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INTEGRAL MANIFOLDS AND THEIR ATTRACTION PROPERTY FOR EVOLUTION EQUATIONS IN ADMISSIBLE FUNCTION SPACES

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Abstract. In this paper we investigate the existence of a center-stable manifold for solutions to the semi-linear evolution equation of the form $u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi,u(\xi))d\xi,\ t \geq s \geq 0$, when its linear part, the evolution family $(U(t,s))_{t\geq s\geq 0}$, has an exponential trichotomy on the half-line and the nonlinear forcing term f satisfies the φ-Lipschitz condition, i.e., $\|f(x)-f(y)\| \leqslant \varphi(t)\|x-y\|$ where $\varphi(t)$ belongs to some class of admissible function spaces on the half-line. Moreover, we consider the existence of unstable manifolds and their attraction property for evolution equations defined on the whole line. Our methods are the Lyapunov-Perron method, the rescaling procedures, and the use of admissible function spaces as in [14, 15].

1. INTRODUCTION

Consider the semi-linear differential equation

$$\frac{dx}{dt} = A(t)x + f(t,x), \quad t \in [0,+\infty),\, x \in X \tag{1.1}$$

where $A(t)$ is a (possibly unbounded) linear operator on a Banach space X for every fixed t, and $f : \mathbb{R}_+ \times X \longrightarrow X$ is a nonlinear operator. When the linear part has an exponential dichotomy (or trichotomy), one tries to find conditions on the nonlinear forcing term f such that Equation (1.1) has an integral manifold (e.g., a stable, unstable or center manifold).

The problem of the existences of the integral manifolds is a matter of great interest of many authors since it brings out the geometric structures of the solutions to semi-linear differential equations. To obtain such existences one needs the characterizations of the exponential dichotomies (or trichotomies) of the linear part in some function spaces. Such characterizations have been used to construct the forms of operators determining

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the manifolds. We refer the reader to [6, 7, 13, 17, 25, 26, 29] for recent contributions to the theory of exponential dichotomy and trichotomy of evolution equations.

There are two main methods to prove the existence of integral manifolds, namely the Hadamard's and Perron's methods. The Hadamard's method has now been generalized to the so-called graph transform method which has been used, e.g., in the works [9, 12, 20] to prove the existence of invariant manifolds. This is a powerful method related to complicated choices of the transforms between graphs representing the involved manifolds. Meanwhile, the Perron's method is extended to the well-known Lyapunov-Perron method. This method is related to the construction of the so-called Lyapunov-Perron equations (or operators) involving the evolution equations under consideration to show the existence of the integral manifolds. It seems to be more natural to use the Lyapunov-Perron method to handle with the flows or semiflows which are generated by semi-linear evolution equations since in this case it is convenient to construct such Lyapunov-Perron equations or operators. We refer the reader to [1, 4, 5, 10, 11, 15, 27] and reference therein for more information on the matter.

To our best knowledge, the most popular condition regarding to the nonlinear part f for the existence of such manifolds is its uniform Lipchitz continuity with a sufficiently small Lipschitz constant (i.e., $\|f(t,x)-f(t,y)\| \leqslant q\|x-y\|$ for q being small enough). However, for equations arising in reaction-diffusion processes, the Lipschitz coefficients may depend on time and may not be small in classical sense. Therefore, one tries to extend the conditions on nonlinear parts such that they describe more exactly such reaction-diffusion processes.

Recently, using Lyapunov-Perron method and the admissibility of function spaces, we have given a more general condition on f for the existence of invariant stable manifolds (see [14]), that is the non-uniform Lipschitz continuity of f, i.e., $\|f(t,x)-f(t,y)\| \leqslant \varphi(t)\|x-y\|$ for φ being a real and positive function which belongs to admissible function spaces defined in Definition 1.3 below. The use of admissible spaces has helped us to construct the invariant manifolds for Equation (1.1) in the case of dichotomic linear part without using the smallness of Lipschitz constants of the nonlinear forcing term in classical sense. Instead, the "smallness" is now understood as sufficient smallness of $\sup_{t\geq 0}\int_t^{t+1}\varphi(\tau)d\tau$ or that of the norm of φ in appropriate function spaces. Consequently, we have obtained the existence of invariant-stable manifolds for the case of dichotomic linear parts under very general conditions on the nonlinear term $f(t,x)$. Using these results and rescaling procedures we shall prove, in the present paper, the existence of center-stable manifolds for the mild solutions of Equation (1.1) in the case of trichotomic linear parts under the same conditions on the nonlinear term $f(t,x)$ as in [14]. Moreover, using the same methods with some modifications relating to dichotomy estimates we can also obtain the existence of unstable manifolds and their attraction property for the evolution equations defined on the whole line. Our main results are contained in Theorems 3.2, 4.6, 4.7, and 4.11.

We also illustrate our results by some examples.

We first recall some notions.

Definition 1.1. A family of bounded linear operators $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ on a Banach space X is a *(strongly continuous, exponentially bounded) evolution family* on the half-line if

(i) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s \geq 0$,

(ii) the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$,

(iii) there are constants $K \geq 1$ and $\alpha \in \mathbb{R}$ such that $\|U(t,s)\| \leqslant Ke^{\alpha(t-s)}$ for $t \geq s \geq 0$.

Then the constant

$$\omega := \inf\{\alpha \in \mathbb{R} : \exists K \geq 1 \text{ such that } \|U(t,s)\| \leqslant Ke^{\alpha(t-s)}, \quad t \geq s \geq 0\}$$

is called the *exponential bound* of $\mathcal{U}$.

This notion arises naturally from well-posed evolution equations of the form

$$\begin{cases} \dfrac{du(t)}{dt} &= A(t)u(t),\ t \geq s \geq 0, \\ u(s) &= x_s \in X, \end{cases} \tag{1.2}$$

where $A(t)$ is (in general) an unbounded linear operator on X for every fixed t. Roughly speaking, when the Cauchy problem (1.2) is well-posed, there exists an evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ solving (1.2), i.e., the solution of (1.2) is given by $u(t) := U(t,s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families and applications to partial differential equations we refer to Pazy [22] (see also Nagel and Nickel [21] for a detailed discussion of well-posedness for non-autonomous abstract Cauchy problems on the whole line $\mathbb{R}$).

We recall some notions on function spaces and refer to Massera and Schäffer [19], Räbiger and Schnaubelt [24] for concrete applications.

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}_+$. The space $L_{1,loc}(\mathbb{R}_+)$ of real-valued locally integrable functions on $\mathbb{R}_+$ (modulo $\lambda-$nullfunctions) becomes a Fréchet space for the seminorms $p_n(f) := \int_{J_n} |f(t)|dt$, where $J_n = [n, n+1]$ for each $n \in \mathbb{N}$ (see [19, Chapt. 2, §20]).

We can now define Banach function spaces as follows.

Definition 1.2. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo $\lambda-$null- functions) is called *a Banach function space* (over $(\mathbb{R}_+, \mathcal{B}, \lambda)$) if

(1) E is Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leqslant |\varphi(\cdot)|,\ \lambda-$a.e., then $\psi \in E$ and $\|\psi\|_E \leqslant \|\varphi\|_E$,

(2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\geq 0} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\geq 0} \|\chi_{[t,t+1]}\|_E > 0$,

(3) $E \hookrightarrow L_{1,loc}(\mathbb{R}_+)$, i.e., for each seminorm p_n of $L_{1,loc}(\mathbb{R}_+)$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \leqslant \beta_{p_n}\|f\|_E$ for all $f \in E$.

We remark that condition (3) in the above definition means that for each compact interval $J \subset \mathbb{R}_+$ there exists a number $\beta_J \geq 0$ such that $\int_J |f(t)|dt \leqslant \beta_J\|f\|_E$ for all $f \in E$.

