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STORNG LEVITIN-POLYAK WELL-POSEDNESS FOR GENERALIZED QUASI-VARIATIONAL INCLUSION PROBLEMS WITH APPLICATIONS

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Abstract. In this paper, we study the strong Levitin-Polyak well-posedness for a class of generalized quasi-variational inclusion problems. We establish some metric characterizations of the strong Levitin-Polyak well-posedness for the generalized quasi-variational inclusion problem. We also prove that under suitable conditions, the strong Levitin-Polyak well-posedness of the generalized quasi-variational inclusion problem is equivalent to the existence and uniqueness of solutions, and that the strong Levitin-Polyak well-posedness of generalized quasi-variational inclusion problem in the generalized sense is equivalent to the existence of solutions. As applications, we obtain some results concerned with Levitin-Polyak well-posedness for several kinds of equilibrium problems.

1. INTRODUCTION

It is well known that well-posedness plays a crucial role in the stability theory for optimization problems. Well-posedness of unconstrained and constrained scalar optimization problems was first introduced and studied by Tykhonov [47] and Levitin and Polyak [23], respectively. Since then, various concepts of well-posedness have been introduced and extensively studied for minimization problems and vector optimization problems. For details, we refer the reader to [2, 6, 7, 14, 15, 19, 38, 40, 44, 50, 51] and the references therein.

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In recent years, the concept of well-posedness has been generalized to several related problems: variational inequality problems [4, 5, 6, 8, 17, 10, 11, 16, 26, 27, 34, 46], saddle point problems [3], Nash equilibrium problems [26, 35, 36, 37, 39], inclusion problems [5, 10, 21, 22], and fixed point problems [5, 10, 21, 22, 43, 49].

Recently, Fang et al. [9] generalized the concept of well-posedness to equilibrium problems and to optimization problems with equilibrium constraints and established some metric characterizations of well-posedness for equilibrium problems and for optimization problems with equilibrium constraints. Kimura et al. [18] further generalized it to vector equilibrium problems. On the other hand, Long et al. [32] generalized the concept of Levitin-Polyak well-posedness to equilibrium problems with functional constraints and obtained some metric characterizations and sufficient conditions for Levitin-Polyak well-posedness of equilibrium problems with functional constraints. Long and Huang [33] further introduced and studied α-well-posedness for sysmetric quasi-equilibrium problems. In 2009, Li and Li [24] introduced and studied Levitin-Polyak well-posedness for vector equilibrium problems. Huang et al. [13] generalized it to vector quasi-equilibrium problems. Li et al. [25] further generalized it to generalized vector quasi-equilibrium problems. Moreover, Li et al. [25] obtained some criteria and metric characterizations of the Levitin-Polyak well-posedness and established the relations between Levitin-Polyak well-posedness of optimization problems and Levitin-Polyak well-posedness of generalized vector quasi-equilibrium problems. Peng et al. [41, 42] discussed Levitin-Polyak well-posedness for some generalized vector equilibrium problems and vector quasi-equilibrium problems with functional constraints.

Very recently, Lin and Chuang [31] further extended the notion of well-posedness to variational inclusion and disclusion problems and for optimization problems with variational inclusion and disclusion problems as constraints. Moreover, Lin and Chuang [31] obtained some necessary and sufficient conditions for well-posedness of the variational inclusion and disclusion problems and for well-posedness of the optimization problems with variational inclusion and disclusion problems as constraints.

On the other hand, the quasi-variational inclusion problem is an important generalization of the variational inclusion problem, which contains lots of important problems as special cases and has many applications, like variational disclusion problems, minimax inequalities, equilibrium problems, saddle point problems, optimization theory, bilevel problems, mathematical program with equilibrium constraint, variational inequalities, fixed point problems, coincidence point problems, Ekeland's variational principle, etc. For details, we refer to [12, 28, 29, 30, 45, 48] and the references therein.

Motivated and inspired by the works mentioned above, in this paper, we shall investigate the strong Levitin-Polyak (for short, LP) well-posedness for generalized quasi-variational inclusion problems (for short, (GQVIP)). We establish some metric characterizations of the strong LP well-posedness for (GQVIP). We also prove that under suitable conditions, the strong LP well-posedness of (GQVIP) is equivalent to the existence and uniqueness of solutions, and that the strong LP well-posedness of (GQVIP) in the generalized sense is equivalent to the existence of solutions. As applications, we obtain some results of LP well-posedness for several kinds of equilibrium problems. The results presented in this paper improve and generalize some known results of Huang et al. [13], Li and Li [24], and Long et al. [32].

2. Preliminaries

In this section, we shall recall some definitions and lemmas used in the sequel.

Definition 2.1. ([1]) Let X and Y be two topological spaces. A multivalued mapping $T : X \to 2^Y$ is said to be

(i) upper semi-continuous (for short, $u.s.c.$) at $x \in X$ if, for each open set V in Y with $T(x) \subseteq V$, there exists an open neighborhood $U(x)$ of x such that $T(x') \subseteq V$ for all $x' \in U(x)$;

(ii) lower semi-continuous (for short, $l.s.c.$) at $x \in X$ if, for each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $T(x') \cap V \neq \emptyset$ for all $x' \in U(x)$;

(iii) $u.s.c.$ (resp. $l.s.c.$) on X if it is $u.s.c.$ (resp. $l.s.c.$) at every point $x \in X$;

(iv) continuous on X if it is both $u.s.c.$ and $l.s.c.$ on X;

(v) closed if the graph of T is closed, i.e., the set $Gr(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$;

(vi) open if the graph of T is open in $X \times Y$.

Lemma 2.1. ([1]). *Let X and Y be two topological spaces, $F : X \to 2^Y$ a multivalued mapping.*

(i) *If F is u.s.c. and closed-valued, then F is closed;*

(ii) *If F is compact-valued, then F is u.s.c. at $x \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x$ and for any net $\{y_\alpha\} \subseteq Y$ with $y_\alpha \in F(x_\alpha)$, there exist $y \in F(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$;*

(iii) *F is $l.s.c.$ at $x \in X$ if and only if for any $y \in F(x)$ and for any net $\{x_\alpha\}$ with $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ with $y_\alpha \in F(x_\alpha)$ such that $y_\alpha \to y$.*

Lemma 2.2. ([29]). *Let X and Y be topological spaces, $G, H : X \multimap Y$ be multivalued mappings. Let $G + H : X \multimap Y$ be defined by $(G + H)(x) := G(x) + H(x)$ for each $x \in X$.*

(i) *If G is $u.s.c.$ with nonempty compact values and H is closed, then $G + H$ is closed;*

(ii) *If G is $l.s.c.$ and H is open, then $G + H$ is open.*

Definition 2.2. ([20]) Let (E, d) be a complete metric sapce. The Kuratowski measure of noncompactness of a subset A of E is defined by

$$\mu(A) = \inf\{\varepsilon > 0 : A \subseteq \cup_{i=1}^{n} A_i, \ \ \text{diam}(A_i) < \varepsilon, \ i = 1, 2, \cdots, n\},$$

where $\text{diam}(A_i)$ denotes the diameter of A_i defined by $\text{diam}(A_i) = \sup\{d(x_1, x_2) : x_1, x_2 \in A_i\}$.

