TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 2, pp. 713-732, April 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

TWO-WEIGHT NORM INEQUALITIES FOR CERTAIN SINGULAR INTEGRALS

R. A. Bandaliev* and K. K. Omarova

Abstract. In this paper we prove the boundedness of certain convolution operator in a weighted Lebesgue space with kernel satisfying the generalized Hörmander's condition. The sufficient conditions for the pair of weights ensuring the validity of two-weight inequalities of a strong type and of a weak type for singular integral with kernel satisfying the generalized Hörmander's condition are found.

0. Introduction

Let $\mathbb{R}^n$ be n-dimensional Euclidean space of points $x=(x_1,\dots,x_n)$, where $n\in\mathbb{N}$. Suppose that ω is a non-negative, Lebesgue measurable and real function defined on $\mathbb{R}^n$, i.e., ω is a weight function defined on $\mathbb{R}^n$. By $L_{p,\,\omega}(\mathbb{R}^n)$ we denote the weighted Lebesgue space of measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L_{p,\,\omega}(\mathbb{R}^n)}=\left(\int\limits_{\mathbb{R}^n}|f(x)|^p\,\omega(x)\,dx\right)^{1/p}<\infty,\ \ 1\le p<\infty.$$

In the case $p=\infty$, the norm on the space $L_{\infty,\omega}(\mathbb{R}^n)$ is defined as

$$\|f\|_{L_{\infty,\omega}(\mathbb{R}^n)}=\|f\|_\infty=ess\sup_{x\in\mathbb{R}^n}|f(x)|.$$

For $\omega=1$ we obtain the nonweighted L_p spaces, i.e., $\|f\|_{L_{p,\,1}(\mathbb{R}^n)}=\|f\|_{L_p(\mathbb{R}^n)}$.

Consider the linear operator T defined for $f\in L_2(\mathbb{R}^n)\cap L_p(\mathbb{R}^n)$ by $\widehat{Tf}(x)=\chi_{[-1,1]}(x)\widehat{f}(x)$, where $\chi_{[-1,1]}$ denotes the characteristic function of the segment

Received September 28, 2010, accepted January 15, 2011.
Communicated by Aram Arutyunov.
2010 *Mathematics Subject Classification*: 46B50, 26D15, 47B38.
Key words and phrases: Weighted Lebesgue space, Singular integral, Kernel, Generalized Hörmander's condition, Boundedness.
*The author is supported by the Science Development Foundation under the President of the Republic of Azerbaijan Project No./01/023.

$[-1,1]$ and $\widehat{f}$ denotes the Fourier transformation of a function f. In fact, T can be constructed from multiplication operators and the Hilbert transform, so the boundedness of T on $L_p(\mathbb{R}^n)$ is just a consequen-ce of the L_p boundedness of the Hilbert transform. It is curious that although the L_p boundedness of T follows from results on singular integrals, it does not follow directly, since the kernel of T has a derivative which does not decay quickly enough at infinity to apply the usual theory. For example, if the kernel of T has the form $\frac{\sin x}{x}$ and e.t.c. On the other hand, singular integrals whose kernels do not satisfy Hörmander's condition have been widely considered (for example, oscillatory and other singular integral) (see [4]). For classical singular integral operators the boundedness properties were proved by A.P.Calderon and A.Zygmund [2] (see also [3]). For power weights the boundedness of classical singular integral operators in weighted Lebesgue spaces was proved by E.Stein in [13].

1. Preliminaries

Definition 1. [6]. A positive measurable and locally integrable function g is said to satisfy the reverse L_∞ condition RL_∞ or $g \in RL_\infty(\mathbb{R}^n)$ if

$$0 < \sup_{x\in B} g(x) \le C\,\frac{1}{|B|}\int\limits_B g(x)\,dx,$$

where B is an arbitrary ball centered at the origin and $C>0$ is a constant independent of B.

Let $K \in L_2(\mathbb{R}^n)$ be a function satisfying the following conditions:

(a) $\left\|\widehat{K}\right\|_\infty \le C$;

(b) $|K(x)| \le \frac{C}{|x|^n}$;

(c) there exist the functions $A_1,\dots,A_m \in L_1^{loc}(\mathbb{R}^n \setminus \{0\})$ and $\Phi = \{\varphi_1,\dots,\varphi_m\}$ such that $\varphi_i \in L_\infty(\mathbb{R}^n)$ and $|det\,[\varphi_j(y_i)]|^2 \in RL_\infty(\mathbb{R}^{nm})$, $y_i \in \mathbb{R}^n$, $i,j = 1,\dots,m$;

(d) for a fixed $\gamma > 0$ and for any $|x| > 2|y| > 0$ the inequality

$$\left|K(x-y) - \sum_{i=1}^m A_i(x)\,\varphi_i(y)\right| \le C\,\frac{|y|^\gamma}{|x-y|^{n+\gamma}} \tag{1.1}$$

is valid, where $C>0$ is a constant independent of x and y. In general, the functions $A_i,\ \varphi_i$ $(i=1,\dots,m)$ are complex-valued.

Remark 1. Let K be a function satisfying condition (1.1). Then the inequality

$$\int\limits_{|x|>2|y|} \left| K(x-y) - \sum_{i=1}^{m} A_i(x)\,\varphi_i(y) \right| dx \le C \tag{1.2}$$

is valid.

Indeed, integrating both sides of inequality (1.1) with respect to the set $|x| > 2|y|$, using the inequality $|x-y| \ge |x| - |y| \ge |x| - \frac{|x|}{2} = \frac{|x|}{2}$ and passing to spherical coordinates in $\mathbb{R}^n$, we have

$$\int\limits_{|x|>2|y|} \left| K(x-y) - \sum_{i=1}^{m} A_i(x)\,\varphi_i(y) \right| dx \le C \int\limits_{|x|>2|y|} \frac{|y|^\gamma}{|x-y|^{n+\gamma}}$$

$$\le 2^{n+\gamma}\, C\, |y|^\gamma \int\limits_{|x|>2|y|} \frac{dx}{|x|^{n+\gamma}} = \frac{2^{n+\gamma}\,\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}\, C\, |y|^\gamma \int\limits_{2|y|}^{\infty} \frac{dt}{t^{\gamma+1}} = \frac{2^{n}\,\pi^{\frac{n}{2}}}{\gamma\,\Gamma\left(\frac{n}{2}+1\right)}\, C = C_1.$$

Therefore condition (1.2) is a weaker condition on the function K than condition (1.1).

Definition 2. [12] It is said that locally integrable weight function ν belongs to $A_p(\mathbb{R}^n)$ if

$$\sup_B \left(\frac{1}{|B|} \int\limits_B \nu(x)\,dx \right) \left(\frac{1}{|B|} \int\limits_B \nu^{1-p'}(x)\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, $1 < p < \infty$ and $p' = \frac{p}{p-1}$. Also, $\nu \in A_1(\mathbb{R}^n)$ if there exists a positive constant C such that for any arbitrary ball $B \subset \mathbb{R}^n$ the inequality

$$\frac{1}{|B|} \int\limits_B \nu(x)\,dx \le C\, ess \inf_{x \in B} \nu(x)$$

holds.

