac{1}{\mu(P)^{1-\alpha}} \int_P f(y)\, d\mu(y) > \frac{t\mu(5Q)^{1-\alpha}}{2^d \mu(P)^{1-\alpha}} \ge 2^{-d} t.$$

Lemma 3.1 is proved. ∎

Proof of Theorem 1.2. We will employ the ideas in [5] with modifications. Let

$$M_\alpha^R f(x) = \sup_{Q \ni x,\, \ell(Q) < R} \frac{1}{\mu(5Q)^{1-\alpha}} \int_Q |f(y)|\, d\mu(y).$$

First, we will prove (1.6) with $M_\alpha f(x)$ replaced by $M_\alpha^R f(x)$. Decompose $\mathbb{R}^d$ in the following way

$$\mathbb{R}^d = \bigcup_{k \in \mathbb{Z}} \Omega_k, \quad \text{with} \quad \Omega_k = \{x \in \mathbb{R}^d :\, 2^k < M_\alpha^R f(x) \le 2^{k+1}\}.$$

Then, for $k \in \mathbb{Z}$ and $x \in \Omega_k$, there exists a cube Q_x^k containing x, such that

$$\frac{1}{\mu(5Q_x^k)^{1-\alpha}} \int_{Q_x^k} |f(y)|\, d\mu(y) > 2^k.$$

By Lemma 3.1, there exists a dyadic cube P_x^k with $Q_x^k \subset 3P_x^k$, $P_x^k \subset 5Q_x^k$, such that

$$\frac{1}{\mu(P_x^k)^{1-\alpha}} \int_{P_x^k} |f(y)|\, d\mu(y) > 2^{-d} 2^k. \tag{3.1}$$

From the definition of M_α^R, every cube Q_x^k has bounded size, and so has every dyadic cube P_x^k. Then, for fixed k, there is a sub-collection of maximal dyadic cubes $\{P_j^k\}_{j\in\mathbb{N}}$ satisfying that every Q_x^k is contained in $3P_j^k$ for some j. We can write

$$\mathbb{R}^d = \bigcup_{k\in\mathbb{Z}} \Omega_k \subset \bigcup_{j,\,k} 3P_j^k.$$

Next, notice that these P_j^k are the maximal cubes for which (3.1) holds. However, for P_i^{k+1},

$$\frac{1}{\mu(P_i^{k+1})^{1-\alpha}} \int_{P_i^{k+1}} |f(y)|\, d\mu(y) > \; 2^{-d} 2^{k+1} > \; 2^{-d} 2^k.$$

Hence, for every i, there exists $j = j(i,\, k)$ such that $P_i^{k+1} \subset P_j^k$. In short, for a fixed k, the cubes $\{P_j^k\}_{j\in\mathbb{N},\, k\in\mathbb{Z}}$ are pairwise disjoint and they are strictly nested for different k.

Let $K > 0$ be a large integer, $\Lambda_K = \{(j,\, k) \in \mathbb{N} \times \mathbb{Z} :\ |k| \le K\}$. Then

$$\begin{aligned}
\mathcal{I}_K &= \int_{\cup_{k=-K}^K \Omega_k} \left(M_\alpha^R f(x)\right)^q u(x)\, d\mu(x) \\
&\le \sum_{(j,\,k)\in\Lambda_K} \int_{3P_j^k} \left(M_\alpha^R f(x)\right)^q u(x)\, d\mu(x) \\
&\le \sum_{(j,\,k)\in\Lambda_K} u\left(3P_j^k\right) \left(2^{k+1}\right)^q \\
&\le C \sum_{(j,\,k)\in\Lambda_K} u\left(3P_j^k\right) \left(\frac{1}{\mu(P_j^k)^{1-\alpha}} \int_{P_j^k} |f(y)|\, d\mu(y) \right)^q \\
&\le C \sum_{(j,\,k)\in\Lambda_K} u\left(3P_j^k\right) \mu\left(P_j^k\right)^{(\alpha-1)q} \ell\left(P_j^k\right)^{nq} \|fv^{\frac{1}{p}}\|_{\bar{\Phi},\, P_j^k}^q \|v^{-\frac{1}{p}}\|_{\Phi,\, P_j^k}^q,
\end{aligned}$$