Definition 2.3. Let A and B be nonempty subsets of a metric space (E, d). The Hausdorff distance $\mathcal{H}(\cdot, \cdot)$ between A and B is defined by

$$\mathcal{H}(A, B) := \max\{e(A, B), e(B, A)\},$$

where $e(A, B) := \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} d(a, b)$. Let $\{A_n\}$ be a sequence of nonempty subsets of E. We say that A_n converges to A in the sense of Hausdorff metric if $\mathcal{H}(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all selection $a_n \in A_n$. For more details on this topic, we refer the reader to [20].

3. Strong LP Well-posedness for (GQVIP)

Let (E, d) be a metric space, $X \subseteq E$ and $X_0 \subseteq X$ be nonempty closed subsets. Let F and Z be Hausdorff topological vector spaces and $Y \subseteq F$ be a nonempty closed subset. Let $K : X \multimap X$, $T : X \multimap Y$ and $G : X \times Y \times X \multimap Z$ be multivalued mappings. Let $e : X \to Z$ be a continuous mapping. Throughout this paper, unless otherwise specified, we use these notations and assumptions.

We consider the following generalized quasi-variational inclusion problem.

(GQVIP): Find $x \in X_0$ such that $x \in K(x)$ and there exists $y \in T(x)$ satisfying

$$0 \in G(x, y, u), \ \forall\, u \in K(x).$$

Denote by S the solution set of (GQVIP).

It is easy to see that (GQVIP) includes many kinds of known variational inclusion problems and generalized equilibrium problems as special cases. The well-posedness concerned with some special cases of (GQVIP) have been studied by several authors (see, for example, [25, 13, 4, 32, 27] and the references therein).

Let $\mathcal{D}$ be a metric space. For each $a \in \mathcal{D}$ and each $r > 0$, we denote by $B(a, r)$ the closed ball centered at a with radius r. When $\mathcal{D} = R$, we denote by $B^+(0, r)$ the closed interval $[0, r]$.

Definition 3.1. A sequence $\{x_n\} \subseteq X$ is said to be a weak LP approximating solution sequence for (GQVIP) if there exist a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ and a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that, for each $n \in N$, $d(x_n, X_0) \leq \varepsilon_n$ and $d(x_n, K(x_n)) \leq \varepsilon_n$ with

$$0 \in G(x_n, y_n, u) + B^+(0, \varepsilon_n) e(x_n), \ \forall\, u \in K(x_n).$$

Definition 3.2. (GQVIP) is said to be strongly LP well-posed if (GQVIP) has a unique solution x, and every weak LP approximating solution sequence for (GQVIP) converges to x, and (GQVIP) is said to be strongly LP well-posed in the generalized sense if (GQVIP) has a nonempty solution set S, and every weak LP approximating solution sequence for (GQVIP) has a subsequence which converges to a point of S.

Define the approximating solution set for (GQVIP) by

$$\begin{aligned}\Omega(\varepsilon) =&\{x \in X : d(x, X_0) \leq \varepsilon,\ d(x, K(x)) \leq \varepsilon \text{ and } \exists\, y \in T(x) \text{ such that}\\ &0 \in G(x, y, u) + B^{+}(0, \varepsilon)e(x),\ \forall\, u \in K(x)\},\ \forall \text{ for each } \varepsilon > 0 \text{ is ginen.}\end{aligned}$$

Clearly, we have

(i) for every $\varepsilon > 0$, $S \subseteq \Omega(\varepsilon)$;

(ii) if $0 < \varepsilon_1 \leq \varepsilon_2$, then $\Omega(\varepsilon_1) \subseteq \Omega(\varepsilon_2)$.

Next, we further consider the properties for $\Omega(\varepsilon)$.

Property 3.1. *Assume that K is closed-valued and T is compact-valued. For each $(x, u) \in X \times X$, if $y \multimap G(x, y, u)$ is closed, then $S = \cap_{\varepsilon>0}\Omega(\varepsilon)$.*

Proof. (i) Clearly, $S \subseteq \cap_{\varepsilon>0}\Omega(\varepsilon)$. Hence, we only need to show that $\cap_{\varepsilon>0}\Omega(\varepsilon) \subseteq S$. Indeed, if $x \in \cap_{\varepsilon>0}\Omega(\varepsilon)$, then, for each $\varepsilon > 0$, $x \in \Omega(\varepsilon)$. Hence, for each $n \in N$, $x \in \Omega(\frac{1}{n})$, and so there exists $y_n \in T(x)$ such that

$$d(x, X_0) \leq \tfrac{1}{n}, \tag{3.1}$$

$$d(x, K(x)) \leq \tfrac{1}{n}, \tag{3.2}$$

$$0 \in G(x, y_n, u) + B^{+}(0, \tfrac{1}{n})e(x),\ \forall\, u \in K(x). \tag{3.3}$$

Note that X_0 and $K(x)$ are closed sets. Then, by (3.1) and (3.2), we have $x \in X_0$ and $x \in K(x)$. Since $\{y_n\} \subseteq T(x)$ and $T(x)$ is a compact set, there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y \in T(x)$ such that $y_{n_k} \to y$ as $k \to \infty$, and so, for each $k \in N$,

$$0 \in G(x, y_{n_k}, u) + B^{+}(0, \frac{1}{n_k})e(x),\ \forall\, u \in K(x). \tag{3.4}$$

For each $u \in K(x)$, by (3.4), for every $k \in N$, there exists $\gamma_k \in B^{+}(0, \frac{1}{n_k})$ such that

$$0 \in G(x, y_{n_k}, u) + \gamma_k e(x).$$

Clearly, $\gamma_k \to 0$ as $k \to \infty$. Since $y \multimap G(x, y, u)$ is closed, we get $0 \in G(x, y, u)$ and so $x \in S$. It follows that $\cap_{\varepsilon>0}\Omega(\varepsilon) \subseteq S$. This completes the proof.