Remark 2. It is clear that from condition RL_∞ implies the well known reverse Hӧlder inequality

$$\left(\frac{1}{|B|} \int\limits_B [g(x)]^{1+\varepsilon}\,dx \right)^{\frac{1}{1+\varepsilon}} \le C \left(\frac{1}{|B|} \int\limits_B g(x)\,dx \right),$$

where $\varepsilon > 0$. It is well known that the reverse Hӧlder condition characterizes the condition $A_p(\mathbb{R}^n)$ (see [5]).

For $1 < p < \infty$ and $f \in L_p(\mathbb{R}^n)$ we put

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)\, f(y)\, dy. \tag{1.3}$$

We will also need the following theorems.

Theorem 1. [14]. *Let $1 < r < \infty$ and $\omega \in A_r(\mathbb{R}^n)$ be a weight function on $\mathbb{R}^n$. Suppose that the kernel of the convolution operator (1.3) satisfies the conditions (a)-(d). Then the following inequality*

$$\|Tf\|_{L_{r,\,\omega}(\mathbb{R}^n)} \le C\, \|f\|_{L_{r,\,\omega}(\mathbb{R}^n)}$$

holds, where the positive constant C is independent of f.

For $r = 1$ there exists a positive constant C such that for any $f \in L_{1,\omega}(\mathbb{R}^n)$ and $\lambda > 0$ the inequality

$$\int_{\{x\in\mathbb{R}^n:\ |Tf(x)|>\lambda\}} \omega(x)\, dx \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, \omega(x)\, dx$$

holds.

Remark 3. Let the kernel of the convolution operator (1.3) satisfy the conditions (a), (c) and (1.2). Then for $\omega = 1$ Theorem 1 was proved in [5].

Example 1.1. If $m = 1$, $A_1(x) = K(x)$ and $\varphi_1(x) \equiv 1$, we get the Hörmander's version of the Calderón-Zygmund Theorem (see [8]).

Example 1.2. Let $m = 2$, $K(x) = \dfrac{\sin x}{x}$, $x \in \mathbb{R} \setminus \{0\}$, $A_1(x) = \dfrac{e^{ix}}{2i\,x}$ $A_2(x) = -\frac{e^{ix}}{2i\,x}$, $\varphi_1(y) = e^{-iy}$ and $\varphi_2(y) = e^{iy}$. Then the conditions (a)-(d) hold.

Theorem 2. [11]. *Let $1 < q < p < \infty$, $u(t)$ and $v(t)$ be positive functions on $(0, \infty)$. Suppose that $F : (0, \infty) \mapsto \mathbb{R}$ be a Lebesgue measurable function.*

1. For the validity of the inequality

$$\left(\int_0^\infty u(t) \left| \int_0^t F(\tau)\, d\tau \right|^q dt \right)^{1/q} \le C_1 \left(\int_0^\infty |F(t)|^p\, v(t)\, dt \right)^{1/p}$$

it is necessary and sufficient that

$$\int_0^\infty \left[\left(\int_t^\infty u(\tau)\, d\tau \right) \left(\int_0^t v^{1-p'}(\tau)\, d\tau \right)^{q-1} \right]^{\frac{p}{p-q}} v^{1-p'}(t)\, dt < \infty,$$

where $C_1 > 0$ is independent of F.

2. *For the validity of the inequality*

$$\left(\int_0^\infty u(t)\left|\int_t^\infty F(\tau)\,d\tau\right|^q dt\right)^{1/q} \le C_2\left(\int_0^\infty |F(t)|^p\,v(t)\,dt\right)^{1/p}$$

it is necessary and sufficient that

$$\int_0^\infty \left[\left(\int_0^t u(\tau)\,d\tau\right)\left(\int_t^\infty v^{1-p'}(\tau)\,d\tau\right)^{q-1}\right]^{\frac{p}{p-q}} v^{1-p'}(t)\,dt < \infty,$$

where $C_2 > 0$ is independent of F.

For $q = 1$ the following Lemma is valid.

Lemma 1. [10]. *Let $p > 1$ and $u(t)$ and $v(t)$ be positive functions on $(0, \infty)$.*

1. *If the pair (u, v) satisfies the condition*

$$\int_0^\infty \left(\int_t^\infty u(\tau)\,d\tau\right)^{p'} v^{1-p'}(t)\,dt < \infty,$$

then there exists a positive constant C_1 such that for an arbitrary function $F : (0, \infty) \mapsto \mathbb{R}$ the inequality

$$\int_0^\infty u(t)\left|\int_0^t F(\tau)\,d\tau\right| dt \le C_1\left(\int_0^\infty |F(t)|^p\,v(t)\,dt\right)^{1/p}$$

holds.

2. *If the pair (u, v) satisfies the condition*

$$\int_0^\infty \left(\int_0^t u(\tau)\,d\tau\right)^{p'} v^{1-p'}(t)\,dt < \infty,$$

then there exists a positive constant C_2 such that for an arbitrary function $F : (0, \infty) \mapsto \mathbb{R}$ the inequality

$$\int_0^\infty u(t)\left|\int_t^\infty F(\tau)\,d\tau\right| dt \le C_2\left(\int_0^\infty |F(t)|^p\,v(t)\,dt\right)^{1/p}$$

holds.

Theorem 3. [9]. *Let* $1 \le q < p < \infty$, $u(x)$ *and* $v(x)$ *be weight functions on* $\mathbb{R}^n$. *Then the condition*

$$\int_{\mathbb{R}^n} [u(x)]^{\frac{p}{p-q}} \, [v(x)]^{-\frac{q}{p-q}} \, dx < \infty$$

is necessary and sufficient for the validity of the inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \, u(x) \, dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p \, v(x) \, dx \right)^{1/p},$$

where $C > 0$ *is independent of* f.

Lemma 2. *Let* $1 \le q < p < \infty$ *and* $\alpha \ge 1$. *Let* u *and* u_1 *be positive increasing functions on* $(0, \infty)$, ψ *be a positive function on* $\mathbb{R}^n$, $\omega = u\psi$ *and* $\omega_1 = u_1\psi$. *If the weight pair* (ω_1, ω) *satisfies the condition*

$$\int_{\mathbb{R}^n} [u_1(\alpha|x|)]^{\frac{p}{p-q}} \, [u(|x|)]^{-\frac{q}{p-q}} \, \psi(x) \, dx < \infty, \tag{1.4}$$

then there exists a constant $C > 0$ *such that for any* $f \in L_{p,\omega}(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \, \omega_1(x) \, dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p \, \omega(x) \, dx \right)^{1/p}.$$

Proof. We have

$$\int_{\mathbb{R}^n} [u_1(\alpha|x|)]^{\frac{p}{p-q}} \, [u(|x|)]^{-\frac{q}{p-q}} \, \psi(x) \, dx \ge \int_{\mathbb{R}^n} [u_1(|x|)]^{\frac{p}{p-q}} \, [u(|x|)]^{-\frac{q}{p-q}} \, \psi(x) \, dx =$$

$$= \int_{\mathbb{R}^n} [u_1(|x|) \, \psi(x)]^{\frac{p}{p-q}} \, [u(|x|) \, \psi(x)]^{-\frac{q}{p-q}} \, dx = \int_{\mathbb{R}^n} [\omega_1(x)]^{\frac{p}{p-q}} \, [\omega(x)]^{-\frac{q}{p-q}} \, dx.$$

By Theorem 3 the proof of Lemma 2 is completed.

