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ZIGZAG AND CENTRAL CIRCUIT STRUCTURE OF $(\{1, 2, 3\}, 6)$-SPHERES

Michel Deza and Mathieu Dutour Sikirić

Abstract. We consider 6-regular plane graphs whose faces have size 1, 2 or 3. In Section 2 a practical enumeration method is given that allowed us to enumerate them up to 53 vertices. Subsequently, in Section 3 we enumerate all possible symmetry groups of the spheres that showed up. In Section 4 we introduce a new Goldberg-Coxeter construction that takes a 6-regular plane graph G_0, two integers k and l and returns two 6-regular plane graphs.

Then in the final section, we consider the notions of zigzags and central circuits for the considered graphs. We introduced the notions of tightness and weak tightness for them and prove an upper bound on the number of zigzags and central circuits of such tight graphs. We also classify the tight and weakly tight graphs with simple zigzags or central circuits.

1. INTRODUCTION

By a (S, k)-*sphere* we call a plane k-regular graph such that any face has size in S.

If G is a 6-regular plane graph, then by Euler formula it satisfies the equality:

$$\sum_{k\geq 1} p_k(3-k) = 6$$

with p_k the number of k-gons, i.e. faces of size k. So, if, moreover, G has only 2- and 3-gonal faces, then it has exactly six 2-gons.

Note that a $(\{2, 3\}, 6)$-sphere with p_3 3-gons has $n = 2 + \frac{p_3}{2}$ vertices. In [6] (Theorem 2.0.1) we proved that for any $n \geq 2$ there exist a $(\{2, 3\}, 6)$-sphere with n vertices. If 1-gons are permitted, then $2p_1 + p_2$ being 6, all possible pairs (p_1, p_2), besides $(0, 6)$, are $(1, 4)$, $(2, 2)$ and $(3, 0)$.

The only possible $(\{s-1, s\}, k)$-spheres have $(s, k) = (6, 3)$ (well-known geometrical *fullerenes*), $(4, 4)$ (considered in [8, 7, 9, 12]) and $(3, 6)$ (the object of this

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paper). $(\{2,3\},6)$-spheres are spherical analog of the 6-regular partition $\{3^6\}$ of the Euclidean plane by regular triangles, with six 2-gons playing role of "defects", disclinations needed to increase the curvature zero to the one of sphere. The problem of existence of plane graphs with a fixed p-vector is an active subject of research, see for example [19].

In Section 2 we expose a practical method for generating $(\{1,2,3\},6)$-spheres. The main idea is to use a reduction to 3-regular graphs for which very efficient programs exist [2]. Then in Section 3 we determine the possible symmetry groups of $(\{1,2,3\},6)$-spheres with i 1-gons. The methods are reasonably simple except for the $(\{1,3\},6)$-spheres for which the Goldberg-Coxeter construction is needed.

In Section 4 we introduce a new Goldberg-Coxeter construction. It takes a 6-regular sphere G_0, two integers k,l and returns two 6-regular spheres G_1, G_2 with $GC_{k,l}(G_0) = \{G_1, G_2\}$. The construction satisfies a multiplicativity property based on the ring of Eisenstein integers. In the case $k = l = 1$ we call the construction *oriented tripling* and we have a more explicit description of it. The Goldberg-Coxeter construction defined here generalizes the one introduced in [14, 3, 8] for 3- or 4-regular plane graphs and allows to describe explicitly all $(\{1,3\},6)$-spheres. It also allows to describe all $(\{2,3\},6)$-spheres of symmetry D_6, D_{6h}, T, T_h, or T_d.

In a plane graph G, a *zigzag* is a circuit of edges such that any two but no three consecutive edges are contained in the same face. A zigzag has necessarily even length. In an *Eulerian* (i.e. degree of any vertex is even) plane graph, a *central circuit* is a circuit of edges such that any edge entering a vertex is followed by the edge opposite to the entering one.

A zigzag is called *simple* if no two edges occur two times and a central circuit is called *simple* if no two vertices occur two times. Let Z and Z' be (possibly, $Z = Z'$) zigzags of a plane graph G and let an orientation be selected on them. An edge e of intersection $Z \cap Z'$ is called of *type I* or *type II*, if Z and Z' traverse e in opposite or same direction, respectively Let C and C' be (possibly, $C = C'$) central circuits of a 6-regular plane graph and let an orientation be selected on them. A vertex v of intersection $C \cap C'$ is called of *type I* or *type II* if C and C' pass by v with orientation shifted by $60°$, respectively, $120°$. We prove in Theorem 13 that the intersection type is always of type II.

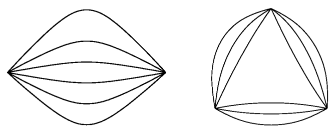
We then introduce the notions of tighness and weak tightness for zigzags and central circuits and we prove upper bound on the maximal number of zigzags and central circuit. The results are summarized in Table 2 and Figure 28. Then we determine completely the weakly tight spheres with simple zigzags or central circuits.

2. Generation Method

In any $(\{2,3\},6)$-sphere, one can collapse its 2-gons into simple edges. By doing so one obtains a graph with vertices of degree at most 6 and with faces of size 3 only.

So, the dual will be a 3-regular graph with faces of size at most 6.

Theorem 1. *With the exception of the following* $(\{2, 3\}, 6)$*-spheres any* $(\{2, 3\}, 6)$*-sphere is obtained from a* $(\{3, 4, 5, 6\}, 3)$*-sphere by adding vertices of degree* 2 *and taking the dual.*



$6 \times K_2$: D_{6h}, $n = 2$ $3 \times K_3$: D_{3h}, $n = 3$

Proof. Let G be a $(\{2, 3\}, 6)$-sphere and let G^* be its dual. Then, by removing from G^* its vertices of degree 2, one gets a 3-regular graph G_1. It can happen that G_1 has no vertices and is reduced to a simple circular edge e. In this case, if one adds six vertices on e and take the dual, one will get the first exceptional graph with 2 vertices. If G_1 has one face F which is a 1-gon, then we have to add 5 vertices of degree 2 on the edge e of F. Necessarily, any face adjacent to F has to be a 1-gon, but this is, clearly, impossible. If F is a 2-gon and F is adjacent to at least one 2-gon, then G_1 is reduced to a graph with two vertices and three edges. The corresponding $(\{2, 3\}, 6)$-sphere is the second exceptional graph. Assume that F is adjacent to F_1, F_2 with F_i being a a_i-gon and $a_i \geq 3$. If one of a_i is 3, then the other is 6 and this gives a 1-gon. Thus, the only possibility is $a_1 = a_2 = 4$. This implies that we have a graph with 4 vertices, two 4-gonal and two 2-gonal faces. But consideration of all possibilities rules out this option. So, G_1 is a 3-regular plane graph with faces of size within $\{3, 4, 5, 6\}$. ∎

The method can be generalized (in Theorem 2) to deal with graphs with 1-gons. Note that for most $(\{3, 4, 5, 6\}, 3)$-spheres one cannot add those vertices of degree 2, in order to get the required spheres, because whenever we add such a 2-gon, we have two faces of size lower than 6 that are adjacent. Graphs admitting such adjacency are relatively rare among $(\{3, 4, 5, 6\}, 3)$-spheres. Some such graphs are the $(\{5, 6\}, 3)$-spheres with the 5-gons organized in pairs, they are part of the class of *face-regular* spheres [6].

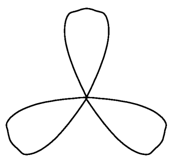
The above theorem gives a method to enumerate $(\{2, 3\}, 6)$-spheres. First enumerate the $(\{3, 4, 5, 6\}, 3)$-spheres using the program `CPF`, which is available from [1] and whose algorithm has been described in [2]. After such enumeration is done, the trick is to add the six vertices of degree 2 in all possibilities. This is relatively easy to do and thus we have an efficient enumeration method. The numbers of graphs are shown in Table 1 for $2 \leq n \leq 53$. We should point out that this algorithm while reasonable for our purpose is very far from being optimal. A better method would be to adapt the

Table 1: Number N_i of $(\{1,2,3\},6)$-spheres with n vertices and $(p_1,p_2)=(i,6-2i)$

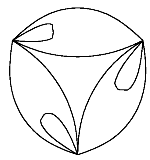
n	N_0	N_1	N_2	N_3	n	N_0	N_1	N_2	N_3	n	N_0	N_1	N_2	N_3
1	0	0	1	1	19	69	36	13	1	37	436	133	24	1
2	1	0	1	0	20	100	34	28	0	38	581	118	37	0
3	1	1	3	1	21	86	46	19	1	39	495	159	32	1
4	3	1	5	1	22	133	33	23	0	40	677	112	59	0
5	2	3	5	0	23	112	62	16	0	41	582	187	26	0
6	7	2	8	0	24	165	44	37	0	42	758	133	53	0
7	5	6	6	1	25	144	57	20	1	43	679	180	27	1
8	12	5	12	0	26	205	54	27	0	44	869	172	53	0
9	10	8	8	1	27	176	75	22	1	45	749	199	43	0
10	19	6	12	0	28	251	61	36	1	46	1000	149	44	0
11	16	14	9	0	29	214	95	19	0	47	868	250	30	0
12	29	11	17	1	30	299	61	40	0	48	1101	182	72	1
13	24	17	10	1	31	265	96	20	1	49	989	235	35	2
14	42	16	16	0	32	360	89	43	0	50	1259	194	57	0
15	35	23	15	0	33	305	111	28	0	51	1076	270	40	0
16	59	18	22	1	34	429	80	33	0	52	1410	210	61	1
17	48	33	12	0	35	375	134	31	0	53	1228	313	33	0
18	79	22	22	0	36	488	105	50	1					

algorithm from [2] although this is not easy to do.