Example 3.1. Let $E = F = Z = R$, $X = Y = [0, +\infty)$ and $X_0 = [0, 1]$. For any $(x, y, u) \in X \times Y \times X$, let

$$e(x) = 1, \quad K(x) = [x, +\infty), \quad T(x) = [\frac{x}{2}, 2x], \quad G(x, y, u) = (-\infty, x - y + u].$$

Then, it is easily to see that all the conditions of Property 3.1 are satisfied. By Property 3.1, $S = \cap_{\varepsilon>0}\Omega(\varepsilon)$. Indeed, by simple computation, we have $S = [0, 1]$ and $\Omega(\varepsilon) = [0, 1 + \varepsilon]$ for all $\varepsilon > 0$, and so $\cap_{\varepsilon>0}\Omega(\varepsilon) = [0, 1] = S$.

Property 3.2. *Assume that K is continuous and closed-valued, T is $u.s.c.$ and compact-valued and G is closed. Then S is a closed subset of X_0; Furthermore, if K is also compact-valued, then for every $\varepsilon > 0$, $\Omega(\varepsilon)$ is a closed subset of X.*

Proof. (i) *Let $x \in clS$. Then, there exists a sequence $\{x_n\}$ in S such that $x_n \to x$ as $n \to \infty$. It follows that, for each $n \in N$, $x_n \in X_0$, $x_n \in K(x_n)$ and there exists some $y_n \in T(x_n)$ such that*

$$0 \in G(x_n, y_n, u), \ \forall\, u \in K(x_n).$$

Since X_0 is closed, we have $x \in X_0$. Moreover, since K is $u.s.c.$ and closed-valued, K is closed and so $x \in K(x)$. Since T is $u.s.c.$ and compact-valued, there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y \in T(x)$ such that $y_{n_k} \to y$ as $k \to \infty$. It follows that, for each $k \in N$,

$$0 \in G(x_{n_k}, y_{n_k}, u), \ \forall\, u \in K(x_{n_k}).$$

For each $u \in K(x)$, since K is $l.s.c.$, there exists a sequence $\{u_k\}$ with $u_k \in K(x_{n_k})$ such that $u_k \to u$ as $k \to \infty$, and so

$$0 \in G(x_{n_k}, y_{n_k}, u_k), \ \forall\, k \in N.$$

By the closedness of G, we get

$$0 \in G(x, y, u).$$

Hence $x \in S$, and this implies that S is a closed subset of X_0.

(ii) *Suppose that K is also compact-valued, we shall show that, for every $\varepsilon > 0$, $\Omega(\varepsilon)$ is a closed subset of X. Indeed, for any $\varepsilon > 0$, if $x \in cl(\Omega(\varepsilon)) \subseteq X$, then there exists a sequence $\{x_n\}$ in $\Omega(\varepsilon)$ such that $x_n \to x$ as $n \to \infty$. It follows that, for each $n \in N$, there exists $y_n \in T(x_n)$ such that*

$$d(x_n, X_0) \le \varepsilon, \tag{3.5}$$

$$d(x_n, K(x_n)) \le \varepsilon, \tag{3.6}$$

$$0 \in G(x_n, y_n, u) + B^+(0, \varepsilon)e(x_n), \ \forall\, u \in K(x_n). \tag{3.7}$$

By (3.5), we have $d(x, X_0) \leq \varepsilon$. *By (3.6), for each* $n \in N$, *there exists* $u_n \in K(x_n)$ *such that*

$$d(x_n, u_n) \leq \varepsilon + \frac{1}{n}. \tag{3.8}$$

Since K *is u.s.c. and compact-valued, there exists a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *and* $u \in K(x)$ *such that* $u_{n_k} \to u$ *as* $k \to \infty$. *It follows that*

$$d(x, u) = \lim_{k\to\infty} d(x_{n_k}, u_{n_k}) \leq \varepsilon.$$

Noting that $u \in K(x)$, *we get*

$$d(x, K(x)) \leq \varepsilon. \tag{3.9}$$

Since T *is* $u.s.c.$ *and compact-valued, there exist a subsequence* $\{y_{n_k}\}$ *of* $\{y_n\}$ *and* $y \in T(x)$ *such that* $y_{n_k} \to y$ *as* $k \to \infty$. *Then, by (3.7), we have, for each* $k \in N$,

$$0 \in G(x_{n_k}, y_{n_k}, u) + B^+(0, \varepsilon)e(x_{n_k}),\ \forall\, u \in K(x_{n_k}).$$

For each $u \in K(x)$, *since* K *is* $l.s.c.$, *there exists a sequence* $\{u_k\}$ *with* $u_k \in K(x_{n_k})$ *such that* $u_k \to u$ *as* $k \to \infty$, *and so*

$$0 \in G(x_{n_k}, y_{n_k}, u_k) + B^+(0, \varepsilon)e(x_{n_k}),\ \forall\, k \in N.$$

Thus, there exists a sequence $\{\gamma_k\} \subseteq B^+(0, \varepsilon)$ *such that*

$$0 \in G(x_{n_k}, y_{n_k}, u_k) + \gamma_k e(x_{n_k}),\ \forall\, k \in N.$$

Observe that $B^+(0, \varepsilon) = [0, \varepsilon] \subseteq R$ *is compact. We may assume that* $\gamma_k \to \gamma \in B^+(0, \varepsilon)$ *as* $k \to \infty$. *Then, by the closedness of G, we get*

$$0 \in G(x, y, u) + \gamma e(x) \subseteq G(x, y, u) + B^+(0, \varepsilon)e(x).$$

Hence $x \in \Omega(\varepsilon)$, *and this implies that* $\Omega(\varepsilon)$ *is a closed subset of* X.

Remark 3.1. In Property 3.2, if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed but not necessarily compact, and the condition "G is closed" can be weakened by "$(x, y) \multimap G(x, y, u)$ is closed".

If E is finite-dimensional, then the assumption that "K is also compact-valued " in Property 3.2 can be removed.

Property 3.3. *Let* E *be finite-dimensional. Assume that* K *is continuous and closed-valued,* T *is* $u.s.c.$ *and compact-valued and* G *is closed. Then* S *and* $\Omega(\varepsilon)$ *are closed subsets for every* $\varepsilon > 0$.

Proof. We can proceed the proof exactly as that of Property 3.2 except for using the assumption that E is finite-dimensional to get (3.9). In fact, since $x_n \to x$, $\{x_n\}$ is bounded. Then, by (3.8), we know that $\{u_n\}$ is also bounded. Since $\{u_n\} \subseteq X \subseteq E$ and E is finite-dimensional, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges to some $u \in X$ as $k \to \infty$. Since K is $u.s.c.$ and closed-valued, K is closed and so $u \in K(x)$. It follows that

$$d(x,u) = \lim_{k\to\infty} d(x_{n_k}, u_{n_k}) \leq \varepsilon.$$

Thus $d(x, K(x)) \leq \varepsilon$, i.e., (3.9) holds.