The following Lemma is proved analogously.

Lemma 3. *Let* $1 \le q < p < \infty$ *and* $\alpha \ge 1$. *Let* u *and* u_1 *be positive decreasing functions on* $(0, \infty)$, ψ *be a positive function on* $\mathbb{R}^n$, $\omega = u\psi$ *and* $\omega_1 = u_1\psi$. *If the weight pair* (ω_1, ω) *satisfies the condition*

$$\int_{\mathbb{R}^n} \left[u_1\left(\frac{|x|}{\alpha} \right) \right]^{\frac{p}{p-q}} \, [u(|x|)]^{-\frac{q}{p-q}} \, \psi(x) \, dx < \infty,$$

then there exists a constant $C > 0$ *such that for any* $f \in L_{p,\omega}(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \, \omega_1(x) \, dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p \, \omega(x) \, dx \right)^{1/p}.$$

2. Main Results

Theorem 4. *Let* $1 < q < p < \infty$ *and the kernel of convolution operator* (1.3) *satisfy the conditions* (a)-(d). *Let* u *and* u_1 *be positive increasing functions on* $(0, \infty)$, $\varphi \in A_q(\mathbb{R}^n)$ *is a radial function,* $\omega = u\varphi$ *and* $\omega_1 = u_1\varphi$. *If the weight pair* (ω_1, ω) *satisfies the condition (1.4) and*

$$\int_0^\infty \left[\left(\int_t^\infty \omega_1(\tau) \, \tau^{n-nq-1} \, d\tau \right) \left(\int_0^t \omega^{1-p'}(\tau) \, \tau^{n-1} \, d\tau \right)^{q-1} \right]^{\frac{p}{p-q}} \omega^{1-p'}(t) \, t^{n-1} dt < \infty,$$

then there exists a constant $C > 0$ *such that for any* $f \in L_{p,\omega}(\mathbb{R}^n)$ *the inequality*

$$\left(\int_{\mathbb{R}^n} |Tf(x)|^q \omega_1(|x|) \, dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p \, \omega(|x|) \, dx \right)^{1/p} \tag{2.1}$$

holds.

Proof. Without loss of generality we may assume that the function u_1 has the form

$$u_1(t) = u_1(0) + \int_0^t \psi(\tau) \, d\tau,$$

where $u_1(0) = \lim_{t \to +0} u(t)$ and ψ is a positive function on $(0, \infty)$. Indeed, for increasing functions on $(0, \infty)$ there exists a sequence of absolutely continuous functions $\varphi_n(t)$ such that $\lim_{n \to \infty} \varphi_n(t) = u_1(t)$, $0 \le \varphi_n(t) \le u_1(t)$ a.e. $t > 0$ and $\varphi_n(0) = u_1(0)$. Furthermore the functions $\varphi_n(t)$ are increasing, and besides

$$\varphi_n(t) = \varphi_n(0) + \int_0^t \varphi_n'(\tau) \, d\tau,$$

where $\lim_{n \to \infty} \varphi_n'(t) = \psi(t)$. Hence, using Fatou's theorem, we obtain estimate (2.1) for any increasing function on $(0, \infty)$.

Let us estimate the left-hand side of inequality (2.1). We have

$$\left(\int_{\mathbb{R}^n} |Tf(x)|^q\,\omega_1(|x|)\,dx\right)^{1/q} = \left(\int_{\mathbb{R}^n} |Tf(x)|^q \left(u_1(0) + \int_0^{|x|} \psi(t)dt\right) \varphi(|x|)\,dt\right)^{1/q}.$$

If $u_1(0) = 0$, then $\left(\int_{\mathbb{R}^n} |Tf(x)|^q\,\omega_1(|x|)\,dx\right)^{1/q} = \left(\int_{\mathbb{R}^n} |Tf(x)|^q \varphi(|x|) \left(\int_0^{|x|} \psi(t)dt\right) dx\right)^{1/q}$.
However, if $u_1(0) > 0$, then

$$\left(\int_{\mathbb{R}^n} |Tf(x)|^q\,\omega_1(|x|)\,dx\right)^{1/q} \le \left(\int_{\mathbb{R}^n} |Tf(x)|^q\,\varphi(|x|)\,u_1(0)dx\right)^{1/q}$$
$$+\left(\int_{\mathbb{R}^n} |Tf(x)|^q\,\varphi(|x|)\left(\int_0^{|x|} \psi(t)dt\right) dx\right)^{1/q} = E_1 + E_2.$$

First we estimate E_1. By Theorem 1 and by Lemma 2, we get

$$\begin{aligned} E_1 &= \left(\int_{\mathbb{R}^n} |Tf(x)|^q\,\varphi(|x|)\,u_1(0)dx\right)^{1/q} = (u_1(0))^{1/q} \left(\int_{\mathbb{R}^n} |Tf(x)|^q\,\varphi(|x|)\,dx\right)^{1/q} \\ &\le (u_1(0))^{1/q} \left(\int_{\mathbb{R}^n} |f(x)|^q\,\varphi(|x|)\,dx\right)^{1/q} \le \left(\int_{\mathbb{R}^n} |f(x)|^q\,\varphi(|x|)\,u_1(|x|)\,dx\right)^{1/q} \\ &= \left(\int_{\mathbb{R}^n} |f(x)|^q\,\omega_1(|x|)\,dx\right)^{1/q} \le C_1 \left(\int_{\mathbb{R}^n} |f(x)|^p\,\omega(|x|)\,dx\right)^{1/p}. \end{aligned}$$