Let now E be a Banach function space and X a Banach space. We set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, X) := \{f : \mathbb{R}_+ \to X : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

(modullo $\lambda-$nullfunctions) endowed with the norm

$$\|f\|_{\mathcal{E}} := \|\|f(\cdot)\|\|_E.$$

Then $\mathcal{E}$ is a Banach space called *the Banach space corresponding to the Banach function space* E.

Definition 1.3. The Banach function space E is called *admissible* if

(1) there is a constant $M \geq 1$ such that for every compact interval $[a, b] \in \mathbb{R}_+$ we have

$$\int_a^b |\varphi(t)|dt \leqslant \frac{M(b-a)}{\|\chi_{[a,b]}\|_E}\|\varphi\|_E \text{ for all } \varphi \in E, \tag{1.3}$$

(2) for $\varphi \in E$ the function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) := \int_t^{t+1} \varphi(\tau)d\tau$ belongs to E.

(3) E is T_τ^+-invariant and T_τ^--invariant , where T_τ^+ and T_τ^- are defined by

$$\begin{aligned} T_\tau^+\varphi(t) &:= \begin{cases} \varphi(t-\tau) & \text{for } t \geq \tau \geq 0 \\ 0 & \text{for } 0 \leqslant t \leqslant \tau, \end{cases} \\ T_\tau^-\varphi(t) &:= \varphi(t+\tau) \text{ for } t \geq 0. \end{aligned} \tag{1.4}$$

Moreover, there are constants N_1, N_2 such that $\|T_\tau^+\| \leqslant N_1,\ \|T_\tau^-\| \leqslant N_2$ for all $\tau \in \mathbb{R}_+$.

Example 1.4. Besides the spaces $L_p(\mathbb{R}_+)$, $1 \leqslant p \leqslant \infty$, and the space

$$\mathbf{M}(\mathbb{R}_+) := \left\{ f \in L_{1,loc}(\mathbb{R}_+) : \sup_{t\geq 0} \int_t^{t+1} |f(\tau)|d\tau < \infty \right\}$$

endowed with the norm $\|f\|_{\mathbf{M}} := \sup_{t\geq 0} \int_t^{t+1} |f(\tau)|d\tau$, many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q},\ 1 < p < \infty,\ 1 < q < \infty$ are admissible.

Remark 1.5. It can be easily seen that, if E is an admissible Banach function space, then $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$.

We now collect some properties of admissible Banach function spaces in the following proposition (see [13, Proposition 2.6]).

Proposition 1.6. *Let E be an admissible Banach function space. Then the following assertions hold.*

(a) *Let $\varphi \in L_{1,loc}(\mathbb{R}_+)$ such that $\varphi \geq 0$ and $\Lambda_1\varphi \in E$, where Λ_1 is defined as in Definition 2.4(ii). For $\sigma > 0$ we define functions $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ by*

$$\Lambda'_\sigma\varphi(t) = \int_0^t e^{-\sigma(t-s)}\varphi(s)ds$$
$$\Lambda''_\sigma\varphi(t) = \int_t^\infty e^{-\sigma(s-t)}\varphi(s)ds$$

Then, $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to E. In particular, if $\sup_{t\geq 0}\int_t^{t+1}|\varphi(\tau)|d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see Remark 1.5)) then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ are bounded. Moreover, denoted by $\|\cdot\|_\infty$ for ess sup-norm, we have

$$\|\Lambda'_\sigma\varphi\|_\infty \leqslant \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1 T_1^+\varphi\|_\infty \text{ and } \|\Lambda''_\sigma\varphi\|_\infty \leqslant \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty$$

(b) *E contains exponentially decaying functions $\psi(t) = e^{-\alpha t}$ for $t \geq 0$ and any fixed constant $\alpha > 0$.*

(c) *E does not contain exponentially growing functions $f(t) = e^{bt}$ for $t \geq 0$ and a constant $b > 0$.*

We would like to note that the translation invariance property from Definition 1.3 cannot be removed since we need this property for the validity of Proposition 1.6 (see the proof of Proposition 1.6 in [13, Proposition 2.6]).

2. Exponential Dichotomy and Stable Manifolds

In this section, we recall some preparatory results obtained in [14] which will be used in the next sections. We now recall the notions of the exponential dichotomy of an evolution family and the $\varphi-$Lipschitz property of the nonlinear term f.

Definition 2.1. An evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ on the Banach space X is said to have an exponential dichotomy on $[0,\infty)$ if there exist bounded linear projections $P(t)$, $t\geq 0$, on X and positive constants N, ν such that

(a) $U(t,s)P(s)=P(t)U(t,s), \quad t\geq s\geq 0,$

(b) the restriction $U(t,s)_| : \mathrm{Ker}P(s)\to \mathrm{Ker}P(t)$, $t\geq s\geq 0$, is an isomorphism (and we denote its inverse by $(U(t,s)_|)^{-1}=U(s,t)_|$ for $t\geq s\geq 0$),

(c) $\|U(t,s)x\|\leqslant Ne^{-\nu(t-s)}\|x\|$ for $x\in P(s)X$, $t\geq s\geq 0$,

(d) $\|U(s,t)_|x\|\leqslant Ne^{-\nu(t-s)}\|x\|$ for $x\in \mathrm{Ker}P(t)$, $t\geq s\geq 0$.

Definition 2.2. Let E be an admissible Banach function space and φ be a positive function belonging to E. A function $f:[0,\infty)\times X\to X$ is said to be φ-Lipschitz if f satisfies

(i) $\|f(t,0)\|\leqslant M\varphi(t)$ for a.e $t\in\mathbb{R}_+$ where M is a fixed constant,

(ii) $\|f(t,x_1)-f(t,x_2)\|\leqslant\varphi(t)\|x_1-x_2\|$ for a.e $t\in\mathbb{R}_+$ and all $x_1,\,x_2\in X$.

To consider the existence of an integral manifold, instead of Equation (1.1) we consider the integral equation

$$u(t)=U(t,s)u(s)+\int_s^t U(t,\xi)f(\xi,u(\xi))d\xi \text{ for } t\geq s\geq 0. \tag{2.1}$$

We note that, if the evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ arises from the well-posed Cauchy problem (1.2) then the function u, which satisfies (2.1) for some given function f, is called a mild solution of the nonlinear problem

$$\begin{cases} \dfrac{du}{dt} &= A(t)u(t)+f(t,u(t)), \quad t\geq s\geq 0,\\ u(s) &= x_s\in X\end{cases}$$

We refer the reader to Pazy [22] for more detailed treatments on the relations between classical and mild solutions of evolution equations.

We now give the definition of an invariant stable manifold for solutions to Equation (2.1).

Definition 2.3. A set $S\subset\mathbb{R}_+\times X$ is said to be an invariant stable manifold for the solutions to Equation (2.1) if for every $t\in\mathbb{R}_+$ the phase space X splits into a direct sum $X=X_0(t)\oplus X_1(t)$ such that

$$\inf_{t\in\mathbb{R}_+} Sn(X_0(t),X_1(t)):=\inf_{t\in\mathbb{R}_+}\inf_{i=0,1}\{\|x_0+x_1\|: x_i\in X_i(t),\ \|x_i\|=1\}>0$$

and if there exists a family of Lipschitz continuous mappings

$$g_t: X_0(t)\to X_1(t), \quad t\in\mathbb{R}_+$$

with the Lipschitz constants being independent of t such that

(i) $S = \{(t, x + g_t(x)) \in \mathbb{R}_+ \times (X_0(t) \oplus X_1(t)) \,|\, t \in \mathbb{R}_+,\, x \in X_0(t)\}$, and we denote by $S_t = \{x + g_t(x) \mid (t, x + g_t(x)) \in S\},\ t \geq 0$,

(ii) S_t is homeomorphic to $X_0(t)$ for all $t \geq 0$,

(iii) to each $x_0 \in S_{t_0}$ there corresponds one and only one solution $u(t)$ of Equation (2.1) on $[t_0, \infty)$ satisfying conditions $u(t_0) = x_0$ and $\operatorname{ess\,sup}_{t \geq t_0} \|u(t)\| < \infty$,

(iv) S is invariant under Equation (2.1) in the sense that, if u is a solution to this equation satisfying conditions $u(t_0) = x_0 \in S_{t_0}$ and $\operatorname{ess\,sup}_{t \geq t_0} \|u(t)\| < \infty$, then $u(s) \in S_s$ for all $s \geq t_0$.