The following theorem shows that the strong LP well-posedness of (GQVIP) can be characterized by considering the behavior of the diameter of the approximating solution set.

Theorem 3.1. *Let E be complete. Assume that K is continuous and compact-valued, T is $u.s.c.$ and compact-valued and G is closed. Then $(GQVIP)$ is strongly LP well-posed if and only if*

(3.10) $$\Omega(\varepsilon) \neq \emptyset,\ \forall\, \varepsilon > 0, \quad \textit{and} \ \ \mathrm{diam}(\Omega(\varepsilon)) \to 0 \ \textit{as} \ \varepsilon \to 0. \qquad \blacksquare$$

Suppose that $(GQVIP)$ is strongly LP well-posed. Then $(GQVIP)$ has a unique solution x, and so $\Omega(\varepsilon) \neq \emptyset$ since $x \in \Omega(\varepsilon)$ for all $\varepsilon > 0$.

Now we shall show that

(3.11) $$\mathrm{diam}(\Omega(\varepsilon)) \to 0 \ \textit{as} \ \varepsilon \to 0.$$

If not, then there exist $r > 0$, sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ as $n \to \infty$ and sequences $\{x_n^1\}$ and $\{x_n^2\}$ with $x_n^1, x_n^2 \in \Omega(\varepsilon_n)$ for each $n \in N$ such that

(3.12) $$d(x_n^1, x_n^2) > r,\ \forall\, n \in N.$$

For each $n \in N$, since $x_n^1, x_n^2 \in \Omega(\varepsilon_n)$, there exist $y_n^1 \in T(x_n^1)$ and $y_n^2 \in T(x_n^2)$ such that

$$d(x_n^1, X_0) \leq \varepsilon_n, \quad d(x_n^1, K(x_n^1)) \leq \varepsilon_n$$
$$\textit{and} \ \ 0 \in G(x_n^1, y_n^1, u) + B^+(0, \varepsilon_n)e(x_n^1),\ \forall\, u \in K(x_n^1)$$

and

$$d(x_n^2, X_0) \leq \varepsilon_n, \quad d(x_n^2, K(x_n^2)) \leq \varepsilon_n$$
$$\textit{and} \ \ 0 \in G(x_n^2, y_n^2, u) + B^+(0, \varepsilon_n)e(x_n^2),\ \forall\, u \in K(x_n^2).$$

Hence $\{x_n^1\}$ *and* $\{x_n^2\}$ *are weak LP approximating solution sequence for (GQVIP). Then, by the strong LP well-posedness of* (*GQVIP*), *they have to converge to the unique solution* x *of (GQVIP), a contradiction to* (*3.12*). *Thus* (*3.11*) *holds.*

Conversely, suppose that condition (*3.10*) *holds. If* $\{x_n\} \subseteq X$ *is a weak LP approximating solution sequence for (GQVIP), then there exist a sequence* $\{\varepsilon_n\}$ *of real positive numbers with* $\varepsilon_n \to 0$ *and a sequence* $\{y_n\}$ *with* $y_n \in T(x_n)$ *such that, for each* $n \in N$,

$$d(x_n, X_0) \leq \varepsilon_n, \quad d(x_n, K(x_n)) \leq \varepsilon_n$$
$$\textit{and} \quad 0 \in G(x_n, y_n, u) + B^+(0, \varepsilon_n)e(x_n), \ \forall\, u \in K(x_n).$$

Hence $x_n \in \Omega(\varepsilon_n)$ *for every* $n \in N$. *By (3.10),* $\{x_n\}$ *is a Cauchy sequence and so it converges to a point* $x \in X$. *Then, by similar arguments as in the second part of the proof of Property 3.2, we can show that* $x \in X_0$, $x \in K(x)$ *and there exists some* $y \in T(x)$ *such that*

$$0 \in G(x, y, u), \ \forall\, u \in K(x).$$

Thus x *is a solution of (GQVIP).*

To complete the proof, it is sufficient to prove that (*GQVIP*) *has a unique solution. If* (*GQVIP*) *has two distinct solutions* x^1 *and* x^2, *it is easy to see that* $x^1, x^2 \in \Omega(\varepsilon)$ *for all* $\varepsilon > 0$. *It follows that*

$$0 < d(x^1, x^2) \leq \operatorname{diam}(\Omega(\varepsilon)), \ \forall\, \varepsilon > 0,$$

a contradiction to (*3.10*). *Therefore,* (*GQVIP*) *has a unique solution.*

Remark 3.2. In Theorem 3.1, (i) if E is finite-dimensional, then the condition "K is continuous and compact-valued" can be weaken by "K is continuous and closed-valued"; (ii) if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed but not necessarily compact, and the condition "G is closed" can be weakened by "$(x, y) \multimap G(x, y, u)$ is closed".

Example 3.2. Let $E = F = Z = R$ and $X = Y = [0, +\infty)$ and $X_0 = [0, 1]$. For every $(x, y, u) \in X \times Y \times X$, let $e(x) = 1$, $K(x) = [0, x]$, $T(x) = [0, x^2]$ and $G(x, y, u) = (-\infty, y - 2x^2 + u]$. Then K is continuous and compact-valued, T is $u.s.c.$ and compact-valued and G is closed. In addition, we have

$$\begin{aligned} S &= \{x \in X : x \in X_0, \ x \in K(x) \text{ and } \exists\, y \in T(x) \\ &\qquad \text{such that } 0 \in G(x, y, u), \ \forall\, u \in K(x)\} \\ &= \{x \in [0,1] : \exists\, y \in [0, x^2] \text{ such that } 0 \in (-\infty, y - 2x^2 + u], \ \forall\, u \in [0, x]\} \\ &= \{x \in [0,1] : \exists\, y \in [0, x^2] \text{ such that } y - 2x^2 + u \geq 0, \ \forall\, u \in [0, x]\} \\ &= \{x \in [0,1] : \exists\, y \in [0, x^2] \text{ such that } y \geq 2x^2 - u, \ \forall\, u \in [0, x]\} \end{aligned}$$

$$
\begin{aligned}
&= \{x \in [0,1] : x^2 \geq 2x^2 - 0\} \\
&= \{x \in [0,1] : x^2 \leq 0\} \\
&= \{0\}
\end{aligned}
$$