where we used the generalization of Hölder inequality. From (1.5), we have

$$\mathcal{I}_K \le C \sum_{(j,\,k)\in\Lambda_K} \ell\left(P_j^k\right)^{\frac{nq}{p}} \|fv^{\frac{1}{p}}\|_{\bar{\Phi},\, P_j^k}^q =: C \int_{\mathcal{Y}} \left(T_K(fv^{\frac{1}{p}})\right)^q d\nu, \tag{3.2}$$

here $\mathcal{Y} = \mathbb{N} \times \mathbb{Z}$, ν is a measure on $\mathcal{Y}$ given by $\nu(j, k) = \ell(P_j^k)^{\frac{nq}{p}}$ and the operator T_K is defined by

$$T_K g(j, k) = \|g\|_{\bar{\Phi}, P_j^k} \chi_{\Lambda_K}(j, k).$$

We next to show that $T_K : L^p(\mathbb{R}^d, \mu) \mapsto L^q(\mathcal{Y}, \nu)$ is bounded independently of K. For a bounded function g with compact support and $\lambda \geq 0$, take

$$F_\lambda = \{(j, k) \in \mathcal{Y} : T_K g(j, k) > \lambda\} = \{(j, k) \in \Lambda_K : \|g\|_{\bar{\Phi}, P_j^k} > \lambda\}.$$

Without loss of generality, we may assume that $\bar{\Phi}(1) = 1$. Write

$$g(x) = g(x)\chi_{\{x: |g(x)| > \frac{\lambda}{2}\}}(x) + g(x)\chi_{\{x: |g(x)| \leq \frac{\lambda}{2}\}}(x) = g_1(x) + g_2(x).$$

Then

$$\begin{aligned} \frac{1}{\ell(Q)^n} \int_Q \bar{\Phi}\left(\frac{2|g_2(x)|}{\lambda}\right) d\mu(x) &\leq \frac{1}{\ell(Q)^n} \int_Q \bar{\Phi}(1)\, d\mu(x) \\ &= \frac{\mu(Q)}{\ell(Q)^n} \leq 1. \end{aligned}$$

So, for every Q, $\|g_2\|_{\bar{\Phi}, Q} \leq \lambda/2$. For Q such that $\|g\|_{\bar{\Phi}, Q} > \lambda$, the triangle inequality gives

$$\begin{aligned} \lambda < \|g\|_{\bar{\Phi}, Q} &= \|g_1 + g_2\|_{\bar{\Phi}, Q} \\ &\leq \|g_1\|_{\bar{\Phi}, Q} + \|g_2\|_{\bar{\Phi}, Q} \\ &\leq \|g_1\|_{\bar{\Phi}, Q} + \frac{\lambda}{2}, \end{aligned}$$

i.e. $\|g_1\|_{\bar{\Phi}, Q} > \lambda/2$. Thus

$$F_\lambda = \{(j, k) \in \Lambda_K : \|g\|_{\bar{\Phi}, P_j^k} > \lambda\} \subset \{(j, k) \in \Lambda_K : \|g_1\|_{\bar{\Phi}, P_j^k} > \frac{\lambda}{2}\} = \widetilde{F}_\lambda.$$

If $(j, k) \in \widetilde{F}_\lambda$, then

$$\frac{1}{\ell(P_j^k)^n} \int_{P_j^k} \bar{\Phi}\left(\frac{2|g_1|}{\lambda}\right) d\mu > 1.$$

So

$$\ell(P_j^k)^n < \int_{P_j^k} \bar{\Phi}\left(\frac{2|g_1|}{\lambda}\right) d\mu.$$

It follows that

$$\begin{aligned} \nu(F_\lambda) &\leq \nu(\widetilde{F}_\lambda) \\ &= \sum_{(j,k) \in \widetilde{F}_\lambda} \ell(P_j^k)^{\frac{nq}{p}} \\ &\leq \sum_{(j,k) \in \widetilde{F}_\lambda} \ell(P_j^k)^{n\left(\frac{q}{p} - 1\right)} \int_{P_j^k} \bar{\Phi}\left(\frac{2|g_1|}{\lambda}\right) d\mu. \end{aligned}$$