Let us estimate the integral E_2. We have

$$\begin{aligned} E_2 &= \left(\int_{\mathbb{R}^n} |Tf(x)|^q\,\varphi(|x|)\left(\int_0^{|x|} \psi(t)dt\right) dx\right)^{1/q} \\ &= \left(\int_{\mathbb{R}^n} |Tf(x)|^q\,\varphi(|x|)\left(\int_0^{\infty} \psi(t)\chi_{\{|x|>t\}}(x)\,dt\right) dx\right)^{1/q} \\ &= \left(\int_0^{\infty} \psi(t)\left(\int_{|x|>t} |Tf(x)|^q\,\varphi(|x|)dx\right) dt\right)^{1/q} \\ &= \left(\int_0^{\infty} \psi(t)\left(\int_{|x|>t} \left|\int_{\mathbb{R}^n} K(x-y)\,f(y)\,dy\right|^q \varphi(|x|)dx\right) dt\right)^{1/q} \end{aligned}$$

$$\leq 2^{1/q'} \left(\int_0^\infty \psi(t) \left(\int_{|x|>t} \left| \int_{|y|>t/2} K(x-y)\, f(y)\, dy \right|^q \varphi(|x|)dx \right) dt \right)^{1/q}$$
$$+2^{1/q'} \left(\int_0^\infty \psi(t) \left(\int_{|x|>t} \left| \int_{|y|\leq t/2} K(x-y)\, f(y)\, dy \right|^q \varphi(|x|)dx \right) dt \right)^{1/q}$$
$$= E_{21} + E_{22}.$$

We estimate E_{21}. Using Theorem 1, we have

$$E_{21} = 2^{1/q'} \left(\int_0^\infty \psi(t) \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} K(x-y)\, f(y)\, \chi_{\{z:\,|z|>t/2\}}(y)\, dy \right|^q \varphi(|x|) \cdot \chi_{\{|x|>t\}}(x)dx \right) dt \right)^{1/q}$$
$$\leq 2^{1/q'} \left(\int_0^\infty \psi(t) \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} K(x-y)\, f(y)\, \chi_{\{z:\,|z|>t/2\}}(y)\, dy \right|^q \varphi(|x|)dx \right) dt \right)^{1/q}$$
$$\leq C_2 \left(\int_0^\infty \psi(t) \left(\int_{\mathbb{R}^n} |f(x)|^q\, \chi_{\{z:\,|z|>t/2\}}(x)\, \varphi(|x|)\, dx \right) dt \right)^{1/q}$$
$$= C_2 \left(\int_{\mathbb{R}^n} |f(x)|^q\, \varphi(|x|) \left(\int_0^{2|x|} \psi(t)\ dt \right) dx \right)^{1/q}$$
$$\leq C_2 \left(\int_{\mathbb{R}^n} |f(x)|^q\, \varphi(|x|)\, u_1(2|x|)\, dx \right)^{1/q}$$
$$= C_2 \left(\int_{\mathbb{R}^n} [|f(x)|^p\, \omega(|x|)]^{\frac{q}{p}}\ u_1(2|x|)\ [\omega(|x|)]^{-\frac{q}{p}}\ \varphi(|x|)\, dx \right)^{1/q}$$
$$C_2 \left(\int_{\mathbb{R}^n} [|f(x)|^p\, \omega(|x|)]^{\frac{q}{p}}\ u_1(2|x|)\ [u(|x|)]^{-\frac{q}{p}}\ [\varphi(|x|)]^{\frac{p-q}{p}}\ dx \right)^{1/q}.$$

Now applying the Hölder's inequality with exponents $\dfrac{p}{q}$ and $\dfrac{p}{p-q}$ and using the condition (1.4) (for $\alpha = 2$), we obtain

$$\left(\int_{\mathbb{R}^n} [|f(x)|^p\, \omega(|x|)]^{\frac{q}{p}}\ u_1(2|x|)\ [u(|x|)]^{-\frac{q}{p}}\ [\varphi(|x|)]^{\frac{p-q}{p}}\ dx \right)^{1/q}$$

$$\leq \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \,\omega(|x|)\,dx\right)^{1/p} \left(\int\limits_{\mathbb{R}^n} [u_1(2|x|)]^{\frac{p}{p-q}} \; [u(|x|)]^{-\frac{q}{p-q}} \; \varphi(|x|)\,dx\right)^{\frac{p-q}{pq}}$$
$$\leq C_3 \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \,\omega(|x|)\,dx\right)^{1/p}.$$

Now we estimate E_{22}. Note that if $|x| > t$ and $|y| \leq \dfrac{t}{2}$, then $|x-y| \geq |x| - |y|$ $\geq |x| - \frac{|x|}{2} = \frac{|x|}{2}$. We get

$$E_{22} = 2^{1/q'} \left(\int\limits_0^\infty \psi(t) \left(\int\limits_{|x|>t} \left|\int\limits_{|y|\leq t/2} K(x-y)\, f(y)\,dy\right|^q \varphi(|x|)dx\right) dt\right)^{1/q}$$
$$\leq C_4 \left(\int\limits_0^\infty \psi(t) \left(\int\limits_{|x|>t} \left(\int\limits_{|y|\leq t/2} \frac{|f(y)|}{|x-y|^n}\,dy\right)^q \varphi(|x|)dx\right) dt\right)^{1/q}$$
$$\leq C_5 \left(\int\limits_0^\infty \psi(t) \left(\int\limits_{|x|>t} \frac{\varphi(|x|)}{|x|^{nq}}\,dx\right) \left(\int\limits_{|y|\leq t/2} |f(y)|\,dy\right)^q dt\right)^{1/q}$$
$$= 2\,C_5 \left(\int\limits_0^\infty \psi(2s) \left(\int\limits_{|x|>2s} \frac{\varphi(|x|)}{|x|^{nq}}\,dx\right) \left(\int\limits_{|y|\leq s} |f(y)|\,dy\right)^q ds\right)^{1/q}$$
$$= C_6 \left(\int\limits_0^\infty \psi(2s) \left(\int\limits_{2s}^\infty \varphi(\tau)\,\tau^{n-nq-1}\,d\tau\right)\right.$$
$$\left.\left(\int\limits_0^s t^{n-1} \left[\int\limits_{|\overline{y}|=1} |f(t\overline{y})|\,d\sigma(\overline{y})\right] dt\right)^q ds\right)^{1/q}.$$

Besides, we have the following estimates:

$$\int\limits_t^\infty \psi(2s) \left(\int\limits_{2s}^\infty \varphi(r)\,r^{n-nq-1}\,dr\right) ds = \frac{1}{2}\int\limits_{2\,t}^\infty \psi(s) \left(\int\limits_s^\infty \varphi(r)\,r^{n-nq-1}\,dr\right) ds$$
$$= \frac{1}{2}\int\limits_{2\,t}^\infty \varphi(r)\,r^{n-nq-1} \left(\int\limits_{2\,t}^r \psi(s)\,ds\right) dr \leq \int\limits_{2\,t}^\infty \varphi(r)\,r^{n-nq-1}\,u_1(r)\,dr$$

$$= \int_{2t}^{\infty} \omega_1(r)\, r^{n-nq-1}\, dr \le \int_{t}^{\infty} \omega_1(r)\, r^{n-nq-1}\, dr.$$