and for every $\varepsilon > 0$,

$$
\begin{aligned}
\Omega(\varepsilon) &= \{x \in X : d(x, X_0) \leq \varepsilon,\ d(x, K(x)) \leq \varepsilon \text{ and } \exists\, y \in T(x) \\
&\qquad \text{such that } 0 \in G(x,y,u) + B^+(0,\varepsilon)e(x),\ \forall\, u \in K(x)\} \\
&= \{x \in [0, 1+\varepsilon] : \exists\, y \in [0, x^2] \\
&\qquad \text{such taht } 0 \in (-\infty, y - 2x^2 + u] + [0,\varepsilon],\ \forall\, u \in [0,x]\} \\
&= \{x \in [0, 1+\varepsilon] : \exists\, y \in [0, x^2] \\
&\qquad \text{such that } 0 \in (-\infty, y - 2x^2 + u + \varepsilon],\ \forall\, u \in [0,x]\} \\
&= \{x \in [0, 1+\varepsilon] : \exists\, y \in [0, x^2] \\
&\qquad \text{such that } y - 2x^2 + u + \varepsilon \geq 0,\ \forall\, u \in [0,x]\} \\
&= \{x \in [0, 1+\varepsilon] : \exists\, y \in [0, x^2] \\
&\qquad \text{such that } y \geq 2x^2 - u - \varepsilon,\ \forall\, u \in [0,x]\} \\
&= \{x \in [0, 1+\varepsilon] : x^2 \geq 2x^2 - 0 - \varepsilon\} \\
&= \{x \in [0, 1+\varepsilon] : x^2 \leq \varepsilon\} \\
&= [0, \sqrt{\varepsilon}].
\end{aligned}
$$

It follows that $\operatorname{diam}(\Omega(\varepsilon)) \to 0$ as $\varepsilon \to 0$. By Theorem 3.1, (GQVIP) is strongly LP well-posed.

For the strong LP well-posedness in the generalized sense, we give the following characterization by considering the Kuratowski measure of noncompact of approximating solution set.

Theorem 3.2. *Let E be complete. Assume that K is continuous and compact-valued, T is $u.s.c.$ and compact-valued and G is closed. Then (GQVIP) is strongly LP well-posed in the generalized sense if and only if*

$$\Omega(\varepsilon) \neq \emptyset,\ \forall\, \varepsilon > 0, \quad \textit{and} \quad \mu(\Omega(\varepsilon)) \to 0 \textit{ as } \varepsilon \to 0. \tag{3.13}$$

Proof. Suppose that (GQVIP) is strongly LP well-posed in the generalized sense. Then S is nonempty. Now we show that S is compact. Indeed, let $\{x_n\}$ be any sequence in S. Then $\{x_n\}$ is a weak LP approximating solution sequence for (GQVIP). By the strong LP well-posedness in the generalized sense of (GQVIP), $\{x_n\}$ has a subsequence which converges to some point of S. Thus S is compact. Clearly, for each $\varepsilon > 0$, $S \subseteq \Omega(\varepsilon)$, and so $\Omega(\varepsilon) \neq \emptyset$.

Now we shall show that

$$\mu(\Omega(\varepsilon)) \to 0 \text{ as } \varepsilon \to 0. \tag{3.14}$$

Observe that for every $\varepsilon > 0$,

$$\mathcal{H}(\Omega(\varepsilon), S) = \max\{e(\Omega(\varepsilon), S), e(S, \Omega(\varepsilon))\} = e(\Omega(\varepsilon), S).$$

Taking into account the compactness of S, we get

$$\mu(\Omega(\varepsilon)) \le 2\mathcal{H}(\Omega(\varepsilon), S) + \mu(S) = 2e(\Omega(\varepsilon), S).$$

To prove (3.14), it is sufficient to show that

$$e(\Omega(\varepsilon), S) \to 0 \text{ as } \varepsilon \to 0. \tag{3.15}$$

If (3.15) does not hold, then there exist $r > 0$, sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ as $n \to \infty$ and sequence $\{x_n\}$ with $x_n \in \Omega(\varepsilon_n)$ for every $n \in N$ such that

$$x_n \notin S + B(0, r), \ \forall n \in N. \tag{3.16}$$

For each $n \in N$, since $x_n \in \Omega(\varepsilon_n)$, there exists $y_n \in T(x_n)$ such that $d(x_n, X_0) \le \varepsilon_n$ and $d(x_n, K(x_n)) \le \varepsilon_n$ with

$$0 \in G(x_n, y_n, u) + B^+(0, \varepsilon_n)e(x_n), \ \forall u \in K(x_n).$$

Hence $\{x_n\}$ is a weak LP approximating solution sequence for (GQVIP). Then, by the strong LP well-posedness in the generalized sense of (GQVIP), $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to some point of S. This contradicts (3.16), and so (3.15) holds.

Conversely, suppose that condition (3.13) holds. Then, by Properties 3.1 and 3.2, $\Omega(\varepsilon)$ is closed for every $\varepsilon > 0$ and $S = \cap_{\varepsilon > 0}\Omega(\varepsilon)$. Since $\mu(\Omega(\varepsilon)) \to 0$ as $\varepsilon \to 0$, by the Kuratowsk theorem ([20], p.412), S is nonempty and compact and

$$e(\Omega(\varepsilon), S) \to 0 \text{ as } \varepsilon \to 0. \tag{3.17}$$

If $\{x_n\}$ is a weak LP approximating solution sequence for (GQVIP), then there exist a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ and a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that, for each $n \in N$, $d(x_n, X_0) \le \varepsilon_n$ and $d(x_n, K(x_n)) \le \varepsilon_n$ with

$$0 \in G(x_n, y_n, u) + B^+(0, \varepsilon_n)e(x_n), \ \forall u \in K(x_n).$$

Hence $x_n \in \Omega(\varepsilon_n)$ for every $n \in N$. Then, by (3.17), $d(x_n, S) \le e(\Omega(\varepsilon_n), S) \to 0$ as $n \to \infty$. Since S is compact, for each $n \in N$, there exists $\bar{x}_n \in S$ such that $d(x_n, \bar{x}_n) = d(x_n, S) \to 0$ as $n \to \infty$. Again from the compactness of S, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ which converges to a point $\bar{x} \in S$. Hence, the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x}$. Therefore, (GQVIP) is strongly LP well-posed in the generalized sense.

Remark 3.3. In Theorem 3.2, (i) if E is finite-dimensional, then the condition "K is continuous and compact-valued" can be weaken by "K is continuous and closed-valued"; (ii) if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed but not necessarily compact, and the condition "G is closed" can be weakened by "$(x, y) \multimap G(x, y, u)$ is closed".