Note that, if we identify $X_0(t) \oplus X_1(t)$ with $X_0(t) \times X_1(t)$, then we can write $S_t = graph(g_t)$ where $graph(g_t)$ is denoted for the graph of the mapping g_t.

Let $\{U(t,s)\}_{t \geq s \geq 0}$ have an exponential dichotomy with the corresponding projections $P(t)$, $t \geq 0$, and the dichotomy constants $N,\ \nu > 0$. Putting $H := \sup_{t \geq 0} \|P(t)\| < \infty$, we can then define the Green's function on the half-line as follows

$$\mathcal{G}(t,\tau) = \begin{cases} P(t)U(t,\tau) & \text{for } t > \tau \geq 0 \\ -U(t,\tau)_{|}(I - P(\tau)) & \text{for } 0 \leqslant t < \tau. \end{cases}$$

Thus, we have

$$\|\mathcal{G}(t,\tau)\| \leqslant N(1+H)e^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau.$$

Next, we recall some related results taken from [14], which will be used in the next sections. The following lemma gives the form of bounded solutions of equation (2.1).

Lemma 2.4. [14, Lem. 4.4]. *Let the evolution family $(U(t,s))_{t \geq s \geq 0}$ have an exponential dichotomy with the corresponding projections $P(t), t \geq 0$, and the dichotomy constants $N, \nu > 0$. Suppose that φ is the positive function which belongs to E. Let $f : \mathbb{R}_+ \times X \to X$ be φ-Lipschitz. Let $u(t)$ be a solution to equation* (2.1) *such that $ess\sup_{t \geq t_0} \|u(t)\| < \infty$ for fixed $t_0 \geq 0$. Then, for $t \geq t_0$ we have that $u(t)$ can be rewritten in the form*

$$u(t) = U(t,t_0)v_0 + \int_{t_0}^{\infty} \mathcal{G}(t,\tau) f(\tau, u(\tau)) d\tau \text{ for some } v_0 \in X_0(t_0) = P(t_0)X, \tag{2.2}$$

where $\mathcal{G}(t,\tau)$ is the Green's function defined as above.

Remark 2.5. Equation (2.2) is called the Lyapunov-Perron equation. By computing directly, we can see that the converse of Lemma 2.4 is also true. This means that, all solutions of Equation (2.2) satisfy Equation (2.1) for $t \geq t_0$.

Theorem 2.6. [14, Thm. 4.6]. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ have an exponential dichotomy with the corresponding projections $P(t), t \geq 0$, and the dichotomy constants $N, \nu > 0$. Suppose that φ is the positive function which belongs to E. Let $f : \mathbb{R}_+ \times X \to X$ be φ-Lipschitz satisfying $k < 1$, where k is defined by the following formula*

$$k := \frac{(1+H)N(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}{1-e^{-\nu}}. \tag{2.3}$$

Then, there corresponds to each $v_0 \in X_0(t_0)$ one and only one solution $u(t)$ of the equation (2.1) *on $[t_0, \infty)$ satisfying the condition $P(t_0)u(t_0) = v_0$ and $ess \sup_{t\geq t_0} \|u(t)\| < \infty$. Moreover, the following estimate is valid for any two solutions $u_1(t), u_2(t)$ corresponding to different values $v_1, v_2 \in X_0(t_0)$:*

$$\|u_1(t) - u_2(t)\| \leqslant C_\mu e^{-\mu(t-t_0)}\|v_1 - v_2\| \text{ for } t \geq t_0, \tag{2.4}$$

where μ is a positive number satisfying $0 < \mu < \nu + \ln(1-k)$, and $C_\mu = \frac{N}{1-k}$.

Theorem 2.7. [14, Thm. 4.7]. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy with the corresponding projections $P(t), t \geq 0$, and the dichotomy constants $N, \nu > 0$. Suppose that $f : \mathbb{R}_+ \times X \to X$ be φ-Lipschitz, where φ is the positive function which belongs to E such that $k < \frac{1}{N+1}$ where k is defined by* (2.3).
Then, there exists an invariant stable manifold S for the solutions of equation (2.1). *Moreover, every two solutions $u_1(t), u_2(t)$ on the manifold S attract each other exponentially in the sense that, there exist positive constants μ and C_μ independent of $t_0 \geq 0$ such that*

$$\|u_1(t) - u_2(t)\| \leqslant C_\mu e^{-\mu(t-t_0)}\|P(t_0)u_1(t_0) - P(t_0)u_2(t_0)\| \text{ for } t \geq t_0. \tag{2.5}$$

3. Exponential Trichotomy and Center-stable Manifolds

In this section, we consider the case that the evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ has an exponential trichotomy on $\mathbb{R}_+$ and the nonlinear forcing term f is φ-Lipschitz. In this case, we will prove that there exists a center-stable manifold for the solutions to Equation (2.1). We first recall the definition of an exponential trichotomy.

Definition 3.1. A given evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ is said to have an exponential trichotomy on the half-line if there are three families of projections $\{P_j(t)\}$, $t \geq 0$, $j = 1, 2, 3$, and positive constants N, α, β with $\alpha < \beta$ such that the following conditions are satisfied:

(i) $\sup_{t\geq 0} \|P_j(t)\| < \infty$, $j =$ 1, 2, 3,

(ii) $P_1(t) + P_2(t) + P_3(t) = Id$ for $t \geq 0$ and $P_j(t)P_i(t) = 0$ for all $j \neq i$.

(iii) $P_j(t)U(t,s) = U(t,s)P_j(s)$ for $t \geq s \geq 0$ and $j = 1, 2, 3$,

(iv) $U(t,s)|_{\mathrm{Im}P_j(s)}$ are homeomorphisms from $\mathrm{Im}P_j(s)$ onto $\mathrm{Im}P_j(t)$, for all $t \geq s \geq 0$ and $j = 2, 3$, respectively; we also denote the inverse of $U(t,s)|_{\mathrm{Im}P_2(s)}$ by $U(s,t)_|$.

(v) For all $t \geq s \geq 0$ and $x \in X$, the following estimates hold:

$$\begin{aligned} \|U(t,s)P_1(s)x\| &\leqslant Ne^{-\beta(t-s)}\|P_1(s)x\| \\ \|U(s,t)_|P_2(t)x\| &\leqslant Ne^{-\beta(t-s)}\|P_2(t)x\| \\ \|U(t,s)P_3(s)x\| &\leqslant Ne^{\alpha(t-s)}\|P_3(s)x\|. \end{aligned}$$

We come to our first main result. It proves the existence of a center-stable manifold for solutions to Equation (2.1).