Theorem 2. *With the exception of the following graphs T_1, T_2 and the spheres of the infinite series depicted in Figures 1, 2, 3 and 4, any $(\{1,2,3\},6)$-sphere with at least one 1-gon is obtained from a $(\{3,4,5,6\},3)$-sphere by taking the dual and then splitting some edges according to following two schemes:*

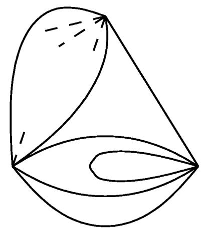


Trifolium T_1: C_{3v}, $n=1$



T_2: C_{3h}, $n=3$

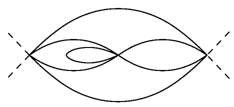
Proof. Let us take a $(\{1,2,3\},6)$-sphere G with at least one 1-gon F in its face-set. Clearly, F cannot be adjacent to another 1-gon. If F is adjacent to a 2-gon, then simple considerations yield that G belongs to the infinite series of Figure 3. So, we can assume in the following that all 1-gons, say $F_1, \dots, F_s$ are adjacent to 3-gons $G_1, \dots, G_s$. If one of the G_i is adjacent to two 2-gons, then we get the sphere B_2 (C_{2h}, $n=2$) depicted in Figure 4. If one of the G_i is adjacent to exactly one 1-gon, then we get the following partial diagram:



Clearly, such diagram extends to one of the graphs of the infinite series depicted in Figure 4.

So, we can now assume that the G_i are adjacent to 3-gons only. If one of the 3-gons adjacent to a G_i turns out to be another G_j, then we get the map C_2 from Figure 4. So, we assume further that those 3-gons are not of the type G_i.

The faces G_i contains two vertices v_i, v'_i with v_i being contained in F_i. If $v_i = v'_i$, then we get the exceptional sphere Trifolium. So, we assume further that $v_i \neq v'_i$. If $v_i = v'_j$ for $i \neq j$, then some easy considerations gives the sphere T_2 as the only possibility. So, let us assume now that the vertex v_i is contained in a 2-gon. Then we have the following local configuration:



From that point, after enumeration of all possibilities we get the infinite series of Figures 1 and 2. So, now we have that all vertices v_i are contained in four 3-gons. This implies that G is obtained from a $(\{3, 4, 5, 6\}, 3)$-sphere by taking the dual and then splitting some edges according to mentioned above schemes. ■

Obviously, the above theorem gives us a method to enumerate the $(\{1, 2, 3\}, 6)$-spheres. The enumeration results are shown in Table 1. Like for $(\{2, 3\}, 6)$-spheres, it would be interesting to have a faster enumeration method.

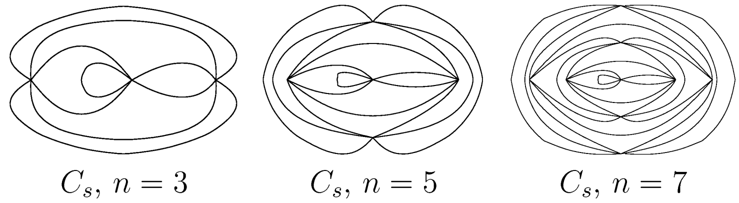


Fig. 1. First terms of an infinite sequence of $(\{1, 2, 3\}, 6)$-spheres R_{2i+1} with $(p_1, p_2) = (1, 4)$.

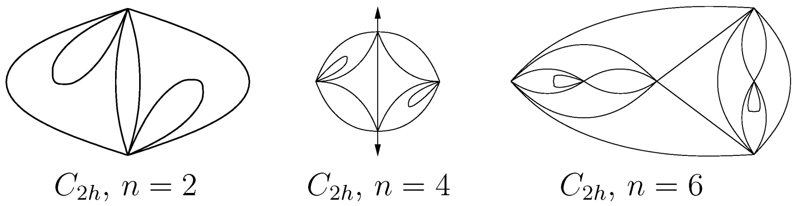


Fig. 2. First terms of an infinite sequence of $(\{1, 2, 3\}, 6)$-spheres S_{2i} with $(p_1, p_2) = (2, 2)$.

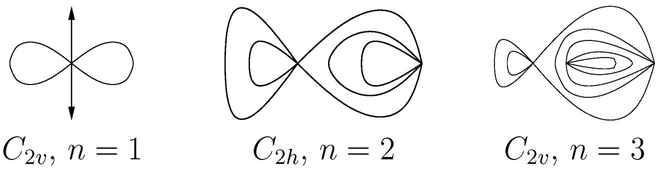


Fig. 3. First terms of an infinite sequence of $(\{1, 2, 3\}, 6)$-spheres A_i with $(p_1, p_2) = (2, 2)$.

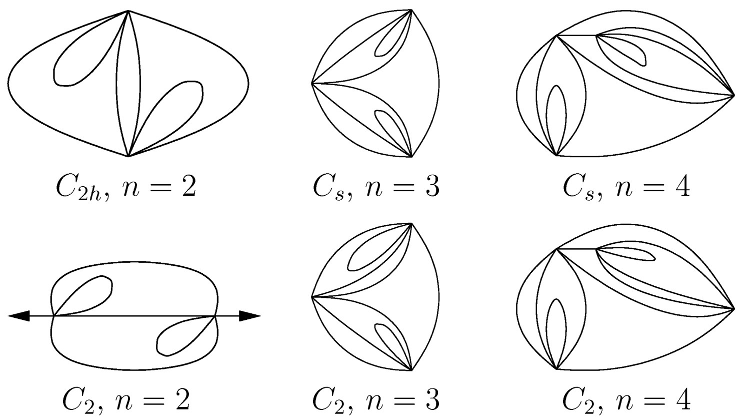


Fig. 4. First terms of two infinite sequences of $(\{1, 2, 3\}, 6)$-spheres B_{i+1}, C_{i+1} with $(p_1, p_2) = (2, 2)$.

3. Symmetry Groups

We now give the possible groups of the considered spheres. Note that we are using the terminology of points groups in chemistry as explained, for example, in [10].

Theorem 3. *The possible symmetry group of a* $(\{2, 3\}, 6)$*-sphere are* C_1, C_2, C_{2h}, C_{2v}, C_3, C_{3h}, C_{3v}, C_i, C_s, D_2, D_{2d}, D_{2h}, D_3, D_{3d}, D_{3h}, D_6, D_{6h}, S_4, S_6, T, T_h *and* T_d. *The minimal possible representatives are given in Figure 5.*

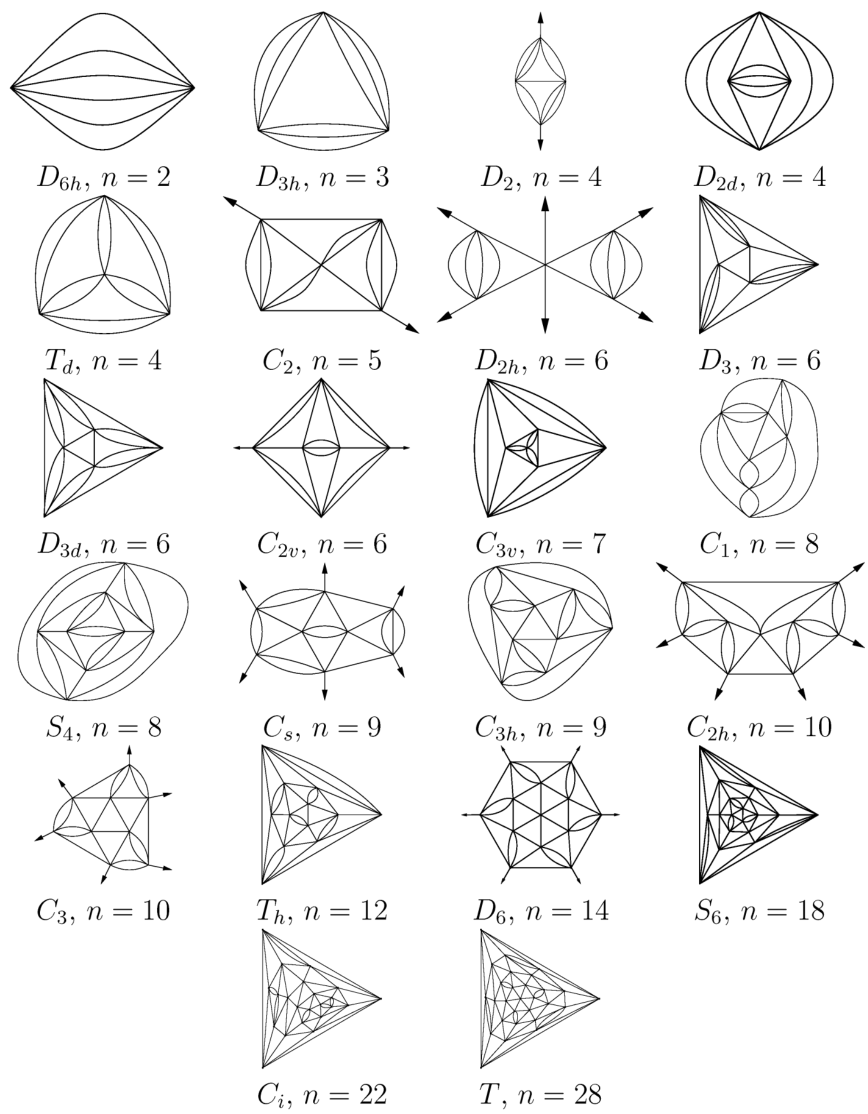


Fig. 5. Minimal representatives for each possible symmetry group of a $(\{2, 3\}, 6)$-sphere.

Proof. The method is to consider the possible axes of symmetry; they are passing through faces, edges or vertices. As a consequence, the possibilities for a k-fold axis of symmetry are 2, 3 or 6. The only groups that could occur, besides 22 given in the theorem, are D_{6d}, C_6, C_{6h} or C_{6v}.