For the strong LP well-posedness in the generalized sense, we also give the following characterization by considering the Hausdorff distance between the solution set and the approximating solution set.

Theorem 3.3. (*GQVIP*) *is strongly LP well-posed in the generalized sense if and only if* S *is nonempty and compact and* $e(\Omega(\varepsilon), S) \to 0$ *as* $\varepsilon \to 0$.

Proof. The proof is similar to that of Theorem 3.2 and so we omit it here.

Example 3.3. Let $E = F = Z = R$, $X = Y = [0, +\infty)$ and $X_0 = [0, 1]$. For every $(x, y, u) \in X \times Y \times X$, let $e(x) = 1$, $K(x) = [\frac{x}{2}, +\infty)$, $T(x) = [x^2, (x+1)^2]$ and $G(x, y, u) = [-(y + x), u - y]$. Then,

$$\begin{aligned} S &= \{x \in X : x \in X_0,\ x \in K(x) \text{ and } \exists\, y \in T(x) \\ &\qquad \text{such that } 0 \in G(x, y, u),\ \forall\, u \in K(x)\} \\ &= \{x \in [0, 1] : \exists\, y \in [x^2, (x + 1)^2] \\ &\qquad \text{such that } 0 \in [-(y + x), u - y],\ \forall\, u \in [\frac{x}{2}, +\infty)\} \\ &= \{x \in [0, 1] : \exists\, y \in [x^2, (x + 1)^2] \\ &\qquad \text{such that } y \le u,\ \forall\, u \in [\frac{x}{2}, +\infty)\} \\ &= \{x \in [0, 1] : x^2 \le \frac{x}{2}\} \\ &= [0, \frac{1}{2}] \end{aligned}$$

and for any $\varepsilon > 0$,

$$\begin{aligned} \Omega(\varepsilon) &= \{x \in X : d(x, X_0) \le \varepsilon,\ d(x, K(x)) \le \varepsilon \text{ and } \exists\, y \in T(x) \\ &\qquad \text{such that } 0 \in G(x, y, u) + B^+(0, \varepsilon)e(x),\ \forall\, u \in K(x)\} \\ &= \{x \in [0, 1 + \varepsilon] : \exists\, y \in [x^2, (x + 1)^2] \\ &\qquad \text{such taht } 0 \in [-(y + x), u - y] + [0, \varepsilon],\ \forall\, u \in [\frac{x}{2}, +\infty)\} \\ &= \{x \in [0, 1 + \varepsilon] : \exists\, y \in [x^2, (x + 1)^2] \\ &\qquad \text{such that } 0 \in [-(y + x), u - y + \varepsilon],\ \forall\, u \in [\frac{x}{2}, +\infty)\} \end{aligned}$$

$$
\begin{aligned}
&= \{x \in [0, 1+\varepsilon] : \exists\, y \in [x^2, (x+1)^2] \\
&\quad \text{such that } y \le u + \varepsilon,\ \forall\, u \in [\frac{x}{2}, +\infty)\} \\
&= \{x \in [0, 1+\varepsilon] : x^2 \le \frac{x}{2} + \varepsilon\} \\
&= \{x \in [0, 1+\varepsilon] : (x - \frac{1}{4})^2 \le \frac{1}{16} + \varepsilon\} \\
&= [0, \frac{1}{4} + \sqrt{\frac{1}{16} + \varepsilon}].
\end{aligned}
$$

Thus, $e(\Omega(\varepsilon), S) \to 0$ as $\varepsilon \to 0$. By Theorem 3.3, (GQVIP) is strongly LP well-posed in the generalized sense. This completes the proof.

The following example illustrates that the compactness condition in Theorem 3.3 is essential.

Example 3.4. Let $E = F = Z = R$, $X_0 = X = Y = [0, +\infty)$. For every $(x, y, u) \in X \times Y \times X$, let $e(x) = 1$, $K(x) = [\frac{x}{2}, x]$, $T(x) = [0, x^2]$ and $G(x, y, u) = [-(y+x), u-y]$. Then, by similar arguments as that of Example 3.3, we have, for each $\varepsilon > 0$, $S = \Omega(\varepsilon) = [0, +\infty)$. Thus, $e(\Omega(\varepsilon), S) \to 0$ as $\varepsilon \to 0$. Let $x_n = n$ for $n = 1, 2, \cdots$. Then, $\{x_n\}$ is a weak LP approximating solution sequence for (GQVIP), which has no convergent subsequence. This implies that (GQVIP) is not strongly LP well-posed in the generalized sense.

The following theorem shows that under suitable conditions, the strong LP well-posedness of (GQVIP) is equivalent to the existence and uniqueness of the solution.

Theorem 3.4. *Let E be finite-dimensional. Assume that*

(i) *K is continuous and closed-valued;*

(ii) *T is $u.s.c.$ and compact-valued;*

(iii) *G is closed;*

(iv) *there exists $\varepsilon > 0$ such that $\Omega(\varepsilon)$ is nonempty and bounded.*

Then (GQVIP) is strongly LP well-posed if and only if (GQVIP) has a unique solution.

Proof. The necessity is obvious. For the sufficiency, suppose that (GQVIP) has a unique solution x. If $\{x_n\}$ is a weak LP approximating solution sequence for (GQVIP), then there exist a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ and a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that, for each $n \in N$, $d(x_n, X_0) \le \varepsilon_n$ and $d(x_n, K(x_n)) \le \varepsilon_n$ with

$$0 \in G(x_n, y_n, u) + B^+(0, \varepsilon_n)e(x_n),\ \forall\, u \in K(x_n).$$

It follows that $x_n \in \Omega(\varepsilon_n)$ for each $n \in N$. Let $\varepsilon > 0$ be such that $\Omega(\varepsilon)$ is nonempty and bounded. Then there exists some $n_0 \in N$ such that $x_n \in \Omega(\varepsilon_n) \subseteq \Omega(\varepsilon)$ for all $n \geq n_0$ and so $\{x_n\}$ is bounded.

Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \to x'$ as $k \to \infty$. Then, by similar arguments as in the second part of the proof of Property 3.2, we can show that $x' \in X_0$, $x' \in K(x')$ and there exists some $y' \in T(x')$ such that $0 \in G(x', y', u)$ for all $u \in K(x')$. Thus x' is a solution of (GQVIP). By the uniqueness of the solution of (GQVIP), we have $x' = x$. Thus, the whole sequence $\{x_n\}$ converges to x and so (GQVIP) is strongly LP well-posed. This completes the proof.