Notice that the dyadic cubes $\{P_j^k\}_{j\in\mathbb{N},\,k\in\mathbb{Z}}$ have bounded size, so we can extract a maximal sub-collection $\{P_i\}$ to get

$$\begin{aligned}
\nu(F_\lambda) &\le \sum_i \sum_{P_j^k\subset P_i} \ell(P_j^k)^{n\left(\frac{q}{p}-1\right)} \int_{P_j^k} \bar{\Phi}\left(\frac{2|g_1|}{\lambda}\right) d\mu \\
&= \sum_i \sum_{m=0}^{\infty} \sum_{\substack{P_j^k\subset P_i \\ \ell(P_j^k)=2^{-m}\ell(P_i)}} \ell(P_j^k)^{n\left(\frac{q}{p}-1\right)} \int_{P_j^k} \bar{\Phi}\left(\frac{2|g_1|}{\lambda}\right) d\mu \\
&= \sum_i \ell(P_i)^{n\left(\frac{q}{p}-1\right)} \sum_{m=0}^{\infty} 2^{-mn(\frac{q}{p}-1)} \sum_{\substack{P_j^k\subset P_i \\ \ell(P_j^k)=2^{-m}\ell(P_i)}} \int_{P_j^k} \bar{\Phi}\left(\frac{2|g_1|}{\lambda}\right) d\mu \\
&\le \sum_i \ell(P_i)^{n\left(\frac{q}{p}-1\right)} \int_{P_i} \bar{\Phi}\left(\frac{2|g_1|}{\lambda}\right) d\mu \sum_{m=0}^{\infty} 2^{-mn(\frac{q}{p}-1)} \\
&\le C \sum_i \ell(P_i)^{n\left(\frac{q}{p}-1\right)} \int_{P_i} \bar{\Phi}\left(\frac{2|g_1|}{\lambda}\right) d\mu.
\end{aligned}$$

Furthermore, for every i, there exists $(j,\,k)\in\widetilde{F}_\lambda$ such that $P_i = P_j^k$. Consequently

$$\begin{aligned}
\nu(F_\lambda) &\le C \sum_i \left(\int_{P_i} \bar{\Phi}\left(\frac{2|g_1|}{\lambda}\right) d\mu\right)^{\frac{q}{p}} \\
&\le C \left(\sum_i \int_{P_i} \bar{\Phi}\left(\frac{2|g_1|}{\lambda}\right) d\mu\right)^{\frac{q}{p}} \\
&\le C \left(\int_{\mathbb{R}^d} \bar{\Phi}\left(\frac{2|g_1|}{\lambda}\right) d\mu\right)^{\frac{q}{p}} \\
&= C \left(\int_{\{x\in\mathbb{R}^d:\,|g(x)|>\frac{\lambda}{2}\}} \bar{\Phi}\left(\frac{2|g|}{\lambda}\right) d\mu\right)^{\frac{q}{p}}.
\end{aligned}$$

Now we can verify

$$\int_{\mathcal{Y}} T_K g(j,\,k)^q\, d\nu(j,\,k) \le C\left(\int_{\mathbb{R}^d} |g(x)|^p\, d\mu(x)\right)^{\frac{q}{p}}, \tag{3.3}$$

where the constant C is independent of K.

To see this, we begin by using the formula of distribution function

$$\begin{aligned}
\int_{\mathcal{Y}} T_K g(j,\,k)^q\, d\nu(j,\,k) &= q\int_0^\infty \lambda^q \nu\left(\{(j,\,k)\in\mathcal{Y}:\, T_K g(j,\,k)>\lambda\}\right) \frac{d\lambda}{\lambda} \\
&= q\int_0^\infty \lambda^q \nu(F_\lambda)\,\frac{d\lambda}{\lambda}.
\end{aligned}$$