Therefore we have

$$\int_0^\infty \left[\left(\int_t^\infty \psi(2s) \left(\int_{2s}^\infty \varphi(r)\, r^{n-nq-1}\, dr \right) ds \right) \left(\int_t^\infty \omega^{1-p'}(\tau)\, \tau^{n-1}\, d\tau \right)^{q-1} \right]^{\frac{p}{p-q}} \omega^{1-p'}(t)\, t^{n-1} dt$$

$$\le \int_0^\infty \left[\left(\int_0^t \omega_1(\tau)\, \tau^{n-nq-1}\, d\tau \right) \left(\int_t^\infty \omega^{1-p'}(\tau)\, \tau^{n-1}\, d\tau \right)^{q-1} \right]^{\frac{p}{p-q}} \omega^{1-p'}(t)\, t^{n-1} dt < \infty.$$

Further taking $F(t) = t^{n-1} \left[\int_{|\overline{y}|=1} |f(t\overline{y})|\, d\sigma(\overline{y}) \right]$, $u(t) = \psi(2t) \int_{2t}^\infty \varphi(r)\, r^{n-nq-1}\, dr$, applying Theorem 2 (part one), and Hölder's inequality, we get

$$C_6 \left(\int_0^\infty \psi(2s) \left(\int_{2s}^\infty \varphi(\tau)\, \tau^{n-nq-1}\, d\tau \right) \left(\int_0^s t^{n-1} \left[\int_{|\overline{y}|=1} |f(t\overline{y})|\, d\sigma(\overline{y}) \right] dt \right)^q ds \right)^{1/q}$$

$$\le C_7 \left(\int_0^\infty \omega(t)\, t^{-(n-1)(p-1)} \left[t^{n-1} \int_{|\overline{y}|=1} |f(t\overline{y})|\, d\sigma(\overline{y}) \right]^p dt \right)^{1/p}$$

$$= C_7 \left(\int_0^\infty \omega(t)\, t^{n-1} \left[\int_{|\overline{y}|=1} |f(t\overline{y})|\, d\sigma(\overline{y}) \right]^p dt \right)^{1/p}$$

$$\le C_8 \left(\int_0^\infty \omega(t)\, t^{n-1} \left[\int_{|\overline{y}|=1} |f(t\overline{y})|^p\, d\sigma(\overline{y}) \right] dt \right)^{1/p} = C_8 \left(\int_{\mathbb{R}^n} |f(y)|^p\, \omega(|x|)\, dx \right)^{1/p}.$$

This completes the proof of Theorem 4.

Example 2.1. Let

$$\omega_1(t) = \begin{cases} t^{q-1} \ln^\beta \frac{1}{t} & for \quad t < e^{-\frac{p}{p-q}} \\ e^{\frac{p(\lambda-q+1)}{p-q}} \left(\frac{p}{p-q} \right)^\beta t^\lambda & for \quad t \ge e^{-\frac{p}{p-q}}, \end{cases}$$

$$\omega(t)=\begin{cases} t^{p-1}\ln^{\gamma}\frac{1}{t} & for \quad t<e^{-\frac{p}{p-q}} \\ e^{\frac{p(\mu-p+1)}{p-q}}\left(\frac{p}{p-q}\right)^{\gamma} t^{\mu} & for \quad t\ge e^{-\frac{p}{p-q}}, \end{cases}$$

where $p-1<\gamma<\dfrac{p(p-1)}{p-q}$, $\beta<\dfrac{q}{p}(\gamma+1)-q-1$, $\beta\neq-1$, $0\le\lambda<\dfrac{q}{p}(\mu+1)-1$ and $\dfrac{q}{p}-1<\mu<p-1$. Then the pair (ω,ω_1) satisfies the condition of Theorem 4 for $n=1$.

Let $\varphi=1$ and $\alpha\ge1$. Then ω and ω_1 are increasing weight functions and

$$\int_0^\infty\left[\left(\int_t^\infty\omega_1(\tau)\,\tau^{n-nq-1}\,d\tau\right)\left(\int_0^t\omega^{1-p'}(\tau)\,\tau^{n-1}\,d\tau\right)^{q-1}\right]^{\frac{p}{p-q}}\omega^{1-p'}(t)\,t^{n-1}dt$$
$$\ge\int_0^\infty\left[\left(\int_{\alpha t}^\infty\omega_1(\tau)\,\tau^{n-nq-1}\,d\tau\right)\left(\int_0^t\omega^{1-p'}(\tau)\,\tau^{n-1}\,d\tau\right)^{q-1}\right]^{\frac{p}{p-q}}\omega^{1-p'}(t)\,t^{n-1}dt$$
$$\ge\int_0^\infty\left[\left(\int_{\alpha t}^\infty\tau^{n-nq-1}\,d\tau\right)\left(\int_0^t\tau^{n-1}\,d\tau\right)^{q-1}\right]^{\frac{p}{p-q}}[\omega_1(\alpha\,t)]^{\frac{p}{p-q}}\,[\omega(t)]^{-\frac{q}{p-q}}\,t^{n-1}dt$$
$$=C\int_0^\infty[\omega_1(\alpha\,t)]^{\frac{p}{p-q}}\,[\omega(t)]^{-\frac{q}{p-q}}\,t^{n-1}dt=C_1\int_{\mathbb{R}^n}[\omega_1(\alpha\,|x|)]^{\frac{p}{p-q}}\,[\omega(|x|)]^{-\frac{q}{p-q}}\,dx.$$

Therefore condition (1.4) holds automatically.

Corollary 1. *Let $1<q<p<\infty$ and the kernel of convolution operator (1.3) satisfy the conditions (a)-(d). Let ω and ω_1 be positive increasing functions on $(0,\infty)$ satisfying the condition*

$$\int_0^\infty\left[\left(\int_t^\infty\omega_1(\tau)\,\tau^{n-nq-1}\,d\tau\right)\left(\int_0^t\omega^{1-p'}(\tau)\,\tau^{n-1}\,d\tau\right)^{q-1}\right]^{\frac{p}{p-q}}\omega^{1-p'}(t)\,t^{n-1}dt<\infty.$$

Then inequality (2.1) holds.

Representing the decreasing function $u_1(t)$ as $u_1(t)=u_1(\infty)+\int_t^\infty\eta(\tau)\,d\tau$, where $u_1(\infty)=\lim_{t\to\infty}u_1(t)$ and η is a positive function on $(0,\infty)$, using Theorem 1 and Theorem 2 (part two), Lemma 2 and arguing as in the proof of Theorem 4, we get the following Theorem.

Theorem 5. *Let $1 < q < p < \infty$ and the kernel of convolution operator (1.3) satisfy the conditions (a)-(d). Let u and u_1 be positive decreasing functions on $(0, \infty)$, $\varphi \in A_q(\mathbb{R}^n)$ be a radial function, $\omega = u\varphi$ and $\omega_1 = u_1\varphi$. If the weight pair (ω_1, ω) satisfies the condition (1.5) and*

$$\int_0^\infty \left[\left(\int_0^t \omega_1(\tau)\, \tau^{n-1}\, d\tau \right) \left(\int_t^\infty \omega^{1-p'}(\tau)\, \tau^{-1-n(p'-1)}\, d\tau \right)^{q-1} \right]^{\frac{p}{p-q}} \omega^{1-p'}(t)\, t^{-1-n(p'-1)} dt < \infty.$$

then there exists a constant $C > 0$ such that for any $f \in L_{p,\omega}(\mathbb{R}^n)$ inequality (2.1) holds.