Theorem 3.2. *Let the evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ have an exponential trichotomy with the corresponding constants N, α, β and projections $\{P_j(t)\}$, $t \geq 0$, $j = 1, 2, 3$, given in Definition 3.1. Suppose that $f : \mathbb{R}_+ \times X \to X$ be φ-Lipschitz, where φ is a positive function which belongs to E satisfying $k < \frac{1}{N+1}$, here k is defined by (2.3). Then, for each fixed $\delta > \alpha$, there exists a center-stable manifold $S = \{(t, S_t) \mid t \in \mathbb{R}_+$ and $S_t \subset X\}$ for the solutions to equation (2.1), which is represented by a family of Lipschitz continuous mapping*

$$g_t : Im(P_1(t) + P_3(t)) \to ImP_2(t),$$

with Lipschitz constants being independent of t such that $S_t = graph(g_t)$ has the following properties:

(*i*) *To each $x_0 \in S_{t_0}$ there corresponds one and only one solution $u(t)$ of equation (2.1) on $[t_0, \infty)$ satisfying $u(t_0) = x_0$ and ess $\sup_{t\geq t_0} e^{-\gamma t}\|u(t)\| < \infty$, where $\gamma = \frac{\delta+\alpha}{2}$,*

(*ii*) *S_t is homeomorphism to $X_1(t) \oplus X_3(t)$ for all $t \geq 0$ where $X_j(t) = P_j(t)X$, $j = 1, 3$,*

(*iii*) *S is invariant under Equation (2.1) in the sense that, if $u(t)$ is the solution to Equation (2.1) satisfying $u(t_0) = x_0 \in S_{t_0}$ and ess $\sup_{t\geq t_0} e^{-\gamma t}\|u(t)\| < \infty$, then $u(s) \in S_s$ for all $s \geq t_0$.*

(*iv*) *For any abitrary solutions $x(\cdot)$ and $y(\cdot)$ which belong to the center-stable manifold, the following estimate holds:*

$$\|x(t) - y(t)\| \leqslant Ce^{\delta(t-t_0)}\|x(t_0) - y(t_0)\| \qquad \textit{for all } t \geq t_0 \geq 0$$

where C is a positive constant independent of t_0, $x(\cdot)$ and $y(\cdot)$.

Proof. Set $P(t) := P_1(t) + P_3(t)$ and $Q(t) := P_2(t) = Id - P(t)$, $t \geq 0$. We have that $P(t)$, $Q(t)$ are bounded linear projections on X. We consider the following rescaling evolution family

$$\widetilde{U}(t,s) = e^{-\gamma(t-s)}U(t,s) \quad \text{for all } t \geq s \geq 0.$$

We now prove that evolution family $\widetilde{U}(t,s)$ has an exponential dichotomy with dichotomy projections $P(t)$, $t \geq 0$. Indeed,

$$\begin{aligned} P(t)\widetilde{U}(t,s) &= e^{-\gamma(t-s)}(P_1(t) + P_3(t))U(t,s) \\ &= e^{-\gamma(t-s)}U(t,s)(P_1(s) + P_3(s)) = \widetilde{U}(t,s)P(s) \end{aligned}$$

Since $U(t,s)|_{\mathrm{Im}P_2(s)}$ is a homeomorphism from Im$P_2(s)$ onto Im$P_2(t)$, we have that $\widetilde{U}(t,s)|_{\mathrm{Ker}P(s)}$ is also homeomorphism from Ker$P(s)$ onto Ker$P(t)$. By the definition of the exponential trichotomy we have

$$\|\widetilde{U}(s,t)_{|}Q(t)x\| \leqslant e^{-(\beta+\gamma)(t-s)}\|Q(t)x\| \quad \text{for all } t \geq s \geq 0.$$

On the other hand,

$$\begin{aligned} \|\widetilde{U}(t,s)P(s)x\| &= e^{-\gamma(t-s)}\|U(t,s)(P_1(s) + P_3(s))x\| \\ &\leqslant Ne^{-\gamma(t-s)}(e^{-\beta(t-s)}\|P_1(s)x\| + e^{\alpha(t-s)}\|P_3(s)x\|) \\ &= Ne^{-\gamma(t-s)}(e^{-\beta(t-s)}\|P_1(s)P(s)x\| + e^{\alpha(t-s)}\|P_3(s)P(s)x\|) \end{aligned}$$

for all $t \geq s \geq 0$ and $x \in X$. Putting $q = \sup\{\|P_j(t)\| : \quad t \geq 0,\, j = 1, 3\}$, we finally get the following estimate

$$\|\widetilde{U}(t,s)P(s)x\| \leqslant 2Nqe^{-\frac{(\delta-\alpha)}{2}(t-s)}\|P(s)x\|.$$

Therefore, $(\widetilde{U}(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy with the projections $(P(t))_{t\geq 0}$ and the dichotomy constants $N_0 := \max\{N, 2Nq\}$ and $\nu = \frac{\delta-\alpha}{2}$.

Put $\widetilde{x}(t) = e^{-\gamma t}x(t)$, $t \geq 0$, and define the mapping $F : \mathbb{R}_+ \times X \to X$ as follows

$$F(t,x) = e^{-\gamma t}f(t, e^{\gamma t}x).$$

Obviously, F is also φ-Lipschitz. Thus, we can rewrite the integral equation (2.1) in the new form

$$\text{(3.1)} \qquad \widetilde{x}(t) = \widetilde{U}(t,s)\widetilde{x}(s) + \int_s^t \widetilde{U}(t,\xi)F(\xi, \widetilde{x}(\xi))d\xi \text{ for all } t \geq s \geq 0.$$

Hence, by Theorem 2.7, we obtain that, if

$$k = \frac{(1+H)N_0}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < \frac{1}{N_0+1}$$

then there exists an invariant stable manifold S for the solutions of the integral equation (3.1). Returning to the integral equation (2.1) by using the relation $x(t) := e^{\gamma t}\widetilde{x}(t)$ we can easily verify the properties of S which are stated in (i), (ii), (iii), and (iv). Thus, S is a center-stable manifold for the solutions of the integral equation (2.1). ∎

We now illustrate our abstract results by some examples.

Example 3.3. Consider equation

$$\frac{dx}{dt} = Ax(t) + f(t,x) \tag{3.2}$$

where A is a generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ and the spectrum $\sigma(A)$ of A is decomposed in three disjoint sets that are $\{\lambda \in \sigma(A)|\ \mathrm{Re}(\lambda) < 0\}$, $\{\lambda \in \sigma(A)|\ \mathrm{Re}(\lambda) > 0\}$ and $\{\lambda \in \sigma(A)|\ \mathrm{Re}(\lambda) = 0\}$ such that $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is of finitely many points. We define the evolution family $U(t,s) = T(t-s)$. By the spectral mapping theorem for analytic semigroups we have that, for fixed $t_0 > 0$, the spectrum of the operator $T(t_0)$ splits into three disjoint sets σ_1, σ_2, σ_3 where $\sigma_1 \subset \{|z| < 1\}$, $\sigma_2 \subset \{|z| > 1\}$ and $\sigma_3 \subset \{|z| = 1\}$. Here, the dichotomy projections are the Riesz projections corresponding to the spectral sets σ_1, σ_2, σ_3, respectively. Then, $(U(t,s))_{t\geq s\geq 0}$ has an exponential trichotomy. By Theorem 3.2, if f be φ-Lipschitz with φ satisfies $\sup_{t\geq 0}\int_t^{t+1}\varphi(\tau)d\tau$ is small enough then there is a center-stable manifold for mild solutions to Equation (3.2).

Example 3.4. For fixed $n \in \mathbb{N}$, consider the equation

$$\begin{aligned} &w_t(t,x) = -w_{xx}(t,x) + n^2 w(t,x) + \varphi(t)\cos(w(t,x)),\ 0 \leqslant x \leqslant \pi,\ t \geq 0,\\ &w(t,0) = w(t,\pi) = 0. \end{aligned} \tag{3.3}$$

where the real function $\varphi(t)$ is defined by

$$\varphi(t) = \begin{cases} n & \text{if } t \in [\frac{2n+1}{2} - \frac{1}{2^{n+c}}\ ,\ \frac{2n+1}{2} + \frac{1}{2^{n+c}}] \text{ for } n = 0,1,2,\cdots,\\ 0 & \text{otherwise.}\end{cases} \tag{3.4}$$

Here, we note that φ can take any arbitrarily large value but we still have that

$$\sup_{t\geq 0}\int_t^{t+1} |\varphi(\tau)|d\tau \leqslant 2\sup_{n\in\mathbb{N}}\int_{\frac{2n+1}{2}-\frac{1}{2^{n+c}}}^{\frac{2n+1}{2}+\frac{1}{2^{n+c}}} n\,dt = \sup_{n\in\mathbb{N}}\frac{n}{2^{n+c-2}} \leqslant \frac{1}{2^{c-1}}.$$

Therefore, $\varphi \in \mathbf{M}(\mathbb{R}_+)$ which is an admissible space.