If a 6-fold axis occurs, then it necessarily passes through two vertices, say, v_1 and v_2. Around this vertex one can add successive rings of triangles as in the classical structure of the triangular lattice. At some point one gets a 2-gon and thus, by the

6-fold symmetry, six 2-gons. Then, one can continue the structure uniquely and the structure is defined uniquely. This completion is the same as the one around v and it implies the existence of a mapping that inverts v with the transformation inverting v_1 and v_2 and the group are D_6, D_{6h} or D_{6d}. The group D_{6d} is ruled out because 2-fold axis passes through the 2-gons. ■

Theorem 4. *The possible symmetry group of a* $(\{1,2,3\},6)$*-sphere with* $p_1 > 0$ *are*

(i) C_1 *or* C_s *if* $p_1 = 1$.

(ii) C_1, C_2, C_i, C_s, C_{2v} *or* C_{2h} *if* $p_1 = 2$.

(iii) C_3, C_{3v} *or* C_{3h} *if* $p_1 = 3$.

The minimal possible representatives are given in Figures 6, 7 and 8.

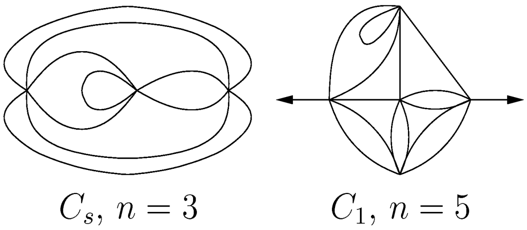


Fig. 6. Minimal representatives for each possible symmetry group of a $(\{1,2,3\},6)$-sphere with $(p_1, p_2) = (1, 4)$.

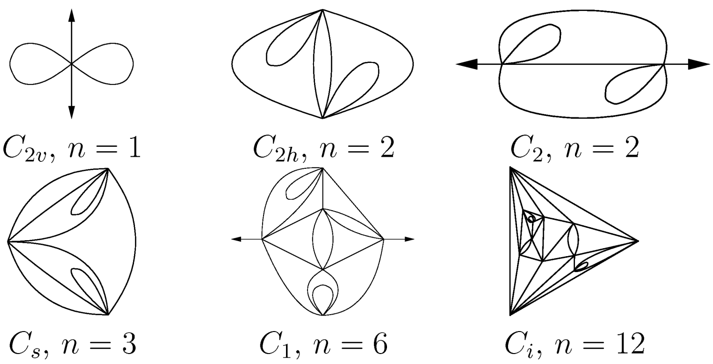
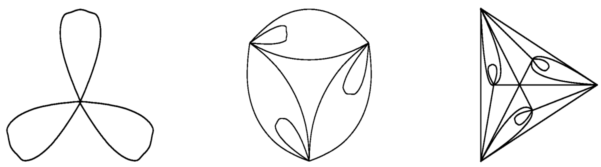


Fig. 7. Minimal representatives for each possible symmetry group of a $(\{1,2,3\},6)$-sphere with $(p_1, p_2) = (2, 2)$.

Proof. For (i), the 1-gon has to be preserved by any symmetry which leaves C_1 and C_s as the only possibilities. They are both realized. For (ii), we proceed in the same way. (iii) is proved in Theorem 10. ■



C_{3v}, $n=1$ C_{3h}, $n=3$ C_3, $n=7$

Fig. 8. Minimal representatives for each possible symmetry group of a $(\{1,3\},6)$-sphere.

An interesting question is to consider whether a $(\{1,2,3\},6)$-sphere can be mapped onto the projective plane $\mathbb{P}^2$. Clearly, this is equivalent to the map having a central inversion. $(\{1,3\},6)$-maps on the projective plane do not exist since no centrally-symmetric $(\{1,3\},6)$-sphere exist. All $(\{2,3\},6)$-maps on the $\mathbb{P}^2$ are antipodal quotients (i.e. with halved p-vector and number of vertices) of $(\{2,3\},6)$-spheres whose groups contain the inversion, i.e. those of symmetry C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, S_6 and T_h. In the next section we will describe explicitly the $(\{2,3\},6)$-sphere of symmetry D_{6h} and T_h.

4. The Goldberg-Coxeter Construction

In [11] a construction is given, generalizing Goldberg-Coxeter construction given in [14, 3] for 3-regular graphs with 6-gonal and 5-, 4-, 3-gonal faces only. For the particular case when G is a geometrical fullerene, there is a large body of literature, see bibliography of [11]. It takes a 3- or 4-regular plane graph G and returns a 3- or 4-regular plane graph. There the first step was to take the dual and get a triangulation or a quadrangulation of the sphere. The respective triangles and squares were subdivided, then put together and the dual was taken. An instrument in this operation was that the Eisenstein and Gaussian integers are best represented on the tiling of the plane by equilateral triangles, respectively, squares. We are able to generalize this construction to the 6-regular case but there are differences.

First, if G is a $(\{2,3\},6)$-sphere, then the dual G^* is a plane graph with faces of size 6 and thus, bipartite. The tessellation of Euclidean plane by regular hexagons is represented on Figure 9. We use there two vectors v_1, v_2 to represent the coordinate of the points. In complex coordinates $v_1 = 1$ and $v_2 = j$ with $j = e^{i\pi/3}$. The lattice $L = \mathbb{Z}v_1 + \mathbb{Z}v_2$ is called the *Eisenstein ring*. The point A is the origin and the point $B(k,l)$ is the point $k+lj$. The points in the bipartite component of A are $L_A = (1+j)L$, while the points in the component of $B(1,0)$ are $L_B = 1 + (1+j)L$. Both sets L_A and L_B are stable under multiplication. We will first define the Goldberg-Coxeter construction for $k + lj \in L_B$. Then we will extend it to any $(k,l) \neq 0$.

Theorem 5. *If $z = k + lj \in L_B$ and G_0 is a 6-regular plane graph with $|G_0|$*

vertices, then it is possible to define a plane graph $G' = GC_z(G_0) = GC_{k,l}(G_0)$ *such that the following holds:*

(i) G' *is a* 6*-regular plane graph with* $|G_0|(k^2 + kl + l^2)$ *vertices.*

(ii) *Every face of* G_0 *corresponds to a face of* G' *with all new faces of* G' *being* 3*-gons.*

(iii) G' *has all rotational symmetries of* G_0 *and all symmetries as well if* $l = 0$ *or* $k = 0$.

(iv) $GC_{1,0}(G_0) = G_0$ *and* $GC_z(G_0) = GC_{zj^2}(G_0)$.

(v) $GC_z(GC_{z'}(G_0)) = GC_{zz'}(G_0)$.

(vi) $GC_z(G_0) = GC_{\overline{z}}(\overline{G_0})$ *where* $\overline{G_0}$ *is the graph that differs from* G_0 *only by a plane symmetry.*

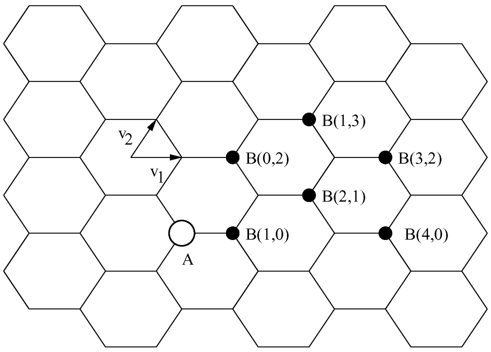
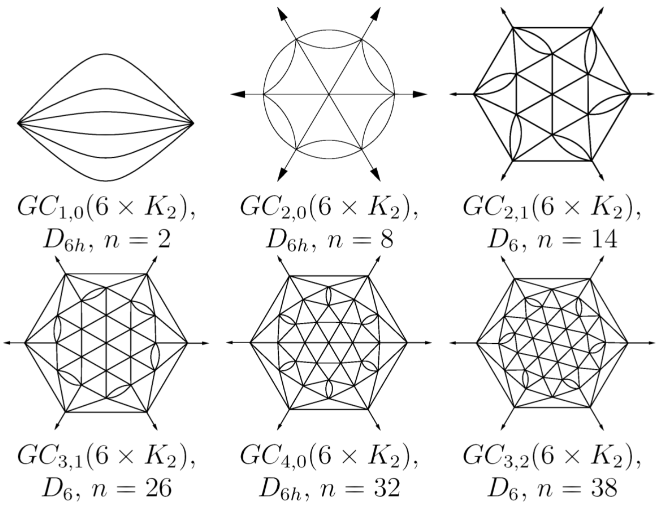


Fig. 9. The tiling by hexagons, the point A and some points in the other bipartite component.



$GC_{1,0}(6 \times K_2)$, D_{6h}, $n = 2$ | $GC_{2,0}(6 \times K_2)$, D_{6h}, $n = 8$ | $GC_{2,1}(6 \times K_2)$, D_6, $n = 14$

$GC_{3,1}(6 \times K_2)$, D_6, $n = 26$ | $GC_{4,0}(6 \times K_2)$, D_{6h}, $n = 32$ | $GC_{3,2}(6 \times K_2)$, D_6, $n = 38$

Fig. 10. Smallest $(\{2,3\},6)$-spheres of symmetry D_6, D_{6h} in terms of the Goldberg-Coxeter construction.

Proof. Let G_0 be a 6-regular graph. The dual G_0^* is a plane graph with all faces being 6-gons. If $z = k+lj \in L_B$, then the point $B(k,l)$ belongs to the same connected component as B.

The point $c = -j^2 z$ is the center of an hexagon and we build around it six points P_q:

$$P_q = c - j^q c \text{ for } 0 \le q \le 5.$$

Those six points form a *master hexagon* that correspond to the original hexagon. Every hexagon of G_0^* can be thus modified and we can arrange them together at the boundary between adjacent hexagons. We can thus obtain another plane graph with 6-regular faces. By taking the dual one more time, we get $GC_{k,l}(G_0)$. Checking the remaining properties is relatively easy. ■

The above theorem is similar to Proposition 3.1 in [11]. But there are some differences. In the 3-regular case, we have $GC_{k,l}(G_0)$ with all symmetries if $k = l$, while here the case $k = l$ is impossible. See in Figure 13 the local structure of the Goldberg-Coxeter construction $GC_{3,2}$ and in Figure 18, $GC_{4,0}$.