Let $\varphi = 1$. Then the following Corollary is valid.

Corollary 2. *Let $1 < q < p < \infty$ and the kernel of convolution operator (1.3) satisfy the conditions (a)-(d). Let ω and ω_1 be positive decreasing functions on $(0, \infty)$ satisfying the condition*

$$\int_0^\infty \left[\left(\int_0^t \omega_1(\tau)\, \tau^{n-1}\, d\tau \right) \left(\int_t^\infty \omega^{1-p'}(\tau)\, \tau^{-1-n(p'-1)}\, d\tau \right)^{q-1} \right]^{\frac{p}{p-q}} \omega^{1-p'}(t)\, t^{-1-n(p'-1)} dt < \infty.$$

Then inequality (2.1) holds.

Remark 4. Note that for $p = q$ Theorem 4 and Theorem 5 were proved in [1]. In the case $p = q$ for some sublinear operator, Theorem 4 and Theorem 5 were proved in [7]. Also, at $\varphi = 1$ for other type singular integral the Theorem 4 and Theorem 5 was proved in [10].

For $q = 1$ a weak $(p, 1)$ type two-weight inequalities are valid.

Theorem 6. *Let $1 < p < \infty$ and the kernel of convolution operator (1.3) satisfy the conditions (a)-(d). Let u be a positive and u_1 be a positive increasing function on $(0, \infty)$, $\varphi \in A_1(\mathbb{R}^n)$ be a radial function, $\omega = u\varphi$ and $\omega_1 = u_1\varphi$. If the weight pair (ω_1, ω) for $q = 1$ satisfies condition (1.4) and*

$$\int_0^\infty \left(\int_t^\infty \frac{\omega_1(\tau)}{\tau}\, d\tau \right)^{p'} \omega^{1-p'}(t)\, t^{n-1} dt < \infty,$$

then there exists a constant $C > 0$ such that for any $f \in L_{p,\omega}(\mathbb{R}^n)$ and $\lambda > 0$

the inequality

$$\int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \omega_1(|x|)\,dx \le \frac{C}{\lambda}\left(\int\limits_{\mathbb{R}^n} |f(x)|^p\,\omega(|x|)\,dx\right)^{1/p} \tag{2.2}$$

holds.

Proof. In first we consider the increasing functions of the form

$$u_1(t) = u_1(0) + \int\limits_0^t \delta(\tau)\,d\tau,$$

where $u_1(0) = \lim\limits_{t\to+0} u(t)$ and δ is a positive function on $(0,\infty)$ (see Theorem 4). We have

$$\int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \omega_1(|x|)\,dx = \int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \varphi(|x|)\left(u_1(0) + \int\limits_0^{|x|}\delta(t)dt\right)dx.$$

If $u_1(0) = 0$, then $\int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \omega_1(|x|)\,dx = \int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \varphi(|x|)$ $\left(\int\limits_0^{|x|}\delta(t)dt\right)dx$. However, if $u_1(0)>0$, then

$$\begin{aligned}\int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \omega_1(|x|)\,dx &= \int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \varphi(|x|)\,u_1(0)dx \\ &+ \int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \varphi(|x|)\left(\int\limits_0^{|x|}\delta(t)dt\right)dx = F_1+F_2.\end{aligned}$$

In first we estimate F_1. By Theorem 1 and by Lemma 2, we have

$$\begin{aligned}F_1 &= \int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \varphi(|x|)\,u_1(0)dx = u_1(0)\int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \varphi(|x|)\,dx \\ &\le u_1(0)\,\frac{C}{\lambda}\int\limits_{\mathbb{R}^n}|f(x)|\,\varphi(|x|)\,dx \le \frac{C_1}{\lambda}\int\limits_{\mathbb{R}^n}|f(x)|\,\varphi(|x|)\,u_1(|x|)dx \\ &= \frac{C_1}{\lambda}\int\limits_{\mathbb{R}^n}|f(x)|\,\omega_1(|x|)\,dx \le \frac{C_2}{\lambda}\left(\int\limits_{\mathbb{R}^n}|f(x)|^p\,\omega(|x|)\,dx\right)^{1/p}.\end{aligned}$$

Let us estimate the integral F_2. We have

$$\begin{aligned}
F_2 &= \int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \varphi(|x|)\left(\int\limits_0^{|x|}\delta(t)dt\right)dx \\
&= \int\limits_{\{x\in\mathbb{R}^n:\,|Tf(x)|>\lambda\}} \varphi(|x|)\left(\int\limits_0^{\infty}\delta(t)\chi_{\{|x|>t\}}(x)dt\right)dx \\
&= \int\limits_0^{\infty}\delta(t)\left(\int\limits_{|x|>t}\chi\left\{x\in\mathbb{R}^n:\ |Tf(x)|>\lambda\right\}\varphi(|x|)\,dx\right)dt \\
&\le \int\limits_0^{\infty}\delta(t)\left(\int\limits_{|x|>t}\chi\left\{x\in\mathbb{R}^n:\ \left|\int\limits_{|y|>\frac{t}{2}}K(x-y)\,f(y)\,dy\right|>\frac{\lambda}{2}\right\}\varphi(|x|)\,dx\right)dt \\
&\quad+\int\limits_0^{\infty}\delta(t)\left(\int\limits_{|x|>t}\chi\left\{x\in\mathbb{R}^n:\ \left|\int\limits_{|y|\le\frac{t}{2}}K(x-y)\,f(y)\,dy\right|>\frac{\lambda}{2}\right\}\varphi(|x|)\,dx\right)dt \\
&= F_{21}+F_{22}.
\end{aligned}$$