We now write Equation (3.3) in an abstract form. To do this, we consider $X = L_2[0,\pi]$ and let $A : X \to X$ be defined by $Ay = -\ddot{y} + n^2 y$ with

$$D(A) = \{y \in X : y \text{ and } \ddot{y} \text{ are absolutely continuous, } \ddot{y} \in X,\ y(0) = y(\pi) = 0\}.$$

Then Equation (3.3) has the form

$$\frac{du}{dt} = Au + f(t,u) \quad \text{ for } u(t) = w(t,\cdot) \text{ where } f(t,u) = \varphi(t)\cos(u).$$

It can be seen that (see [8]) that A is the generator of an analytic semigroup $(T(t))_{t\geq 0}$, and we can compute the spectrum of A as

$$\sigma(A) = \{-1+n^2, -4+n^2, \dots, 0, \dots, -(1+k)^2+n^2, \dots\}.$$

Obviously, f is φ-Lipschitz with $\varphi(t)$ being defined as above. Using Example 3.3 we obtain that, if $\sup\limits_{t\geq 0}\int_t^{t+1}\varphi(\tau)d\tau$ (which is less than or equal $\frac{1}{2^{c-1}}$) is sufficient small (or c is sufficiently large), then there is a center-stable manifold for mild solutions to Equation (3.3).

4. Unstable Manifolds for Equations Defined on the Whole Line

In this section we prove the existence of unstable manifolds for evolution equations defined on the whole line $\mathbb{R}$ under the conditions that the evolution family $(U(t,s))_{t\geq s}$ has an exponential dichotomy (on the whole line) and the nonlinear term $f(t,x)$ is $\varphi-$Lipschitz.

Firstly, we recall the concepts of admissible Banach function spaces, exponential dichotomy, and some other notions defined on the whole line.

If we replace the half-line $\mathbb{R}_+$ by the whole line $\mathbb{R}$, then we have the similar notions of admissible spaces on the whole line with slight changes as follows:

(1) In Definition 1.3, the translation semigroups T_τ^+ and T_τ^- for $\tau\in\mathbb{R}_+$ should be replaced by T_τ^+ and T_τ^- defined for $\tau\in\mathbb{R}$ as

$$\begin{aligned} T_\tau^+\varphi(t) &:= \varphi(t-\tau) \text{ for } t\in\mathbb{R}, \\ T_\tau^-\varphi(t) &:= \varphi(t+\tau) \text{ for } t\in\mathbb{R}. \end{aligned} \tag{4.1}$$

(2) In Proposition 1.6 (a), the functions Λ'_σ and Λ''_σ should be replaced by

$$\begin{aligned} \Lambda'_\sigma\varphi(t) &:= \int_t^\infty e^{-\sigma|t-s|}\varphi(s)ds, \\ \Lambda''_\sigma\varphi(t) &:= \int_{-\infty}^t e^{-\sigma|s-t|}\varphi(s)ds. \end{aligned}$$

(3) In Proposition 1.6 (b) and (c) the functions $\psi(t) = e^{-\alpha t}$ ($t\geq 0$, and fixed $\alpha>0$) should be replaced by $\psi(t) = e^{-\alpha|t|}$, $t\in\mathbb{R}$ and fixed $\alpha>0$; and the functions $f(t) := e^{bt}$ for $t\geq 0$ and any fixed constant $b>0$ should be replaced by $f(t) := e^{b|t|}$, $t\in\mathbb{R}$, and fixed $b>0$.

We denote the admissible Banach function space of the functions defined on $\mathbb{R}$ by $E_{\mathbb{R}}$.

Definition 4.1. An evolution family $\{U(t,s)\}_{t\geq s}$ on the Banach space X is said to have an exponential dichotomy on $\mathbb{R}$ if there exist bounded linear projections $P(t)$, $t \in \mathbb{R}$, on X and positive constants N, ν such that

(a) $U(t,s)P(s) = P(t)U(t,s), \quad t \geq s,$

(b) the restriction $U(t,s)_| : \mathrm{Ker}P(s) \to \mathrm{Ker}P(t)$, $t \geq s$, is an isomorphism (and we denote its inverse by $(U(t,s)_|)^{-1} = U(s,t)_|$ for $t \geq s$),

(c) $\|U(t,s)x\| \leqslant Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)X$, $t \geq s$,

(d) $\|U(s,t)_| x\| \leqslant Ne^{-\nu(t-s)}\|x\|$ for $x \in \mathrm{Ker}P(t)$, $t \geq s$.

For an evolution family $\{U(t,s)\}_{t\geq s}$ having an exponential dichotomy on the whole line, we can define the Green's function on $\mathbb{R}$ as follows:

$$\mathcal{G}(t,\tau) = \begin{cases} P(t)U(t,\tau) & \text{for } t \geq \tau \\ -U(t,\tau)_|(I - P(\tau)) & \text{for } t < \tau \end{cases} \tag{4.2}$$

Thus, we have

$$\|\mathcal{G}(t,\tau)\| \leqslant N(1+H)e^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau$$

where $H = \sup_{t\in\mathbb{R}} \|P(t)\| < \infty$.

Definition 4.2. Let $E_{\mathbb{R}}$ be an admissible Banach function space and φ be a positive function belonging to $E_{\mathbb{R}}$. A function $f : \mathbb{R} \times X \to X$ is said to be φ-Lipschitz if f satisfies

(i) $\|f(t,0)\| \leqslant M\varphi(t)$ for a.e. $t \in \mathbb{R}$,

(ii) $\|f(t,x_1) - f(t,x_2)\| \leqslant \varphi(t)\|x_1 - x_2\|$ for a.e $t \in \mathbb{R}$ and all $x_1, x_2 \in X$.

In this section, we consider the following equation

$$\frac{du}{dt} = A(t)u + f(t,u), \quad t \in \mathbb{R},\ u \in X \tag{4.3}$$

where $A(t)$, $t \in \mathbb{R}$, are unbounded operators on X which generate a dichotomic evolution family $\{U(t,s)\}_{t\geq s}$ defined on $\mathbb{R}$; and the nonlinear forcing term $f : \mathbb{R}\times X \to X$ is φ-Lipschitz.

As usual, we consider the mild solutions of Equation (4.3), that is the solutions to the following integral equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi,u(\xi))d\xi \text{ for } t \geq s. \tag{4.4}$$

The existence of an invariant stable manifold for the solutions of Equation (4.4) has been proved in Theorem 2.7. In this section, we will prove the existence and attraction property of an invariant unstable manifold for the solutions to this equation. We start by the definition of such a manifold.

Definition 4.3. A set $\mathbf{U} \subset \mathbb{R} \times X$ is said to be *an invariant unstable manifold* for the solutions to the integral equation (4.4) if for every $t \in \mathbb{R}$ the phase spaces X splits into a direct sum $X = X_0(t) \oplus X_1(t)$ such that

$$\inf_{t\in\mathbb{R}} Sn(X_0(t), X_1(t)) := \inf_{t\in\mathbb{R}} \inf_{i=0,1} \{\|x_0 + x_1\| : x_i \in X_i(t),\ \|x_i\| = 1\} > 0$$

and there exists a family of Lipschitz continuous mappings

$$g_t : X_1(t) \to X_0(t), \quad t \in \mathbb{R}$$

with the Lipschitz constants being independent of t such that

(i) $\mathbf{U} = \{(t, x + g_t(x)) \in \mathbb{R} \times (X_1(t) \oplus X_0(t)) \mid t \in \mathbb{R},\ x \in X_1(t)\}$, and we denote by $\mathbf{U}_t = \{x + g_t(x) : (t, x + g_t(x)) \in \mathbf{U}\}$.