By the calculation of $\nu(F_\lambda)$, we have

$$\begin{aligned}\int_{\mathcal{Y}} T_K g(j,\,k)^{\,q}\, d\nu(j,\,k) &\le C \sum_{s\in\mathbb{Z}} \int_{2^s}^{2^{s+1}} \left(\lambda^p \int_{\{x\in\mathbb{R}^d:\, |g(x)|>\frac{\lambda}{2}\}} \bar{\Phi}\left(\frac{2|g|}{\lambda}\right) d\mu \right)^{\frac{q}{p}} \frac{d\lambda}{\lambda} \\ &\le C \sum_{s\in\mathbb{Z}} \left(\left(2^{s+1}\right)^p \int_{\{x\in\mathbb{R}^d:\, |g(x)|>\frac{2^s}{2}\}} \bar{\Phi}\left(\frac{2|g|}{2^s}\right) d\mu \right)^{\frac{q}{p}} \\ &\le C \left(\sum_{s\in\mathbb{Z}} \left(2^{s+1}\right)^p \int_{\{x\in\mathbb{R}^d:\, |g(x)|>\frac{2^s}{2}\}} \bar{\Phi}\left(\frac{2|g|}{2^s}\right) d\mu \right)^{\frac{q}{p}}.\end{aligned}$$

Note that

$$\begin{aligned}&\sum_{s\in\mathbb{Z}} \left(2^{s+1}\right)^p \int_{\{x\in\mathbb{R}^d:\, |g(x)|>\frac{2^s}{2}\}} \bar{\Phi}\left(\frac{2|g(x)|}{2^s}\right) d\mu(x) \\ &= \sum_{s\in\mathbb{Z}} \int_{2^s}^{2^{s+1}} \left(2^{s+1}\right)^p \int_{\{x\in\mathbb{R}^d:\, |g(x)|>\frac{2^s}{2}\}} \bar{\Phi}\left(\frac{2|g(x)|}{2^s}\right) d\mu(x)\, \frac{d\lambda}{2^s} \\ &\le \int_0^\infty (2\lambda)^p \int_{\{x\in\mathbb{R}^d:\, |g(x)|>\frac{\lambda}{4}\}} \bar{\Phi}\left(\frac{4|g(x)|}{\lambda}\right) d\mu(x)\, \frac{2d\lambda}{\lambda} \\ &= 2^{p+1} \int_{\mathbb{R}^d} \int_0^{4|g(x)|} \lambda^p\, \bar{\Phi}\left(\frac{4|g(x)|}{\lambda}\right) \frac{d\lambda}{\lambda}\, d\mu(x) \\ &= 2^{p+1} \int_{\mathbb{R}^d} \int_1^\infty \left(\frac{4|g(x)|}{t}\right)^p \bar{\Phi}(t)\, \frac{dt}{t}\, d\mu(x) \\ &= 2^{3p+1} \left(\int_{\mathbb{R}^d} |g(x)|^p\, d\mu(x) \right) \left(\int_1^\infty \frac{\bar{\Phi}(t)}{t^p}\, \frac{dt}{t} \right) \\ &\le C \int_{\mathbb{R}^d} |g(x)|^p\, d\mu(x),\end{aligned}$$

where we used the fact that $\bar{\Phi} \in B_p$. We have proved the estimate (3.3) with the constant C independent of K. Combining (3.2) and (3.3) leads to that

$$\mathcal{I}_K \le C \int_{\mathcal{Y}} T_K\left(f v^{\frac{1}{p}}\right)^q d\nu \le C \left(\int_{\mathbb{R}^d} |f|^p v\, d\mu \right)^{\frac{q}{p}}.$$

The uniformity in K of this estimate and the Monotone Convergence Theorem imply that

$$\int_{\mathbb{R}^d} \left(M_\alpha^R f(x)\right)^q u(x)\, d\mu(x) \le C \left(\int_{\mathbb{R}^d} |f(x)|^p v(x)\, d\mu(x) \right)^{\frac{q}{p}},$$

where the constant C is independent of R. Letting $R \to \infty$, using the Monotone Convergence Theorem again, we complete the proof of Theorem 1.2.

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Weihong Wang, Chaoqiang Tan and Zengjian Lou
Department of Mathematics
Shantou University
Guangdong Shantou 515063
P. R. China
E-mail: 08whwang@stu.edu.cn
cqtan@stu.edu.cn
zjlou@stu.edu.cn