The following theorem shows that under suitable conditions, the strong LP well-posedness in the generalized sense of (GQVIP) is equivalent to the existence of the solution.

Theorem 3.5. *Let E be finite-dimensional. Assume that*

(i) *K is continuous and closed-valued;*

(ii) *T is $u.s.c.$ and compact-valued;*

(iii) *G is closed;*

(iv) *there exists $\varepsilon > 0$ such that $\Omega(\varepsilon)$ is nonempty and bounded.*

Then $($GQVIP$)$ is strongly LP well-posed in the generalized sense if and only if $($GQVIP$)$ has a nonempty solution set S.

Proof. The necessity is obvious . For the sufficiency, suppose that (GQVIP) has a nonempty solution set S. If $\{x_n\}$ is a weak LP approximating solution sequence for (GQVIP), then there exist a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ and a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that, for each $n \in N$,

$$d(x_n, X_0) \leq \varepsilon_n, \quad d(x_n, K(x_n)) \leq \varepsilon_n$$
$$\text{and} \quad 0 \in G(x_n, y_n, u) + B^+(0, \varepsilon_n)e(x_n), \ \forall\, u \in K(x_n).$$

It follows that $x_n \in \Omega(\varepsilon_n)$ for each $n \in N$. Let $\varepsilon > 0$ be such that $\Omega(\varepsilon)$ is nonempty and bounded. Then there exists some $n_0 \in N$ such that $x_n \in \Omega(\varepsilon_n) \subseteq \Omega(\varepsilon)$ for all $n \geq n_0$. Thus $\{x_n\}$ is bounded, and so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$ as $k \to \infty$. Then, by similar arguments as in the second part of the proof of Property 3.2, we can show that $x \in X_0$, $x \in K(x)$ and there exists some $y \in T(x)$ such that $0 \in G(x, y, u)$ for all $u \in K(x)$. Thus, x is a solution of (GQVIP). It follows that (GQVIP) is strongly LP well-posed in the generalized sense. This completes the proof.

4. Applications

In this section, we shall apply the theorems obtained in Section 3 to present some results of LP well-posedness for variational equilibrium problems.

4.1. LP well-posedness for scalar equilibrium problems

Let $g : X \times Y \times X \to R$ be a function. We consider the following scalar equilibrium problems.

(EP): Find $x \in X_0$ such that $x \in K(x)$ and there exists $y \in T(x)$ satisfying

$$g(x, y, u) \geq 0, \ \forall\, u \in K(x).$$

Denote by $S_{(EP)}$ the solution set of (EP).

Definition 4.1. A sequence $\{x_n\} \subseteq X$ is said to be an LP approximating solution sequence for (EP) if there exist a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ and a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that, for each $n \in N$, $d(x_n, X_0) \leq \varepsilon_n$ and $d(x_n, K(x_n)) \leq \varepsilon_n$ with

$$g(x_n, y_n, u) + \varepsilon_n \geq 0, \ \forall\, u \in K(x_n).$$

Definition 4.2. (EP) is said to be LP well-posed if (EP) has a unique solution x, and every LP approximating solution sequence for (EP) converges to x, and (EP) is said to be LP well-posed in the generalized sense if (EP) has a nonempty solution set $S_{(EP)}$, and every LP approximating solution sequence for (EP) has some subsequence which converges to some point of $S_{(EP)}$.

Remark 4.1. If E is a normed space, $K(x) = X_0 = \{x \in X : h(x) \in D\}$ (D is a nonempty closed subset of a metric space $\mathcal{D}$ and $h : X \to \mathcal{D}$ is a continuous mapping) and $g(x, y, u) = g(x, u)$ for all $(x, y, u) \in X \times Y \times X$, then LP well-posedness in the generalized sense of (EP) reduces to type I LP well-posedness of explicit constrained equilibrium problem of Long et al. [32]. Furthermore, if $g(x, u) = \langle A(x), u - x\rangle$ ($A : X \to E^*$ is a vector-valued mapping with E^* is the dual of E), then LP well-posedness in the generalized sense of (EP) reduces to type I LP well-posedness of constrained variational inequality problem of Huang et al. [17].

Define the approximating solution set for (EP) by

$$\begin{aligned}\Omega_{(EP)}(\varepsilon) =& \{x \in X : d(x, X_0) \leq \varepsilon, \ d(x, K(x)) \leq \varepsilon \ \text{ and } \exists\, y \in T(x) \text{ such that} \\ & g(x, y, u) + \varepsilon \geq 0, \ \forall\, u \in K(x)\}, \quad \forall\, \varepsilon > 0.\end{aligned}$$

Let $Z = R$, $e(x) = 1$ and $G(x, y, u) = g(x, y, u) - R_+$ for all $(x, y, u) \in X \times Y \times X$. Then, (GQVIP) reduces to (EP), and so $S = S_{(EP)}$, $\Omega(\varepsilon) = \Omega_{(EP)}(\varepsilon)$ for all $\varepsilon > 0$, and Definitions 3.1 and 3.2 reduce to Definitions 4.1 and 4.2, respectively. Indeed, $S = S_{(EP)}$ is obvious. For each $\varepsilon > 0$, we have

$$G(x, y, u) + B^+(0, \varepsilon)e(x) = g(x, y, u) - R_+ + [0, \varepsilon] = g(x, y, u) + \varepsilon - R_+,$$

and so

$$0 \in G(x,y,u) + B^+(0,\varepsilon)e(x) \Longleftrightarrow g(x,y,u) + \varepsilon \geq 0.$$

From this, it is easily to see that $\Omega(\varepsilon) = \Omega_{(EP)}(\varepsilon)$ for all $\varepsilon > 0$, and Definitions 3.1 and 3.2 coincide with Definitions 4.1 and 4.2, respectively. Furthermore, if g is $u.s.c.$, then it is easy to prove that G is closed, and so, by Theorems 3.1-3.5, we can obtain the following results of LP well-posedness for (EP).