(ii) $\mathbf{U}_t$ is homemorphic to $X_1(t)$ for all $t \in \mathbb{R}$,

(iii) to each $x_0 \in \mathbf{U}_{t_0}$ there corresponds one and only one solution $u(t)$ to Equation (4.4) on $(-\infty, t_0]$ satisfying conditions $u(t_0) = x_0$ and $\operatorname{ess\,sup}_{t\leqslant t_0} \|u(t)\| < \infty$,

(iv) $\mathbf{U}$ is invariant under Equation (4.4) in the sense that, if u is a solution of (4.4) satisfying conditions $u(t_0) = x_0 \in \mathbf{U}_{t_0}$ and $\operatorname{ess\,sup}_{t\leqslant t_0} \|u(t)\| < \infty$, then $u(s) \in \mathbf{U}_s$ for all $s \leqslant t_0$.

The following lemma gives the structures of solutions to Equation (4.4) which are essentially bounded on a negative half-line.

Lemma 4.4. *Let the evolution family $\{U(t,s)\}_{t\geq s}$ have an exponential dichotomy with the corresponding projections $P(t)$, $t \in \mathbb{R}$, and the dichotomy constants N, $\nu > 0$. Suppose that φ is the positive function which belongs to $E_{\mathbb{R}}$. Let $f : \mathbb{R} \times X \to X$ be φ-Lipschitz. Let $x(t)$ be a solution to Equation (4.4) such that ess $\sup_{t\leqslant t_0} \|x(t)\| < \infty$ for some fixed t_0. Then, for $t \leqslant t_0$ we have that $x(t)$ can be rewritten in the form*

$$x(t) = U(t, t_0)_| v_1 + \int_{-\infty}^{t_0} \mathcal{G}(t,\tau) f(\tau, x(\tau)) d\tau \tag{4.5}$$

where $v_1 \in X_1(t_0) = (I - P(t_0))X$ and $\mathcal{G}(t,\tau)$ is the Green's function defined by Formula (4.2).

Proof. Put

$$y(t) = \int_{-\infty}^{t_0} \mathcal{G}(t,\tau) f(\tau, x(\tau)) d\tau \quad \text{for all } t \leqslant t_0.$$

We have that

$$
\begin{aligned}
\|y(\cdot)\|_\infty &\leqslant N(1+H)\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\|f(\tau,x(\tau))\|d\tau\\
&\leqslant \left[\int_{-\infty}^{t} e^{-\nu(t-\tau)}\varphi(\tau)d\tau+\int_{t}^{t_0} e^{-\nu(\tau-t)}\varphi(\tau)d\tau\right]\\
&\qquad (1+H)N(\|x(\cdot)\|_\infty+M)\\
&\leqslant (1+H)N(\|x(\cdot)\|_\infty+M)\frac{N_2\|\Lambda_1\varphi\|_\infty+N_1\|\Lambda_1T_1^+\varphi\|_\infty}{1-e^{-\nu}}<\infty.
\end{aligned}
$$

By computing directly, it is straightforward to see that $y(\cdot)$ satisfies the integral equation

$$
y(t_0)=U(t_0,t)y(t)+\int_t^{t_0} U(t_0,\tau)f(\tau,x(\tau))d\tau \quad \text{for } t\leqslant t_0.
$$

On the other hand,

$$
x(t_0)=U(t_0,t)x(t)+\int_t^{t_0} U(t_0,\tau)f(\tau,x(\tau))d\tau \quad \text{for } t\leqslant t_0.
$$

Then $x(t_0)-y(t_0)=U(t_0,t)[x(t)-y(t)]$, $t\leqslant t_0$.

For $s\leqslant t$ we have that

$$
\begin{aligned}
P(t)x(t) &= P(t)U(t,s)x(s)+P(t)\int_s^t U(t,\tau)f(\tau,x(\tau))d\tau\\
&= U(t,s)P(s)x(s)+\int_s^t U(t,\tau)P(\tau)f(\tau,x(\tau))d\tau,
\end{aligned}
$$

and

$$
\|U(t,s)P(s)x(s)\|\leqslant e^{-\nu(t-s)}\|P(s)x(s)\|\leqslant He^{-\nu(t-s)}\|x(\cdot)\|_\infty.
$$

Therefore, letting $s\to-\infty$ we obtain that

$$
P(t)x(t)=\int_{-\infty}^t U(t,\tau)P(\tau)f(\tau,x(\tau))d\tau=P(t)y(t).
$$

Thus, $x(t)-y(t)\in \mathrm{Ker}P(t)$, $t\leqslant t_0$. This leads to $x(t_0)-y(t_0)=U(t_0,t)[x(t)-y(t)]\in \mathrm{Ker}P(t_0)$. Putting $v_1=x(t_0)-y(t_0)$ we have that $x(t)=U(t,t_0)_|v_1+y(t)$. Therefore, $x(t)$ satisfies Equation (4.5). ■

Remark 4.5. By computing directly, we can see that the converse of Lemma 4.4 is also true. It means, all solutions of Equation (4.5) satisfied Equation (4.4) for all $t\leqslant t_0$.

Theorem 4.6. *Let the evolution family $\{U(t,s)\}_{t\geq s}$ have an exponential dichotomy with the corresponding projections $P(t)$, $t\in\mathbb{R}$, and the dichotomy constants N, $\nu>0$. Suppose that φ is the positive function which belongs to $E_{\mathbb{R}}$. Let $f:\mathbb{R}\times X\to X$ be φ-Lipschitz satisfying*

(4.6) $$k := \frac{(1+H)N}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < 1.$$

Then, there corresponds to each $v_1 \in X_1(t_0)$ one and only one solution $x(t)$ to Equation (4.4) on $(-\infty, t_0]$ satisfying the conditions $(I - P(t_0))x(t_0) = v_1$ and ess $\sup_{t\leqslant t_0}\|x(t)\| < \infty$. *Moreover, the following estimate is valid for any two solutions $x_1(t), x_2(t)$ corresponding to different initial values $x_1(0) = v_1$, $x_2(0) = v_2$ (v_1, $v_2 \in X_1(t_0)$):*

(4.7) $$\|x_1(t) - x_2(t)\| \leqslant C_\mu e^{-\mu(t_0-t)}\|v_1 - v_2\| \qquad \textit{for all } t \leqslant t_0$$

where μ is positive number satisfying

$$0 < \mu < \nu + \ln(1-k), \;\; \textit{and } C_\mu = \frac{N}{1-k}.$$

Proof. For each $v_1 \in X_1(t_0)$, we consider an operator

$$T : L_\infty((-\infty, t_0], X) \to L_\infty((-\infty, t_0], X)$$
$$(Tx)(t) = U(t,t_0)_| v_1 + \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)f(\tau, x(\tau))d\tau.$$

For $x(\cdot) \in L_\infty((-\infty, t_0], X)$ we have that $\|f(t, x(t))\| \leqslant \varphi(t)(M + \|x(t)\|)$. Thus,

$$\|(Tx)(t)\| \leqslant Ne^{-\nu(t_0-t)}\|v_1\| + (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)(M + \|x(\tau)\|)d\tau$$

$$\|Tx\|_\infty \leqslant N\|v_1\| + \frac{(1+H)N(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}{1-e^{-\nu}}(M + \|x(\cdot)\|_\infty)$$

Therefore, $Tx \in L_\infty((-\infty, t_0], X)$. We consider

$$\begin{aligned}\|Tx - Ty\|_\infty &\leqslant (1+H)N\int_{-\infty}^{t_0} e^{-\nu|t-\tau|}\varphi(\tau)d\tau\|x(\cdot) - y(\cdot)\|_\infty \\ &\leqslant \frac{(1+H)N(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}{1-e^{-\nu}}\|x(\cdot) - y(\cdot)\|_\infty \\ &= k\|x(\cdot) - y(\cdot)\|_\infty.\end{aligned}$$

Sine $k < 1$ we obtain that T is a contraction. Then, there is a unique function $x(t) \in L_\infty((-\infty, t_0], X)$ such that $Tx = x$. By Lemma 4.4 and Remark 4.5 we obtain that $x(t)$ is a unique solution of Equation (4.4). The estimate (4.7) can be obtained by the same arguments as in the proof of Theorem 2.7. ∎

We now come to our next result on the existence of an unstable manifold for solutions to Equation (4.4).