Theorem 6. *The* $(\{2,3\},6)$*-spheres of symmetry* D_6*,* D_{6h} *are obtained as* $GC_{k,l}(6\times K_2)$ *with* $k + lj \in L_B$*.*

Proof. Let us take a $(\{2,3\},6)$-sphere G of symmetry D_6 or D_{6h} and let us take the dual G^*. The 6-fold axis passes through a 6-gon F and the 2-gons of G correspond to 6-regular vertices. But the position of those 2-gons define a master hexagon around F and thus we get exactly the structure of a graph $GC_{k,l}$ $(6\times K_2)$. ■

In Theorem 5 we have defined the Goldberg-Coxeter construction $GC_{k,l}$ for $k+lj \in L_B$. Now we want to define it for any $k, l \neq 0$. For that we first introduce the notion of *oriented tripling*.

Definition 7. If G is a 6-regular plane graph, then its dual G^* is bipartite. For each such bipartite class C we define a graph $Or_C(G)$ with the following properties:

(i) $Or_C(G)$ is a 6-regular plane graph with 3 times as many vertices.

(ii) Each vertex of G corresponds to 3 vertices of $Or_C(G)$ and 4 triangular faces.

(iii) Every symmetry of G preserving C also occur as symmetry of $Or_C(G)$.

The local configuration of the operation is shown in Figure 11. For every face F of G, we orient the edges of F counter-clockwise. Thus for every bipartite class C of G^* we get an orientation of the edges of G. Around a vertex v and its six adjacent vertices, there are three vertices w to which the edge $\{v, w\}$ is oriented from v to w. They are the vertices $1, 3, 5$ in Figure 11.

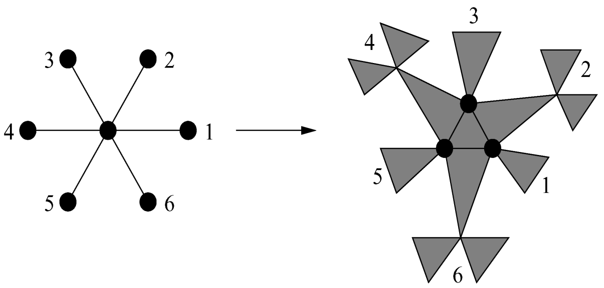
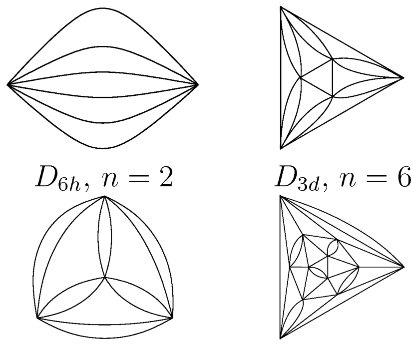


Fig. 11. Local configuration around a vertex of the oriented tripling operation.



D_{6h}, $n = 2$ D_{3d}, $n = 6$

T_d, $n = 4$ T_h, $n = 12$

Fig. 12. Two examples of $(\{2, 3\}, 6)$-spheres with unique oriented tripling.

So, if G has two inequivalent bipartite components C_1 and C_2, then $Or_{C_1}(G)$ and $Or_{C_2}(G)$ are not necessarily isomorphic and the smallest such example is shown on Figure 15. In Figure 12 we give two examples of the action of the oriented tripling when the obtained graph is unique.

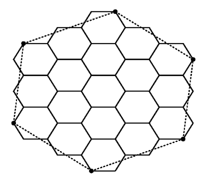
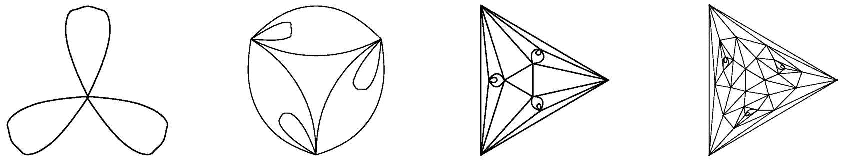


Fig. 13. Local structure of the Goldberg-Coxeter construction $GC_{3,2}$.

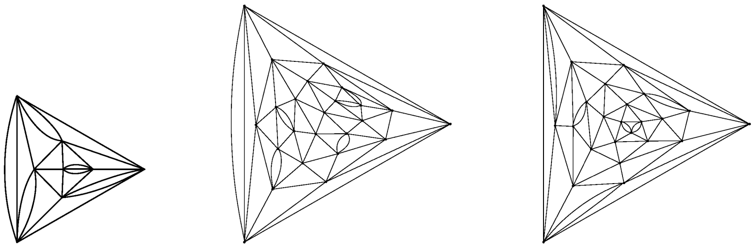
For the Trifolium T_1, we can define a sequence T_i of graphs with T_{i+1} obtained by applying the oriented tripling to T_i. The first 4 terms are shown in Figure 14.

We now introduce the Goldberg-Coxeter construction in the general case. For a sphere G, denote by $Tr(G)$ the truncation of G, i.e. the sphere obtained by replacing every vertex of degree k of G by a k-gonal face.



C_{3v}, $n = 1$ C_{3h}, $n = 3$ C_{3v}, $n = 9$ C_{3h}, $n = 27$

Fig. 14. First terms of the infinite sequence of $(\{1, 3\}, 6)$-spheres T_i.



G: C_1, $n = 8$ $Or_{C_1}(G)$: C_1, $n = 24$ $Or_{C_2}(G)$: C_1, $n = 24$

Fig. 15. Smallest $(\{2, 3\}, 6)$-sphere having two non-isomorphic oriented triplings.

We will also use the following result: if G is a 3-regular sphere with faces of even size, then it is possible to color the faces of G so that any two adjacent faces have different colors. Such a coloring is unique up to permutation of the colors. If G_0 is a graph with vertices of even degree, then its dual is bipartite and the three colors in $Tr(G_0)$ come from the vertices of G_0 and the two classes of faces in G_0.

Theorem 8. *For a* 6*-regular plane graph* G_0 *and two integers* k, l *with* $k, l \neq 0$, *we can define two* 6*-regular spheres* G_1, G_2 *with* $GC_{k,l}(G_0) = \{G_1, G_2\}$. *This will satisfy to the following properties:*

(*i*) $Tr(G_i) = GC_{k,l}(Tr(G_0))$ *for* $i = 1, 2$ *with* $GC_{k,l}$ *being the Goldberg-Coxeter construction for* 3*-regular spheres.*

(*ii*) G_1 *and* G_2 *are* 6*-regular plane graphs with* $|G_0|(k^2 + kl + l^2)$ *vertices.*

(*iii*) *Every face of* G_0 *corresponds to a face of* G_1 *and* G_2 *with all new faces of* G_1 *and* G_2 *being* 3*-gons.*

(*iv*) $GC_{1,1}(G_0) = \{Or_C(G_0), Or_{C'}(G_0)\}$.

(*v*) *If* $k + lj \in L_B$, *then* $G_1 = G_2$.

(*vi*) *If* $GC_{k,l}(G_0) = \{G_1, G_2\}$, *then* $GC_{k',l'}(G_1) = GC_{k_p,l_p}(G_0)$ *with* $k_p + l_p j = (k + lj)(k' + l'j)$.

Proof. Let us take a 3-coloring in white, red, blue of the faces of $Tr(G_0)$ with white corresponding to the faces coming from vertices of G_0. The 3-regular sphere $GC_{k,l}(Tr(G_0))$ has the faces of G_0 and some 6-gonal faces, thus all its faces are of even size and we can find a 3-coloring of them.

One can see directly that all the white faces of $Tr(G_0)$ have the same color in $GC_{k,l}(Tr(G_0))$; we color them white. If $k \equiv l \pm 1 \pmod 3$ (this contains the case $k + lj \in L_B$), then the faces of $Tr(G_0)$ coming from faces of G_0 will not be white in $GC_{k,l}(Tr(G_0))$. Thus by shrinking the white faces, we get a graph, which is actually the $GC_{k,l}(G_0)$ defined in Theorem if $k + lj \in L_B$.

If $k \equiv l \pmod 3$ then all faces of $Tr(G_0)$ will correspond to white faces in $GC_{k,l}(Tr(G_0))$. The remaining 6-gonal faces have color red and blue. This gives two set of faces that can be shrunk and thus two possible graphs. All properties follow easily. ■

Theorem 9. (i) *Any* $k + lj \neq 0$ *can be written as* $k + lj = (1 + j)^s(k' + l'j)j^u$ *with* $s \geq 0$*,* $u \in \{0, 1\}$ *and* $k' + l'j \in L_B$*.*

(ii) *The sphere* $GC_{k,l}(G_0)$ *is obtained by applying the oriented tripling* s *times and then the Goldberg-Coxeter construction from Theorem 5.*

Proof. (i) The ring of Eisenstein integers is a unique factorization domain. That is every $k + lj \neq 0$ can be factorized by into the relevant primes. The condition $k \equiv l \pmod 3$ is equivalent to $k + lj$ being divisible by $1 + j$. Thus by repeated application of this we can write

$$k + lj = (1 + j)^s(k_2 + l_2 j) \text{ with } k_2 \equiv l_2 \pm 1 \pmod 3.$$

If $k_2 \equiv l_2 + 1 \pmod 3$, then we are done, otherwise we divide by j.

(ii) follows from the multiplicativity property (vi) of Theorem 8. ■

This idea of using the truncation and resulting 3-regular spheres was, perhaps, used for the first time in [15]. This idea could in principle be applied to the enumeration of the $(\{2, 3\}, 6)$-spheres, since the $(\{4, 6\}, 3)$-spheres can be obtained from the program `CPF`. But the truncation multiplies the number of vertices by 6 and this makes this method uncompetitive to the one of Section 2.