By Theorem 1 we get

$$\begin{aligned}
F_{21} &= \int\limits_0^{\infty}\delta(t)\Bigg(\int\limits_{\mathbb{R}^n}\chi\left\{x\in\mathbb{R}^n:\ \left|\int\limits_{|y|>\frac{t}{2}}K(x-y)\,f(y)\,dy\right|>\frac{\lambda}{2}\right\} \\
&\qquad \varphi(|x|)\,\chi_{\{|x|>t\}}(x)dx\Bigg)\,dt \\
&\le \int\limits_0^{\infty}\delta(t)\left(\int\limits_{\mathbb{R}^n}\chi\left\{x\in\mathbb{R}^n:\ \left|\int\limits_{|y|>\frac{t}{2}}K(x-y)\,f(y)\,dy\right|>\frac{\lambda}{2}\right\}\varphi(|x|)\,dx\right)dt \\
&= \int\limits_0^{\infty}\delta(t)\left(\int\limits_{\left\{x\in\mathbb{R}^n:\ \left|Tf\chi_{\{|x|>t/2\}}(x)\right|>\lambda/2\right\}}\varphi(|x|)\,dx\right)dt \\
&\le \frac{C_2}{\lambda}\int\limits_0^{\infty}\delta(t)\left(\int\limits_{\mathbb{R}^n}\chi_{\left\{|x|>\frac{t}{2}\right\}}(x)|f(x)|\,\varphi(|x|)\,dx\right)dt \\
&= \frac{C_2}{\lambda}\int\limits_{\mathbb{R}^n}|f(x)|\,\varphi(|x|)\left(\int\limits_0^{2\,|x|}\delta(t)\,dt\right)dx \le \frac{C_2}{\lambda}\int\limits_{\mathbb{R}^n}|f(x)|\,\varphi(|x|)\,u_1(2|x|)\,dx
\end{aligned}$$

$$= \frac{C_2}{\lambda} \int\limits_{\mathbb{R}^n} [|f(x)|^p \, \omega(|x|)]^{\frac{1}{p}} \; [\omega(|x|)]^{-\frac{1}{p}} \; u_1(2|x|) \, \varphi(|x|) \, dx$$

$$= \frac{C_2}{\lambda} \int\limits_{\mathbb{R}^n} [|f(x)|^p \, \omega(|x|)]^{\frac{1}{p}} \; u_1(2|x|) \; [u(|x|)]^{-\frac{1}{p}} \; [\varphi(|x|)]^{\frac{1}{p'}} \; dx.$$

Now, applying the Hölder's inequality and using condition (1.4) (for $\alpha = 2$ and $q = 1$), we obtain

$$\frac{C_2}{\lambda} \int\limits_{\mathbb{R}^n} [|f(x)|^p \, \omega(|x|)]^{\frac{1}{p}} \; u_1(2|x|) \; [u(|x|)]^{-\frac{1}{p}} \; [\varphi(|x|)]^{\frac{1}{p'}} \; dx$$

$$\le \frac{C_2}{\lambda} \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \, \omega(|x|) \, dx \right)^{1/p} \left(\int\limits_{\mathbb{R}^n} [u_1(2|x|)]^{p'} \; [u(|x|)]^{-\frac{1}{p-1}} \; \varphi(|x|) \, dx \right)^{1/p'}$$

$$\le C_3 \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \, \omega(|x|) \, dx \right)^{1/p} .$$

Now we estimate F_{22}. Note that if $|x| > t$ and $|y| \le \dfrac{t}{2}$, then $|x-y| \ge |x|-|y| \ge |x| - \frac{|x|}{2} = \frac{|x|}{2}$. We have

$$F_{22} = \int\limits_0^\infty \delta(t) \left(\int\limits_{|x|>t} \varphi(|x|) \, \chi \left\{ x \in \mathbb{R}^n : \left| \int\limits_{|y| \le \frac{t}{2}} K(x-y) \, f(y) \, dy \right| > \frac{\lambda}{2} \right\} dx \right) dt$$

$$\le \frac{2}{\lambda} \int\limits_0^\infty \delta(t) \left(\int\limits_{|x|>t} \varphi(|x|) \left| \int\limits_{|y| \le \frac{t}{2}} K(x-y) \, f(y) \, dy \right| dx \right) dt$$

$$\le \frac{C_5}{\lambda} \int\limits_0^\infty \delta(t) \left(\int\limits_{|x|>t} \varphi(|x|) \left(\int\limits_{|y| \le \frac{t}{2}} \frac{|f(y)|}{|x-y|^n} \, dy \right) dx \right) dt$$

$$\le \frac{C_6}{\lambda} \int\limits_0^\infty \delta(t) \left(\int\limits_{|x|>t} \frac{\varphi(|x|)}{|x|^n} \, dx \right) \left(\int\limits_{|y| \le \frac{t}{2}} |f(y)| \, dy \right) dt$$

$$= \frac{2\, C_6}{\lambda} \int\limits_0^\infty \delta(2t) \left(\int\limits_{|x|>2t} \frac{\varphi(|x|)}{|x|^n} \, dx \right) \left(\int\limits_{|y| \le t} |f(y)| \, dy \right) dt$$

$$= \frac{C_7}{\lambda} \int\limits_0^\infty \delta(2t) \left(\int\limits_{2t}^\infty \frac{\varphi(r)}{r} \, dr \right) \left(\int\limits_0^t s^{n-1} \left[\int\limits_{|\overline{y}|=1} |f(s\overline{y})| \, d\sigma(\overline{y}) \right] ds \right) dt$$

Besides we have the following estimates:

$$\int_{s}^{\infty} \delta(2t) \left(\int_{2t}^{\infty} \frac{\varphi(r)}{r}\, dr \right) dt = \frac{1}{2} \int_{2\,s}^{\infty} \delta(t) \left(\int_{t}^{\infty} \frac{\varphi(r)}{r}\, dr \right) dt =$$

$$= \frac{1}{2} \int_{2\,s}^{\infty} \frac{\varphi(r)}{r} \left(\int_{2\,s}^{r} \delta(t)\, dt \right) dr \le \int_{2\,s}^{\infty} \frac{\varphi(r)\, u_1(r)}{r}\, dr =$$

$$= \int_{2\,s}^{\infty} \frac{\omega_1(r)}{r}\, dr \le \int_{s}^{\infty} \frac{\omega_1(r)}{r}\, dr.$$

Therefore we have

$$\int_{0}^{\infty} \left(\int_{s}^{\infty} \delta(2t) \left(\int_{2t}^{\infty} \frac{\varphi(r)}{r}\, dr \right) dt \right)^{p'} \omega^{1-p'}(s)\, s^{n-1}\, ds \le$$

$$\le \int_{0}^{\infty} \left(\int_{s}^{\infty} \frac{\omega_1(\tau)}{\tau}\, d\tau \right)^{p'} \omega^{1-p'}(s)\, s^{n-1}\, dt < \infty.$$

Further taking $F(t) = \int\limits_{0}^{t} s^{n-1} \left[\int\limits_{|\overline{y}|=1} |f(s\overline{y})|\, d\sigma(\overline{y}) \right] ds,\ u(t) = \delta(2s) \int\limits_{2s}^{\infty} \frac{\varphi(r)}{r}\, dr$ and applying Lemma 1 (part one) and Hölder's inequality, we get

$$\frac{C_7}{\lambda} \int_{0}^{\infty} \delta(2t) \left(\int_{2t}^{\infty} \frac{\varphi(r)}{r}\, dr \right) \left(\int_{0}^{t} s^{n-1} \left[\int_{|\overline{y}|=1} |f(s\overline{y})|\, d\sigma(\overline{y}) \right] ds \right) dt$$

$$\le \frac{C_8}{\lambda} \left(\int_{0}^{\infty} \omega(t)\, t^{-(n-1)(p-1)} \left[t^{n-1} \int_{|\overline{y}|=1} |f(t\overline{y})|\, d\sigma(\overline{y}) \right]^{p} dt \right)^{1/p}$$

$$= \frac{C_8}{\lambda} \left(\int_{0}^{\infty} \omega(t)\, t^{n-1} \left[\int_{|\overline{y}|=1} |f(t\overline{y})|\, d\sigma(\overline{y}) \right]^{p} dt \right)^{1/p}$$

$$\le \frac{C_9}{\lambda} \left(\int_{0}^{\infty} \omega(t)\, t^{n-1} \left[\int_{|\overline{y}|=1} |f(t\overline{y})|^{p}\, d\sigma(\overline{y}) \right] dt \right)^{1/p}$$

$$= \frac{C_8}{\lambda} \left(\int_{\mathbb{R}^n} |f(x)|^{p}\, \omega(|x|)\, dx \right)^{1/p}.$$

This completes the proof of Theorem 6.