Theorem 4.7. *Let the evolution family $\{U(t,s)\}_{t\geq s}$ have an exponential dichotomy with the corresponding projections $P(t)$, $t \in \mathbb{R}$, and the dichotomy constants N, $\nu > 0$. Suppose that $f : \mathbb{R} \times X \to X$ be φ-Lipschitz, where φ is the positive function which belongs to $E_{\mathbb{R}}$ satisfying $k < \frac{1}{N+1}$, here k defined as in Theorem 4.6. Then, there exists an invariant unstable manifold* **U** *for the solutions of equation (4.4). Moreover, any two solutions $x_1(\cdot)$ and $x_2(\cdot)$ on* **U** *attract each other exponentially in the sense that they satisfy the following estimate*

$$\begin{aligned} &\|x_1(t) - x_2(t)\| \\ &\leqslant C_\mu e^{-\mu(t_0-t)}\|(Id - P(t_0))(x_1(t_0) - x_2(t_0))\| \ \text{ for all } t \leqslant t_0 \end{aligned} \tag{4.8}$$

with μ, C_μ are positive constants independent of t_0.

Proof. For each $t \in \mathbb{R}$ we define $g_t : X_1(t) \to X_0(t)$ as follows:

$$g_t(y) = \int_{-\infty}^{t} \mathcal{G}(t,\tau) f(\tau, x(\tau)) d\tau$$

where $x(\cdot)$ is the unique solution in $L_\infty((-\infty, t], X)$ of Equation (4.4) on $(-\infty, t]$ satisfying condition $(Id - P(t))x(t) = y$. The existence of the solution $x(\cdot)$ is asserted in Theorem 4.6. The existence of an unstable manifold S can be proved by the same way as in Theorem 2.7. Below, we will prove that manifold S is invariant under Equation (4.4).

Let $x(t)$ be a solution to Equation (4.4) in $L_\infty((-\infty, t_0], X)$ satisfying $x(t_0) = x_0 \in S(t_0)$. By Lemma 4.4 we have that

$$x(s) = U(s,t_0)_| x_0 + \int_{-\infty}^{t_0} \mathcal{G}(s,\tau) f(\tau, x(\tau)) d\tau \quad \text{ for all } s \leqslant t_0.$$

Put $w_s = U(s,t_0)_| x_0 + \int_s^{t_0} \mathcal{G}(s,\tau) f(\tau, x(\tau)) d\tau$. We obtain that $w_s \in \ker P(s)$ and

$$x(s) = w_s + \int_{-\infty}^{s} \mathcal{G}(s,\tau) f(\tau, x(\tau)) d\tau.$$

On the other hand, for $t \leqslant s$ we have that

$$\begin{aligned} U(t,s)_| w_s &+ \int_{-\infty}^{s} \mathcal{G}(t,\tau) f(\tau, x(\tau)) d\tau = U(t,s)_| U(s,t_0)_| x_0 \\ &+ U(t,s)_| \int_s^{t_0} \mathcal{G}(s,\tau) f(\tau, x(\tau)) d\tau + \int_{-\infty}^{s} \mathcal{G}(t,\tau) f(\tau, x(\tau)) d\tau \\ = U(t,t_0)_| x_0 &+ \int_s^{t_0} \mathcal{G}(t,\tau) f(\tau, x(\tau)) d\tau + \int_{-\infty}^{s} \mathcal{G}(t,\tau) f(\tau, x(\tau)) d\tau \\ = U(t,t_0)_| x_0 &+ \int_{-\infty}^{t_0} \mathcal{G}(t,\tau) f(\tau, x(\tau)) d\tau = x(t) \end{aligned}$$

Thus, $x(s) = w_s + g_s(w_s)$. This leads to $x(s) \in S_s$ for all $s \leqslant t_0$. ∎

Next, we will show the attraction property of the unstable manifold. To do so, we need the concept of (ϵ, w)-suitable property of a function in the following definition.

Definition 4.8. Given numbers $\epsilon,\ w > 0$. A real function $g(\cdot)$ is called (ϵ, w)-suitable if there exist positive numbers μ, η such that $\eta e^{\mu} < \epsilon$ and

$$\int_s^t g(\tau) e^{\int_s^\tau g(u)du} \leqslant \eta e^{(\mu-\omega)(t-s)}.$$

In our strategy, the number w in the above definition will be the exponential bound of the evolution family $(U(t,s))_{t\geq s}$.

Put $X(t,s) : y \mapsto X(t,s)y = U(t,s)y + \int_s^t U(t,\tau) f(\tau, x(\tau)) d\tau$ for all $t \geq s$, where $x(t)$ is the unique solution to the integral equation

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\tau) f(\tau, x(\tau)) d\tau \text{ for all } t \geq s,$$

with the initial condition $x(s) = y$.

Proposition 4.9. *Define* $\phi(t,s) := X(t,s) - U(t,s)$. *If the function* $N\varphi(\cdot)$ *is* $(\frac{\epsilon}{N}, w)$*-suitable where* w *is the exponential bound of the evolution family* $(U(t,s))_{t\geq s}$*, then there exists a pair of positive numbers* μ, η *such that* $\eta e^{\mu} < \epsilon$ *and*

$$\|\phi(t,s)x(s) - \phi(t,s)y(s)\| \leqslant \eta e^{\mu(t-s)} \|x(s) - y(s)\| \textit{ for all } t \geq s \in \mathbb{R},$$

where $x(\cdot), y(\cdot)$ *are two arbitrary solutions to Equation* (4.4)*.*

Proof. Firstly, we will show that

$$\|x(t) - y(t)\| \leqslant N e^{\omega(t-s)} \|x(s) - y(s)\| e^{\int_s^t N\varphi(u)du} \text{ for all } t \geq s.$$

Put $z(t) := \|x(t) - y(t)\|$ for all $t \in \mathbb{R}$, we can rewrite the above inequality as follows

$$z(t) \leqslant N e^{\omega(t-s)} z(s) e^{\int_s^t N\varphi(u)du} \text{ for all } t \geq s.$$

Indeed, since $x(\cdot), y(\cdot)$ are two solutions of Equation (4.4) we have that

$$\begin{aligned} z(t) = \|x(t) - y(t)\| \leqslant & \|U(t,s)(x(t) - y(t))\| \\ & + \|\int_s^t U(t,\tau)(f(\tau, x(\tau)) - f(\tau, y(\tau)))\| d\tau \\ \leqslant & N e^{\omega(t-s)} + \int_s^t N e^{\omega(t-\tau)} \varphi(\tau) \|x(\tau) - y(\tau)\| d\tau \\ \leqslant & N e^{\omega(t-s)} + \int_s^t N e^{\omega(t-\tau)} \varphi(\tau) z(\tau) d\tau \text{ for all } t \geq s. \end{aligned}$$

Set $\gamma(t) := z(t)e^{-\omega t}$, then

$$\gamma(t) \leqslant N\gamma(s) + \int_s^t N\varphi(\tau)\gamma(\tau)d\tau \text{ for all } t \geq s.$$

Applying Gronwall's inequality we obtain that

$$\gamma(t) \leqslant N\gamma(s)e^{\int_s^t N\varphi(\tau)d\tau} \text{ for all } t \geq s.$$

Thus,

$$z(t) \leqslant Ne^{\omega(t-s)}z(s)e^{\int_s^t N\varphi(u)du} \text{ for all } t \geq s.$$