We cannot say much in general for the symmetry groups of $GC_{k,l}(G_0)$. This is essentially the same situation as for the oriented tripling. What happens is that for 3-regular graphs, Goldberg-Coxeter construction $GC_{k,l}$ preserve all symmetries if $k = 0$ or $k = l$ and only rotational symmetries otherwise. Thus we get the automorphism group Γ of $GC_{k,l}(Tr(G_0))$. If Γ preserves the set of faces of color red and blue, then Γ is a group of symmetries of $GC_{k,l}(G_0)$, otherwise the stabilizer of the red faces is a group of symmetries of $GC_{k,l}(G_0)$. But some accidental symmetries can occur and we have thus to work on a case-by-case basis.

Theorem 10. *Let G be a $(\{1,3\},6)$-sphere. The following holds:*

(i) *$G = GC_{k,l}(Trifolium)$ with $0 \leq l \leq k$ and has k^2+kl+l^2 vertices.*

(ii) *G has symmetry C_{3v} if $k=0$, C_{3h} if $k=l$ and C_3 otherwise.*

(iii) *G is obtained as $GC_{k,l}(T_i)$ with $k+lj \in L_B$, where $(T_i)_{i\geq 1}$ is the infinite series of 6-regular graphs obtained by starting from Trifolium.*

Proof. Let us take a $(\{1,3\},6)$-sphere. Then $Tr(G)$ is a $(\{2,6\},3)$-sphere. Either from [15] or [18], we know that such spheres are obtained as $GC_{k,l}(3\times K_2)$ with $GC_{k,l}$ denoting here the 3-regular Goldberg-Coxeter construction. Since the faces of $GC_{k,l}(3\times K_2)$ are of even size, it is possible to define a 3-coloring of those faces. The 2-gonal faces should not be in all 3 different colorings. This can occur only if $k \equiv l \pmod 3$. So, $k+lj$ can be factorized as $(1+j)(k'+l'j)$ and we get

$$\begin{aligned} Tr(G) &= GC_{k,l}(3\times K_2)\\ &= GC_{k',l'}(GC_{1,1}(3\times K_2)\\ &= GC_{k',l'}(Tr(Trifolium)). \end{aligned}$$

Thus we have proved (i).

The symmetry of $GC_{k,l}(3\times K_2)$ is D_{3h} if $k=0$ or $k=l$ and D_3 otherwise. If $k\equiv l \pmod 3$, then all 2-gons of $GC_{k,l}(3\times K_2)$ are in the same color, say white. The 3-gonal faces that are not white are of two possible colors: red and blue. In order for a symmetry of $Tr(G) = GC_{k,l}(3\times K_2)$ to induce a symmetry of G, it is necessary and sufficient that it preserves all 3 colors of the coloring. This reduces by a factor of 2 the symmetry group and we get C_3, C_{3h} and C_{3v} as possible groups.

Statement (iii) follows from Theorem 9 (ii). ■

Note that the possible number of vertices of $(\{2,6\},3)$-spheres was already determined in [15].

Denote by $K_2 \times Tetrahedron$ the Tetrahedron with edges doubled.

Theorem 11. (i) *Any $(\{2,3\},6)$-sphere of symmetry T, T_h or T_d is obtained as $GC_{k,l}(K_2\times Tetrahedron)$.*

(ii) *The $(\{2,3\},6)$-spheres of symmetry T_d, respectively T_h, are of the form $GC_{k,0}$ $(K_2\times Tetrahedron)$, respectively $GC_{k,k}(K_2\times Tetrahedron)$.*

Proof. Take a $(\{2,3\},6)$-sphere G of symmetry T, T_d or T_h and consider their truncation $Tr(G)$. It is a $(\{4,6\},3)$-sphere which contains a subgroup T of symmetry. By Theorem 6.2 of [5], this implies that the symmetry group of $Tr(G)$ is O or O_h. By [11] Theorem 5.2, $Tr(G)$ is described by the Goldberg-Coxeter construction applied to the cube, i.e. $Tr(G) = GC_{k,l}(Cube)$. We need now to determinate which graphs $GC_{k,l}(Cube)$ are of the form $Tr(G)$. For that we need to consider the 3-coloring of the faces. Thus the 4-gonal faces are colored by at most 2 colors. This implies that $k \equiv l \pmod 3$. In turn, this give us that $k + lj = (1 + j)(k' + l'j)$, which then gives

$$\begin{aligned} Tr(G) &= GC_{k,l}(Cube) \\ &= GC_{k',l'}(GC_{1,1}(Cube)) \\ &= GC_{k',l'}(Tr(K_2 \times Tetrahedron)). \end{aligned}$$

(i) follows from Theorem 8.

If a $(\{2, 3\}, 6)$-sphere is of symmetry T_d or T_h, then the symmetry group of the truncation is O_h and such spheres are described as $GC_{k,0}(Cube)$ and $GC_{k,k}$ $(Cube)$. ■

Theorem 12. *The number of* $(\{1, 2, 3\}, 6)$*-spheres with* i 1*-gons and less than* n *vertices grows like* $O(n^{4-i})$.

Proof. Take G a $(\{1, 2, 3\}, 6)$-sphere with n vertices and i 1-gons. Then $Tr(G)$ is a $(\{2, 4, 6\}, 3)$-sphere with i 2-gons, $6 - 2i$ 4-gons and $6n$-vertices. Thus, the number of faces of size 2 or 4 is $6-i$. The 3-regular plane graphs whose faces have size at most 6 and the set of faces of size less than 6 is fixed are described by the parametrization theory of Thurston [18]. By using it, [16] obtained some upper bound on the number of geometric fullerenes. The proof applies just as well for the other classes of graphs and give the required upper bound. ■

Note that in principle, Thurston's theory allows to say more. First it gives that the $(\{2, 4, 6\}, 3)$-spheres with i 2-gons. are described by $4 - i$ Eisenstein integers. Not all such spheres correspond to $(\{1, 2, 3\}, 6)$-spheres with i 1-gons. For that some congruence have to be satisfied.

5. Zigzags and Central Circuits

For a plane graph G and a zigzag or central circuit, if we change the orientation, then the type of intersection does not change. Thus, to a zigzag or central circuit of length l with α_1 and α_2 intersections of type I and II, we attribute the symbol l_{α_1,α_2} and we define the *z-vector*, respectively, *c-vector* ${l_1}_{\alpha_{1,1},\alpha_{1,2}}^{m_1}, \dots, {l_p}_{\alpha_{p,1},\alpha_{p,2}}^{m_p}$ to be the vector enumerating such lengths with multiplicities m_i.

Theorem 13. *For a* 6*-regular plane graph, it is possible to find an orientation on the zigzags and central circuits such that any edge, respectively, vertex of intersection is of type II.*

Proof. Let us take a 6-regular plane graph G. Since every vertex is of even degree and G is planar, the dual graph G^* is bipartite. Let us take one color c of the faces of

G and orient the edges of the face of color c in such a way that they turn clockwise around the face (see Figure 17). It is apparent that such orientation carries over to the zigzags and central circuits of G and that with this orientation all the intersection are of type II. ■

For zigzags, this result is not new, see for example [5, 17].

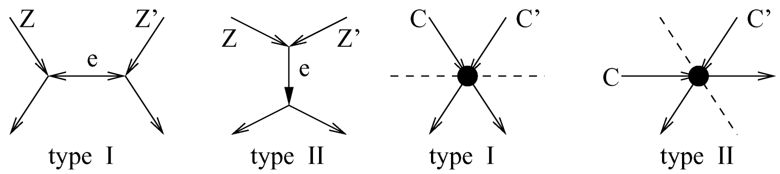


Fig. 16. Intersection types of zigzags and central circuits.

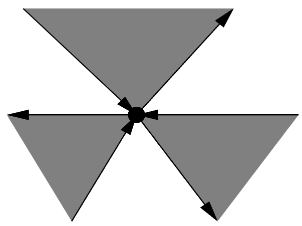


Fig. 17. The orientation of the edges of the face.

Theorem 14. *Let us take a* 6*-regular plane graph* G *with z-vector* $\dots, l_{i\,\alpha_i,\beta_i}^{a_i}, \dots$ *and c-vector* $\dots, k_{j\,\alpha'_j,\beta'_j}^{b_j}, \dots$. *Then the z-vector and c-vector of* $GC_{1+4u,0}(G)$ *are*

$$\dots, \{l_i(1+3u)\}_{\alpha_i,\beta_i}^{a_i(1+u)}, \dots, \dots, \{2k_j(1+3u)\}_{\alpha'_j,\beta'_j}^{2ub_j}, \dots$$

and

$$\dots, \left\{l_i\frac{1+3u}{2}\right\}_{\alpha_i,\beta_i}^{ua_i}, \dots, \dots, \{k_j(1+3u)\}_{\alpha'_j,\beta'_j}^{b_j(1+2u)}, \dots$$

Proof. The proof uses the Goldberg-Coxeter construction previously built. One goes into the dual and subdivides the hexagons. The picture in Figure 18 shows that any central circuit of G corresponds to $1+2u$ central circuits (named B in the figure) and that the zigzags in A on one side correspond to zigzags in G. The result follows similarly for z-vector. ■

Theorem 15. (i) *If a* $(\{1,2,3\},6)$*-sphere has a* 1*-gon, then it has at least one self-intersecting central circuit and one self-intersecting zigzag.*

(ii) *For a* $(\{1,3\},6)$*-spheres, all central circuits and zigzags are self-intersecting.*

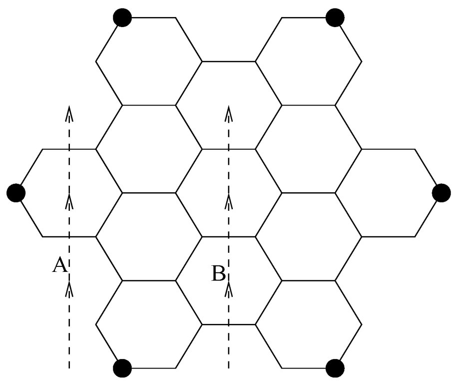


Fig. 18. The local structure of the Goldberg-Coxeter construction $GC_{4,0}$.