Example 2.2. Let

$$\omega(t)=\begin{cases}\dfrac{1}{t}\,\ln^{\beta}\dfrac{1}{t} & for \quad t<e^{2\,\beta}\\ e^{-2\,\beta(\lambda+1)}\,(-2\beta)^{\beta}t^{\lambda} & for \quad t\geq e^{2\,\beta},\end{cases}$$

$$\omega_1(t)=\begin{cases}\dfrac{1}{t}\,\ln^{\gamma}\dfrac{1}{t} & for \quad t<e^{2\beta}\\ e^{-2\,\beta(\mu+1)}\,(-2\beta)^{\gamma}\,t^{\mu} & for \quad t\geq e^{2\beta},\end{cases}$$

where $\mu>p(\lambda+1)-1,\ -1<\lambda<0,\ \beta<-1$ and $\gamma>p(\beta+2)+1$. Then the pair (ω,ω_1) satisfies the condition of Theorem 6.

For $\varphi=1$ the following Corollary is valid.

Corollary 3. *Let $1<q<p<\infty$ and the kernel of convolution operator (1.3) satisfy the conditions (a)-(d). Let ω be a positive and ω_1 be a positive increasing function on $(0,\,\infty)$ satisfying the condition*

$$\int\limits_0^{\infty}\left(\int\limits_t^{\infty}\frac{\omega_1(\tau)}{\tau}\,d\tau\right)^{p'}\omega^{1-p'}(t)\,t^{n-1}dt<\infty.$$

Then inequality (2.2) holds.

Representing the decreasing function $u_1(t)$ as $u_1(t)=u_1(\infty)+\int\limits_t^{\infty}\eta(\tau)\,d\tau$, where $u_1(\infty)=\lim\limits_{t\to\infty}u_1(t)$ and η is a positive function on $(0,\,\infty)$, using Theorem 1 (for $r=1$) and Lemma 1 (part two), Lemma 3 and arguing as in the proof of Theorem 6, we get the following Theorem.

Theorem 7. *Let $1<p<\infty$ and the kernel of convolution operator (1.3) satisfy the conditions (a)-(d). Let u be a positive and u_1 be a positive decreasing function on $(0,\,\infty)$, $\varphi\in A_1\,(\mathbb{R}^n)$ be a radial function, $\omega=u\varphi$ and $\omega_1=u_1\varphi$. Suppose that the weight pair $(\omega_1,\,\omega)$ for $q=1$ satisfies condition (1.4) and*

$$\int\limits_0^{\infty}\left(\int\limits_0^{t}\omega_1(\tau)\tau^{n-1}\,d\tau\right)^{p'}\omega^{1-p'}(t)\,t^{n(1-p')-1}dt<\infty.$$

Then inequality (2.2) holds.

Analogously for $\varphi=1$ the following Corollary is valid.

Corollary 4. *Let* $1 < q < p < \infty$ *and the kernel of convolution operator* (*1.3*) *satisfy the conditions* (a)-(d). *Let* ω *be a positive and* ω_1 *be a positive decreasing function on* $(0, \infty)$ *satisfying the condition*

$$\int_0^\infty \left(\int_0^t \omega_1(\tau)\tau^{n-1}\, d\tau \right)^{p'} \omega^{1-p'}(t)\, t^{n(1-p')-1} dt < \infty.$$

Then inequality (2.2) holds.

Remark 5. Note that for other type singular integral at $\varphi = 1$, Theorem 6 and Theorem 7 were proved in [10]. For some sublinear operator at $p = q = 1$, Theorem 6 and Theorem 7 were proved in [7].

REFERENCES

1. R. A. Bandaliev, Two-weight inequalities for convolution operators in Lebesgue spaces, *Mat. Zametki*, **1** (80) (2006), 3-10 (in Russian); English translation: *Math. Notes,* **80(1)** (2006), 3-10.
2. A. P. Calderòn and A. Zygmund, On the existence of certain singular integrals, *Acta Math.*, **88** (1952), 85-139.
3. A. P. Calderòn and A. Zygmund, On singular integrals, *Amer. J. Math.*, **78(2)** (1956), 289-309.
4. K. Davis and Y. Chang, *Lectures on Bochner-Riesz means*, London Math. Soc., Lecture Note Ser. 114, Cambridge Univ. Press, 1987.
5. J. Garsia-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Math. Studies, Amsterdam, 1985, p. 116.
6. D. J. Grubb and C. N. Moore, A variant of Hörmander's condition for singular integrals, *Colloq. Math.*, **73(2)** (1997), 165-172.
7. V. S. Guliyev, Two-weight inequalities for singular integrals satisfying a variant of Hörmander condition, *Journal of Function Spaces and Appl.*, **7(1)** (2009), 43-54.
8. L. Hörmander, Estimates for translation invariant operators in L_p spaces, *Acta Math.*, **104** (1960), 93-140.
9. V. P. Kabaila, On the embedding $L_p(\mu)$ into $L_q(\nu)$, *Litovsky Mat. Sb.*, **21** (1981), 143-148 (in Russian).
10. V. Kokilashvili and A. Meskhi, Two-weight inequalities for singular integrals defined on homogeneous groups, *Proc. A. Razmadze Math. Inst.*, **112** (1997), 57-90.
11. V. G. Maz'ya, *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.

12. B. Muckenhoupt, Weighted norm inequalities for Hardy maximal function, *Trans. Amer. Math. Soc.*, **165** (1972), 207-226.

13. E. M. Stein, Note on singular integral, *Proc. Amer. Math. Soc.*, **8(2)** (1957), 250-254.

14. R. Trujillo-Gonzàles, Weighted norm inequalities for singular integrals operators satisfying a variant of Hörmander condition, *Comment. Math. Univ. Carolinae*, **44(1)** (2003), 137-152.

R. A. Bandaliev
Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan
Baku, Az 1141
Azerbaijan
E-mail: bandalievr@rambler.ru

K. K. Omarova
Institute of Cybernetics of National Academy of Sciences of Azerbaijan
Baku, Az 1141
Azerbaijan
E-mail: konul-kamal@rambler.ru