From this it follows that

$$\begin{aligned}
\|\phi(t,s)x(s) - \phi(t,s)y(s)\| &= \|\int_s^t U(t,\tau)(f(\tau,x(\tau)) - f(\tau,y(\tau)))\| \\
&\leqslant \int_s^t Ne^{\omega(t-\tau)}\varphi(\tau)\|x(\tau) - y(\tau)\|d\tau \\
&\leqslant \int_s^t Ne^{\omega(t-\tau)}\varphi(\tau)Ne^{\omega(\tau-s)}z(s)e^{\int_s^\tau N\varphi(u)du}d\tau \\
&\leqslant \int_s^t N^2e^{\omega(t-s)}\varphi(\tau)z(s)e^{\int_s^\tau N\varphi(u)du}d\tau \\
&\leqslant Ne^{\omega(t-s)}\|x(s) - y(s)\| \int_s^t N\varphi(\tau)e^{\int_s^\tau N\varphi(u)du}d\tau.
\end{aligned}$$

Thus, the proposition follows from the $(\frac{\epsilon}{N}, w)$-suitable property of the function $N\varphi(\cdot)$. ■

Putting $\tilde{M} := \sup_{t\in\mathbb{R}} \|P(t)\| + \sup_{t\in\mathbb{R}} \|Q(t)\| < \infty$ and choosing the positive number h such that $Ne^{-\nu h} < \frac{1}{2}$ we next prove the following lemma as a primary step in order to show the attraction property of the unstable manifold.

Lemma 4.10. *Under the assumptions and notations of the previous proposition, if* $\eta e^{\mu h} \leqslant \frac{1}{2\tilde{M}}$*, then there exists a constant* $0 < \tilde{\delta} < 1$ *such that for all* $t, s \in \mathbb{R}$ *satisfying* $t - s = h$*, we have the inequality*

(4.9) $\quad \|P(t)X(t,s)x - g_t(Q(t)X(t,s)x)\| \leqslant \tilde{\delta}\|P(s)x - g_s(Q(s)x)\|$ *for all* $x \in X$.

Proof. We recall from the previous proposition that

$$\|\phi(t,s)x - \phi(t,s)y\| \leqslant \eta e^{\mu(t-s)}\|x - y\| \text{ for all } t \geq s \in \mathbb{R} \text{ and } x, y \in X$$

where $\phi(t,s) = X(t,s) - U(t,s)$.

Now, we prove that (4.9) holds. For the simplicity of notations we put $F := X(t,s),\ S := U(t,s),\ \Phi := \phi(t,s)$, we then have

$$\|P(t)F(x) - g_t(Q(t)F(x))\| \leqslant \|P(t)F(x) - g_t(Q(t)F(Q(s)x + g_s(Q(s)x)))\| \\ + \|g_t(Q(t)F(Q(s)x + g_s(Q(s)x))) - g_t(Q(t)F(x))\|.$$

Consider the first term of the right hand side. Using the property $graph(g_t) = X(t,s)graph(g_s)$ for all $t \geq s$ and the dichotomy property of the family $\mathcal{U}$ we deduce that

$$\begin{aligned} &\|P(t)F(x) - g_t(Q(t)F(Q(s)x + g_s(Q(s)x)))\| \\ &= \|P(t)F(x) - P(t)F(Q(s)x + g_s(Q(s)x))\| \\ &\leqslant \|P(t)S(x) - P(t)S(Q(s)x + g_s(Q(s)x))\| \\ &\quad + \|P(t)\Phi(x) - P(t)\Phi(Q(s)x + g_s(Q(s)x))\| \\ &\leqslant \|P(t)S(P(s)x - g_s(Q(s)x))\| \\ &\quad + \|P(t)(\Phi(x) - \Phi(Q(s)x + g_s(Q(s)x)))\| \\ &\leqslant Ne^{-\nu h}\|P(s)x - g_s(Q(s)x)\| + \sup_t \|P(t)\|\eta e^{\mu h}\|P(s)x - g_s(Q(s)x)\| \\ &\leqslant (Ne^{-\nu h} + \sup_t \|P(t)\|\eta e^{\mu h})\|P(s)x - g_s(Q(s)x)\|. \end{aligned} \tag{4.10}$$

On the other hand, using the contraction property of g_t proved in the previous section, we have

$$\begin{aligned} &\|g_t(Q(t)F(Q(s)x + g_s(Q(s)x))) - g_t(Q(t)F(x))\| \\ &\leqslant \|Q(t)F(Q(s)x + g_s(Q(s)x))) - Q(t)F(x)\| \\ &\leqslant \|Q(t)S(Q(s)x + g_s(Q(s)x))) - Q(t)S(x)\| \\ &\quad + \|Q(t)\Phi(Q(s)x + g_s(Q(s)x))) - Q(t)\Phi(x)\| \\ &\leqslant \sup_t \|Q(t)\|\eta e^{\mu h}\|P(s)x - g_s(Q(s)x)\| \end{aligned} \tag{4.11}$$

Now, combining (4.10) and (4.11), we get

$$\begin{aligned} &\|P(t)F(x) - g_t(Q(t)F(x))\| \\ &\leqslant [Ne^{-\nu h} + (\sup_{t\in\mathbb{R}} \|P(t)\| + \sup_{t\in\mathbb{R}} \|Q(t)\|)\eta e^{\mu h}]\|P(s)x - g_s(Q(s)x)\|. \end{aligned}$$

The quantity $\tilde{\delta} := Ne^{-\nu h} + (\sup_{t\in\mathbb{R}} \|P(t)\| + \sup_{t\in\mathbb{R}} \|Q(t)\|)\eta e^{\mu h}$ is less than 1 if $\eta e^{\mu h} < \frac{1}{2\tilde{M}}$. This implies the assertion of the lemma. ∎

From (4.9), we deduce that

$$d(x(t), \mathbf{U}_t) \leqslant \tilde{\delta} d(x(s), \mathbf{U}_s) \text{ for all } t \geq s \in \mathbb{R}, \quad t - s = h,$$

where $x(\cdot)$ is any solution of the integral equation (4.4).
Hence, if we put

$$\begin{cases} \tilde{\eta} := & \dfrac{-ln\tilde{\delta}}{h}, \\ \tilde{K} := & \tilde{\delta}^{-h} \sup\limits_{r\in[0,1]} d(x(r), \mathbf{U}_r) \leqslant \tilde{\delta}^{-h} \sup\limits_{r\in[0,1]} \|x(r)\| < \infty, \end{cases}$$

then we have that $0 < \tilde{\eta} < 1$, and

$$d(x(t), \mathbf{U}_t) \leqslant \tilde{K}\tilde{\eta}^{t-s} d(x(s), \mathbf{U}_s) \text{ for all } t \geq s \in \mathbb{R}.$$

Thus, we obtain the attraction property of an unstable manifold which is stated in the next theorem.

Theorem 4.11. *Let the evolution family $(U(t,s))_{t\geq s}$ have an exponential dichotomy with the corresponding projections $(P(t))_{t\in\mathbb{R}}$ and the dichotomy constants $N, \nu > 0$. Suppose that $f : \mathbb{R} \times X \to X$ be φ-Lipschitz, where φ is the positive function which belongs to $E_{\mathbb{R}}$ such that $k < 1$, where k is defined by Formula* (4.6)*. Then, there exists an unstable manifold* ***U*** *for the solutions to Equation* (4.4)*. Moreover, this manifold exponentially attracts all orbits of solutions to Equation* (4.4)*, i.e., for any solution $x(\cdot)$ of* (4.4) *and $s \in \mathbb{R}$ there are constants $\tilde{K} > 0$ and $0 < \tilde{\eta} < 1$ such that*

$$d(x(t), \boldsymbol{U}_t) \leqslant \tilde{K}\tilde{\eta}^{t-s} d(x(s), \boldsymbol{U}_s) \textit{ for all } t \geq s \in \mathbb{R}.$$

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