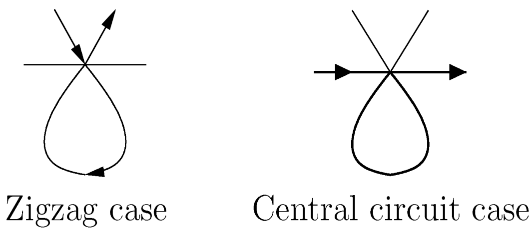


Fig. 19. Self-intersection induced by a 1-gon.

Proof. (i) The self-intersection is evident from Figure 19.

(ii) If a central circuit is simple in a $(\{1,2,3\},6)$-sphere G, then it splits G into two domains D_1 and D_2. If one denotes $n_{i,j}$ the number of faces of size i into the domain D_j, then one has obviously $2n_{1,j}+n_{2,j}=3$. So, if $n_{2,j}=0$, then there is no solution. The proof for zigzags is the same. ■

A z-, respectively *c-railroad* is the circuit of 3-gons bounded by two parallel zigzags, respectively central circuits. See Figure 20 for an illustration.

A $(\{1,2,3\},6)$-sphere is called *z-tight* if for any zigzag there is at least one 1-gon or 2-gon on each of its side of the sphere. It is called *z-weakly tight* if for any zigzag there is no zigzag parallel to it. We define the corresponding notions for central circuits. See Figures 21 and 22 for some illustration of those notions.

The notion of tightness was introduced in [8] for 4-regular plane graphs (see also [7]) and in [4, 5] for 3-regular plane graphs. For 4-regular plane graphs, central circuits were used. A central circuit is then called *reducible* if on one of its side there are only

4-gons. This sequence of 4-gons can be completely eliminated to get a reduced graph. For a 3-regular plane graph, a zigzag is called *reducible* if on one side there is only 6-gons. We can reduce the graph by eliminating those 6-gons only if the zigzag is simple. Moreover, there are several possibilities for this reduced graph while in the 4-regular case, the reduction is uniquely defined.

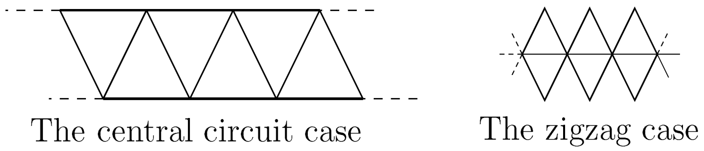


Fig. 20. A c-railroad and a z-rairoadss bounded by two central circuits, respectively, zigzags.

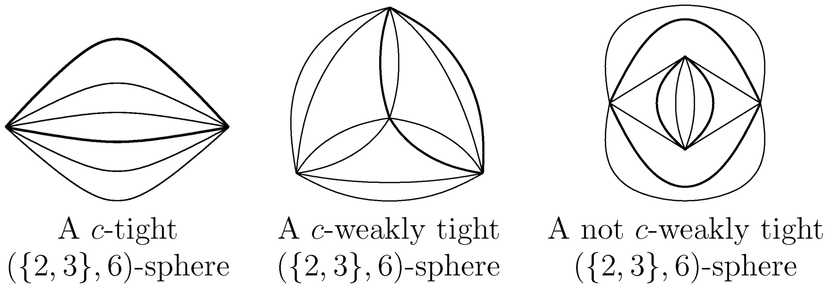


Fig. 21. Illustration of the notions of c-tightness.

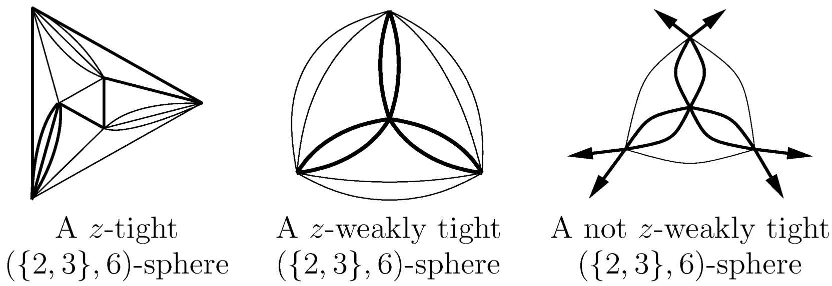


Fig. 22. Illustration of the notions of z-tightness.

For a $(\{1, 2, 3\}, 6)$-sphere G, let $s(G) = p_1(G) + 2p_2(G)$, where $p_i(G)$ is the number of i-gonal faces. If $p \neq 3$ a p-gon is called *incident to* a zigzag or central circuit if it share an edge with it. It is called *weakly incident* if it is not incident to it but still prevent the existence of a railroad.

Theorem 16. *For a* $(\{1,2,3\},6)$*-sphere G we have:*

(i) *If G is z-, respectively c-tight, then it has at most* $\frac{s(G)}{2}$ *zigzags, respectively central circuits.*

(ii) *If G is z-, respectively c-weakly tight then it has at most* $s(G)$ *zigzags, respectively central circuits.*

Proof. Suppose that G is c-tight, then for any central circuit C there is at least one face of of size different from 3 on each side. Since the number of sides is $s(G)$ and there are two sides per central circuit, this gives (i). The zigzag case is identical.

If G is c-weakly tight, then a p-gon for $p \neq 3$ is incident, respectively weakly incident, to at most p central circuits. Since it is weakly tight on each side of central circuits, there should be at least one incident or weakly incident central circuit. Thus the maximal number of central circuits is $s(G)$ and the proof for zigzags is identical. ■

For $(\{1,2,3\},6)$-spheres with i 1-gons ($i=0,1,2,3$), this gives the upper bounds of $(6,4,3,1)$ for tightness and $(12,9,6,3)$ for weak tightness.

Conjecture 17. (i) Any $(\{1,2,3\},6)$-sphere has the number of vertices v and the number of zigzags of the same parity.

(ii) The z-vector of a $(\{1,2,3\},6)$-sphere is the doubling of its c-vector if and only if v and the number of central circuits are of the same parity.

Theorem 18. *For a* $(\{1,3\},6)$*-sphere it holds:*

(i) *It cannot be c-, or z-tight.*

(ii) *Every central circuit correspond in a unique way to a zigzag of doubled length.*

(iii) *If it is c- or z-weakly tight, then the number of central circuits, zigzags is* 1 *or* 3.

Proof. By Theorem 4, all $(\{1,3\},6)$-spheres have symmetry C_3, C_{3v} or C_{3h}. Hence they have a 3-fold axis of rotation and hence the 1-gons belong to a single orbit under the group. The faces of a 6-regular plane graph are partitioned in two classes, say F_1, F_2, since their dual graph is bipartite. Clearly, the 1-gons are all in one partition class, say, F_1. A c-, z-circuit has two sides, and the faces in those sides all belong to the same partition class. Thus on one side of any zc-circuit, there is only 3-gons and so, (i) holds.

For a central circuit C, denote by $t_1, \ldots, t_N$ the triangles on the side of F_2. Clearly, the set of edges of triangles t_i not contained in C, defines a zigzag and (ii) holds.

If C is a central circuit in a c-weakly tight $(\{1,3\},6)$-sphere G, then on the side of F_1 there is a 1-gon and there is at most 3 central circuits. 2 is excluded by the group action. ■

Table 1. The maximal number of zigzags and central circuits for both notions of tightness and 4 types of spheres. Bold numbers are definite answer, while intervals give the possible range

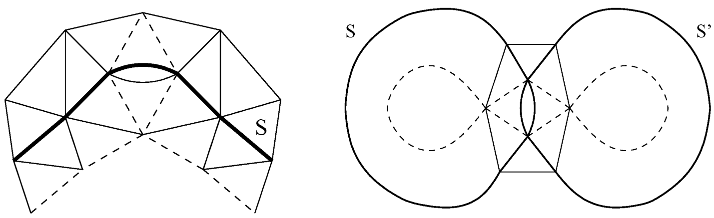
	z-tig.	z-w. tig.	c-tig.	c-w. tig.
$(\{2,3\},6)$-spheres	**6**	**9**	**6**	$[8,9]$
$(\{1,2,3\},6)$-spheres, $p_1=1$	$[3,4]$	$[5,7]$	$[3,4]$	$[5,7]$
$(\{1,2,3\},6)$-spheres, $p_1=2$	**3**	**5**	**3**	$[4,5]$
$(\{1,3\},6)$-spheres	**0**	**3**	**0**	**3**

Theorem 19. *Table 2 for the maximal number of zigzags and central circuits and both notions of weak tightness and tightness holds.*

Proof. For $(\{1,3\},6)$-spheres, Theorem 18 resolves the question. The existence of specific graphs in Figure 28 shows the lower bounds that are indicated. Theorem 16 shows the required upper bounds for z-tightness and c-tightness.

For the notion of weak tightness, we have to provide something more. Let G be a c-weakly tight $(\{1,2,3\},6)$-sphere with central circuits $C_1, \ldots, C_l$. The number of 1-gons and 2-gons is $p_1 = i$, $p_2 = 6-2i$. We obtain $2l$ sides since every central circuits has two sides. A side S is called *lonely* if it is incident or weakly incident to only one 2-gon.

If a side S is incident to exactly one 2-gon, then Figure 23(a) shows that there is a side of parallel central circuit that is weakly incident two times to this 2-gon. Moreover, if it is incident exactly two times then there is another lonely side, see Figure 23(b). A similar structure show up if a side is weakly incident to a 2-gon.



(a) One lonely side S (b) Two lonely sides S and S'

Fig. 23. Local structure around a side S incident to a 2-gon.

Call n_{1a} the number of lonely sides in the first case and n_{1b} the number of lonely sides in the second case. Call n_{1c} the number of sides incident or weakly incident to exactly one 1-gon. Also let n_2 be the number of sides incident to exactly two i-gons (identical or not). Let n_3 be the number of sides incident to at least 3 i-gons (identical or not). Obviously, $l = \frac{1}{2}(n_{1a} + n_{1b} + n_{1c} + n_2 + n_3)$.

In case (a), a lonely side S is incident to at least 3 i-gons so $n_{1a} \leq n_3$. Clearly, $n_{1c} \leq 2i$. Every 2-gon can be incident to 0, 1 or 2 lonely sides so $n_{1a} + \frac{n_{1b}}{2} \leq 6 - 2i$. By an enumeration of incidence we get

$$n_{1a} + n_{1b} + n_{1c} + 2n_2 + 3n_3 \leq s(G) = 2i + 4(6 - 2i) = 24 - 6i.$$

Denote by $\mathcal{P}_i$ the 5-dimensional polytope defined by these inequalities and $n_{1a}, \ldots, n_3 \geq 0$. We optimize the quantity l over $\mathcal{P}_i$ by using cdd [13], which uses exact arithmetic, and found the optimal value to be $9 - 2i$ for $i \leq 0 \leq 3$. The proof for zigzags is identical. ■

In the rest of this section, we give a local Euler formula for central circuits in order to enumerate the $(\{2, 3\}, 6)$-spheres which are c-weakly tight and with simple central circuits. The method for zigzags is very similar.

Let G be a $(\{2, 3\}, 6)$-sphere. Consider a patch A in G, which is bounded by t arcs (i.e., sections of central circuits) belonging to central circuits (different or coinciding).

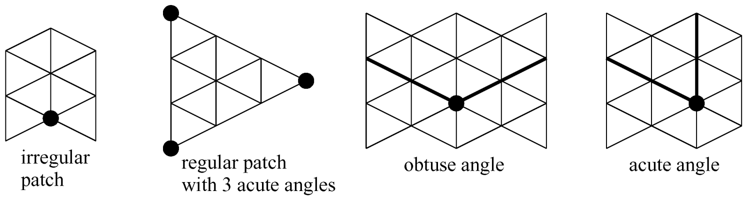


Fig. 24. Examples of patches and their angles.

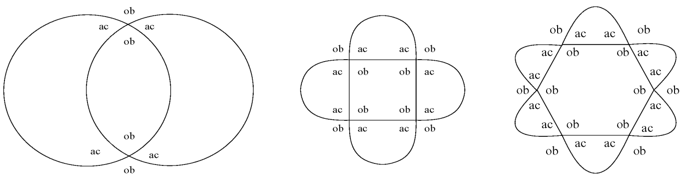
We admit also 0-gonal patch A, i.e., just the interior of a simple central circuit. Suppose that the patch A is *regular*, i.e., the continuation of any of its bounding arcs (on the central circuit, to which it belongs) lies outside of the patch (see Figure 24). Let $p_2'(A)$ be the number of 2-gonal faces in A.

There are two types of intersections of arcs on the boundary of a regular patch: either intersection in an edge of the boundary, or intersection in a vertex of the boundary. Let us call these types of intersections *obtuse* and *acute*, respectively (see Figure 24); denote by t_{ob} and t_{ac} the respective number of obtuse and acute intersections. Clearly, $t_{ob} + t_{ac} = t$, where t is the number of arcs forming the patch. The following formula can easily be verified:

(1) $$6 - t_{ob} - 2t_{ac} = 2p_2'(A).$$

Theorem 20. *The intersection of every two simple central circuits, respectively zigzags, of a* $(\{2, 3\}, 6)$*-sphere, if non-empty, has one of the following forms (and so, its size is* 2, 4 *or* 6*):*

Proof. Let us consider the central circuit case, the zigzag case being identical. Define H to be the graph, whose vertices are edges of intersection between simple central circuits C and C', with two vertices being adjacent if they are linked by a path belonging to one of C, C'. The graph H is a plane 4-regular graph and C, C' define two central circuits in H. Since C and C' are simple, the faces of H are t-gons with even t.



Applying formula (1) to a t-gonal face F of H, we obtain that the number $p_2'(F)$ of 2-gons in F satisfies $6 - t_{ob} - 2t_{ac} = 2p_2'(F)$. So, the numbers t_{ob} and t_{ac} are even, since $t = t_{ob} + t_{ac}$. Also, $6 - t_{ob} - 2t_{ac} \geq 0$. So, $t \leq 6$.

We obtain the following five possibilities for the faces of H: 2-gons with two acute angles, 2-gons with two obtuse angles, 4-gons with four obtuse angles, 4-gons with two acute and two obtuse angles, 6-gons with six obtuse angles.

Take an edge e of a 6-gon in H and consider the sequence (possibly, empty) of adjacent 4-gons of H emanating from this edge. This sequence will stop at a 2-gon or a 6-gon; the case-by-case analysis of angles yields that this sequence has to stop at a 2-gon (see Figure 25(a)).

Take an edge of a 2-gon in H and consider the same construction. If the angles are both obtuse, then the construction is identical and the sequence will terminate at a 2-gon or a 6-gon. If the angles are both acute, then cases b), c) of Figure 25 are possible.

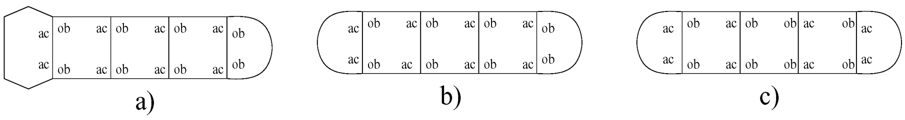


Fig. 25. Three cases for sequence of 4-gons.

In the first case, all 4-gons contain two obtuse angles and two acute angles; so, the sequence of 4-gons finishes with an edge of two obtuse angles. In the second case, there is a 4-gon, whose angles are all obtuse; this 4-gon is unique in the sequence and its position is arbitrary. Every pair of opposite edges of a 4-gon belongs to a sequence of 4-gons considered above. So, all angles of a 4-gon are the same, i.e., obtuse. This fact restricts the possibilities of intersections to the three cases of the theorem. ∎

The proof of this theorem is similar to the one of Theorem 6.4 given in [5].

Theorem 21. *The only weakly tight* $(\{2,3\},6)$*-spheres, having only simple zigzags, respectively simple central circuits, are the ones of Figure 26 and 27.*

Proof. Let us consider first the central circuit case. By Theorem 20, every two simple central circuits intersect in at most six vertices. If a $(\{2,3\},6)$-sphere has t central circuits, this gives an upper bound of $6\frac{t(t-1)}{2}$ on the number of vertices of intersection. Since any vertex can be the intersection of only 3 central circuits we get the upper bound of $t(t-1)$ on the number of vertices. If one uses the upper bound of Table on t for weakly tight $(\{2,3\},6)$-spheres, then one gets $t \leq 9$ and the upper bound 72 on the number of vertices, which is too large for the enumeration done in Table 1. If one looks at the proof of Theorem 19, then one sees that a lonely side implies a self-intersection of a parallel central circuit. So, there is no lonely sides in $(\{2,3\},6)$-spheres with only simple central circuits. This gives the upper bound $t \leq 6$ on the number of central circuits and then 30 on the number of vertices.

For zigzags we have the upper bound $3t(t-1)$ on the number of edges and this gives the same upper bound of $t(t-1)$ on the number of vertices. The enumeration result shown in the Figures follow from the determination results of Section 2. ■

An interesting problem is to determine all $(\{2,3\},6)$-spheres with simple zigzags and/or central circuits. Theorem 14 implies that the number of such spheres is infinite.

A $(\{1,2,3\},6)$-sphere is called z- or *c-knotted* if it has only one zigzag of central circuit. Conjecture 17 implies that a z-knotted sphere has odd v.

Conjecture 22. (i) The possible symmetries of a z- or c-knotted $(\{1,2,3\},6)$-sphere, except tripled triangle (only z-knotted) D_{3h} and Trifolium C_{3v}, are: C_3 if $p_1 = 3$; C_1, C_2 if $p_1 = 2$; C_1 if $p_1 = 1$ and C_1, C_2, C_3, D_2, D_3 if $p_1 = 0$.

(ii) The $(\{2,3\},6)$-spheres with only simple central circuits have symmetry T_d, T_h, D_{6h}, D_{3d}, D_{2d}, D_{2h}, D_3, C_{2h} and C_{3v}.

(iii) A $(\{1,2,3\},6)$-sphere of symmetry D_{6h}, T_h, T_d have only simple central circuits and zigzags.

(iv) The $(\{2,3\},6)$-spheres of symmetry T_d have $v = 4x^2$ vertices, c-vector $(3x)^{4x}$ and z-vector $(6x)^{4x}$.

(v) The $(\{2,3\},6)$-spheres of symmetry T_h have $12x^2$ vertices, c-vector $(6x)^{6x}$ and z-vector $(12x)^{6x}$.

(vi) The symmetries occuring for odd v (i.e. not only for even v), are: for $p_1 = 0$, C_1, C_2, C_{2v}, C_3, C_{3h}, C_{3v}, C_s, D_{2h}, D_3 or D_{3h}; for $p_1 = 1$, C_1 or C_s; for $p_1 = 2$, C_1, C_2, C_s, C_{2v} or C_{2h}; for $p_1 = 3$, C_3, C_{3v} or C_{3h}.

(vii) A c-knotted sphere is also z-knotted if and only if v is odd.

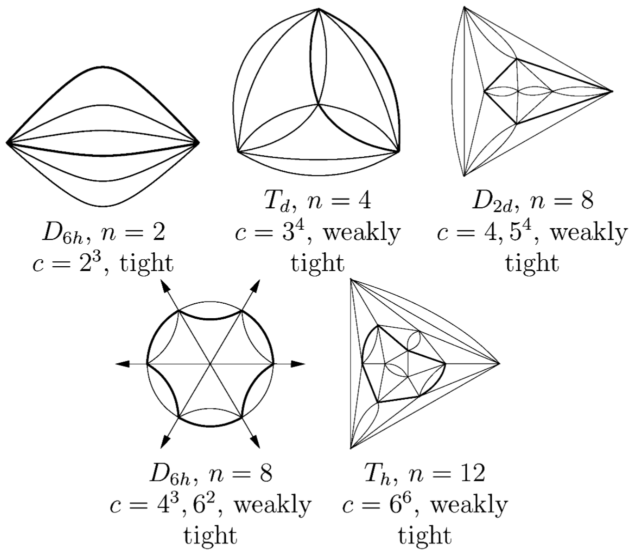


Fig. 26. The c-weakly tight $(\{2,3\},6)$-spheres with simple central circuits.

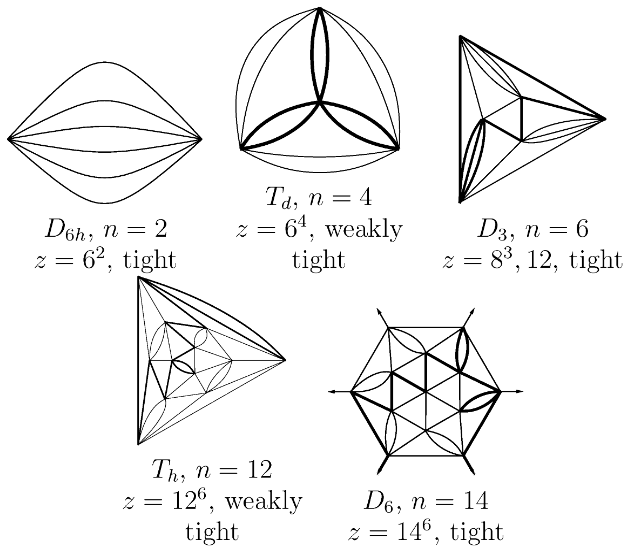


Fig. 27. The z-weakly tight $(\{2,3\},6)$-spheres with simple zigzags.

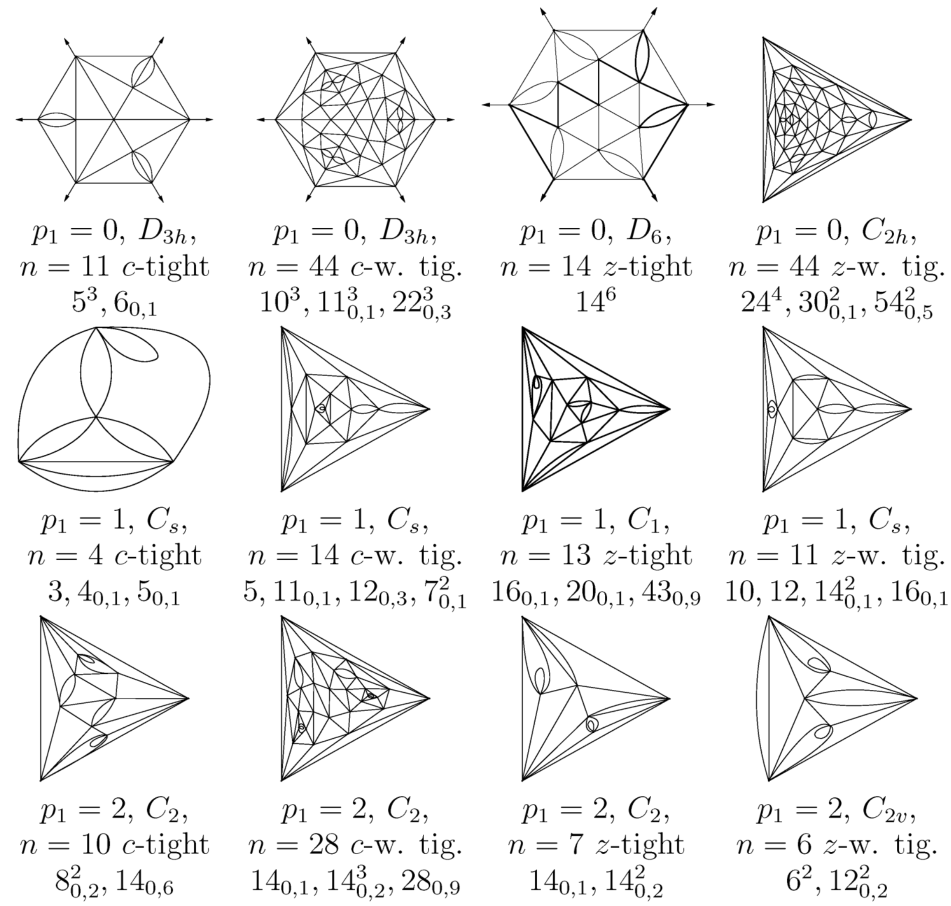
Conjecture 23. Let $f_i(v)$ denote the maximal number of central circuits in a $(\{1,2,3\},6)$-sphere with i 1-gons and v vertices. We conjecture:

(i) $f_2(v) = v + 1$. It is realized exactly by the series (one for each $v \geq 1$) having symmetry C_{2h} and $c = 1^v, (2v)_{0,v}$.

(ii) $f_1(v) = \frac{v-1}{2} + 1$, $\frac{v-1}{2} + 2$ for $v \equiv 3, 1 \pmod 4$. It is realized exactly by the series (one for each odd $v \geq 3$, all of symmetry C_s) with $c = 2^{(v-1)/2}, (2v+1)_{0,v+2}$

if $v \equiv 3 \pmod 4$ and $2^{(v-1)/2}, v_{0,\frac{v-1}{4}}, (v+1)_{0,\frac{v+1}{4}}$ if $v \equiv 1 \pmod 4$.

For even v, $f_1(v) = \lfloor \frac{v-1}{3} \rfloor + 2$. It is realized for $v \geq 4$ by series of symmetry C_s (two spheres for $v \equiv 2 \pmod 6$ and unique for other even v) with $c = 3^{\lfloor \frac{v-1}{3} \rfloor}$, $(\frac{v}{2}+2+3\lfloor \frac{v}{18} \rfloor)_{0,2\lfloor \frac{v}{18} \rfloor+1}, (v+1+3z(v))_{0,4z(v)+1}$, where $z(v) = 2\lfloor \frac{v}{18} \rfloor+1$ if $v \equiv 6, 8, 10 \pmod{18}$ and $z(v) = 2\lfloor \frac{v+6}{18} \rfloor$ if v is other even number.



$p_1 = 0$, D_{3h}, $n = 11$ c-tight, $5^3, 6_{0,1}$

$p_1 = 0$, D_{3h}, $n = 44$ c-w. tig. $10^3, 11^3_{0,1}, 22^3_{0,3}$

$p_1 = 0$, D_6, $n = 14$ z-tight, 14^6

$p_1 = 0$, C_{2h}, $n = 44$ z-w. tig. $24^4, 30^2_{0,1}, 54^2_{0,5}$

$p_1 = 1$, C_s, $n = 4$ c-tight, $3, 4_{0,1}, 5_{0,1}$

$p_1 = 1$, C_s, $n = 14$ c-w. tig. $5, 11_{0,1}, 12_{0,3}, 7^2_{0,1}$

$p_1 = 1$, C_1, $n = 13$ z-tight, $16_{0,1}, 20_{0,1}, 43_{0,9}$

$p_1 = 1$, C_s, $n = 11$ z-w. tig. $10, 12, 14^2_{0,1}, 16_{0,1}$

$p_1 = 2$, C_2, $n = 10$ c-tight, $8^2_{0,2}, 14_{0,6}$

$p_1 = 2$, C_2, $n = 28$ c-w. tig. $14_{0,1}, 14^3_{0,2}, 28_{0,9}$

$p_1 = 2$, C_2, $n = 7$ z-tight, $14_{0,1}, 14^2_{0,2}$

$p_1 = 2$, C_{2v}, $n = 6$ z-w. tig. $6^2, 12^2_{0,2}$

Fig. 28. The smallest weakly tight and tight $(\{1,2,3\},6)$-spheres with the maximal known number of zigzags and central circuits.

(iii) $f_0(v) = \frac{v}{2}+1, \frac{v}{2}+2$ for $v \equiv 0, 2 \pmod 4$. It is realized exactly by the series (one for each $v \geq 6$) of symmetry D_{2d} with $c = 2^{\frac{v}{2}}, 2v_{0,v}$ if $v \equiv 0 \pmod 4$ and of symmetry D_{2h} with $c = 2^{\frac{v}{2}}, (v_{0,\frac{v-2}{4}})^2$ if $v \equiv 2 \pmod 4$.

(iv) For odd v, f_0 is $\lfloor \frac{v}{3} \rfloor+3$ if $v \equiv 2, 4, 6, \pmod 9$ and $\lfloor \frac{v}{3} \rfloor+1$, otherwise. Define t_v by $\frac{v-t_v}{3} = \lfloor \frac{v}{3} \rfloor$. $f_0(v)$ is realized by the series of symmetry C_{3v} if $v \equiv 1 \pmod 3$

and D_{3h}, otherwise. c-vector is $3^{\lfloor \frac{v}{3} \rfloor}, (2\lfloor \frac{v}{3} \rfloor + t_v)^3_{0, \lfloor \frac{v-2t_v}{9} \rfloor}$ if $v \equiv 2,4,6 \pmod 9$ and $3^{\lfloor \frac{v}{3} \rfloor}, (2v + t_v)_{0, v+2t_v}$, otherwise.

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Michel Deza
École Normale Supérieure
Paris
E-mail: Michel.Deza@ens.fr

Mathieu Dutour Sikirić
Rudjer Bosković Institute
Bijenicka 54, 10000 Zagreb
Croatia
E-mail: mdsikir@irb.hr