Theorem 4.1. *Let E be complete. Assume that K is continuous and compact-valued, T is $u.s.c.$ and compact-valued and g is $u.s.c.$. Then (EP) is LP well-posed if and only if*

$$\Omega_{(EP)}(\varepsilon) \neq \emptyset,\ \forall\, \varepsilon > 0, \quad \textit{and} \quad \mathrm{diam}(\Omega_{(EP)}(\varepsilon)) \to 0 \textit{ as } \varepsilon \to 0.$$

Theorem 4.2. *Let E be complete. Assume that K is continuous and compact-valued, T is $u.s.c.$ and compact-valued and g is $u.s.c.$. Then (EP) is LP well-posed in the generalized sense if and only if*

$$\Omega_{(EP)}(\varepsilon) \neq \emptyset,\ \forall\, \varepsilon > 0, \quad \textit{and} \quad \mu(\Omega_{(EP)}(\varepsilon)) \to 0 \textit{ as } \varepsilon \to 0.$$

Theorem 4.3. *(EP) is well-posed in the generalized sense if and only if $S_{(EP)}$ is nonempty and compact and $e(\Omega_{(EP)}(\varepsilon), S_{(EP)}) \to 0$ as $\varepsilon \to 0$.*

Theorem 4.4. *Let E be finite-dimensional. Assume that*

(i) *K is continuous and closed-valued;*

(ii) *T is $u.s.c.$ and compact-valued;*

(iii) *g is $u.s.c.$;*

(iv) *there exists $\varepsilon > 0$ such that $\Omega_{(EP)}(\varepsilon)$ is nonempty and bounded.*

Then (EP) is LP well-posed if and only if (EP) has a unique solution.

Theorem 4.5. *Let E be finite-dimensional. Assume that*

(i) *K is continuous and closed-valued;*

(ii) *T is $u.s.c.$ and compact-valued;*

(iii) *g is $u.s.c.$;*

(iv) *there exists $\varepsilon > 0$ such that $\Omega_{(EP)}(\varepsilon)$ is nonempty and bounded.*

Then (EP) is LP well-posed in the generalized sense if and only if (EP) has a nonempty solution set $S_{(EP)}$.

Remark 4.2. In Theorems 4.1 and 4.2, (i) if E is finite-dimensional, then the condition "K is continuous and compact-valued" can be weaken by "K is continuous and closed-valued"; (ii) if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed but not necessarily compact, and the condition "g is $u.s.c.$" can be weakened by "$(x,y) \multimap g(x,y,u)$ is $u.s.c.$".

Remark 4.3. By Remark 4.2 (ii), we know that Theorems 4.2 and 4.3 generalize Theorems 3.4 and 3.1 of Long et al. [32], respectively.

4.2. LP well-posedness for weak vector equilibrium problems

Let $C : X \multimap Z$ be a multivalued mapping such that for any $x \in X$, $C(x)$ is a proper, closed and convex cone in Z with nonempty interior $intC(x)$. Let $e : X \to Z$ be a continuous mapping such that for any $x \in X$, $e(x) \in intC(x)$. Let $H : X \times Y \times X \multimap Z$ be a multivalued mapping.

We consider the following weak vector equilibrium problems.

(WVEP): Find $x \in X_0$ such that $x \in K(x)$ and there exists $y \in T(x)$ satisfying

$$H(x, y, u) \nsubseteq -intC(x),\ \forall\, u \in K(x).$$

Denote by $S_{(WVEP)}$ the solution set of (WVEP).

Definition 4.3. A sequence $\{x_n\} \subseteq X$ is said to be an LP approximating solution sequence for (WVEP) if there exist a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ and a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that, for each $n \in N$, $d(x_n, X_0) \leq \varepsilon_n$ and $d(x_n, K(x_n)) \leq \varepsilon_n$ with

$$H(x_n, y_n, u) + \varepsilon_n e(x_n) \nsubseteq -intC(x_n),\ \forall\, u \in K(x_n).$$

Definition 4.4. (WVEP) is said to be LP well-posed if (WVEP) has a unique solution x, and every LP approximating solution sequence for (WVEP) converges to x, and (WVEP) is said to be LP well-posed in the generalized sense if (WVEP) has a nonempty solution set $S_{(WVEP)}$, and every LP approximating solution sequence for (WVEP) has some subsequence which converges to some point of $S_{(WVEP)}$.

Remark 4.4. (i) if $X_0 = X$ and for every $(x, y, u) \in X \times Y \times X$, $H(x, y, u) = H(x, u)$, then (WVEP) reduces to (GVQEP 1) of Li et al. [25]. Therefore, LP approximating solution sequence for (WVEP) reduces to LP approximating solution sequence for (GVQEP 1) of Li et al. [25] and LP well-posedness in the generalized sense of (WVEP) reduces to LP well-posedness of (GVQEP 1) of Li et al. [25];

(ii) If $X = E$ and for every $(x, y, u) \in X \times Y \times X$, $K(x) = X_0$ and $H(x, y, u) = g(x, u)$ ($g : E \times E \to Z$ is a vector-valued mapping), then (WVEP) reduces to (VEP) of Li and Li [24]. Hence, LP approximating solution sequence for (WVEP) reduces to type I LP approximating solution sequence for (VEP) of Li and Li [24] and LP well-posedness in the generalized sense of (WVEP) reduces to type I LP well-posedness of (VEP) of Li and Li [24];

(iii) If E and Z are real Banach space, $X_0 = X$ and for every $(x, y, u) \in X \times Y \times X$, $C(x) = C$ (C is a pointed, closed and convex cone with $intC \neq \emptyset$), $e(x) = e$ ($e \in intC$ is a fixed point) and $H(x, y, u) = g(x, u)$ ($g : X \times X \to Z$

is a vector-valued mapping), then (WVEP) reduces to (VQE) of Huang et al. [13]. Thus, LP approximating solution sequence for (WVEP) reduces to approximating sequence for (VQE) of Huang et al. [13] and LP well-posedness in the generalized sense of (WVEP) reduces to well-posedness of (VQE) of Huang et al. [13].

Now we define the approximating solution set for (WVEP) by

$$\Omega_{(WVEP)}(\varepsilon) =\{x \in X : d(x, X_0) \le \varepsilon,\ d(x, K(x)) \le \varepsilon \text{ and } \exists\, y \in T(x) \text{ such that } H(x, y, u) + \varepsilon e(x) \not\subseteq -intC(x),\ \forall\, u \in K(x)\}, \quad \forall\, \varepsilon > 0.$$

Let

$$\begin{aligned} G(x, y, u) &= H(x, y, u) - [-intC(x)]^c \\ &= H(x, y, u) + [intC(x)]^c, \quad \forall (x, y, u) \in X \times Y \times X. \end{aligned}$$

Then, (GQVIP) reduces to (WVEP), and so $S = S_{(WVEP)}$, $\Omega(\varepsilon) = \Omega_{(WVEP)}(\varepsilon)$ for all $\varepsilon > 0$, and Definitions 3.1 and 3.2 reduce to Definitions 4.3 and 4.4, respectively. Indeed, $S = S_{(WVEP)}$ is trival. Next, we show that, for each $\varepsilon > 0$,

$$H(x, y, u) + [intC(x)]^c + B^+(0, \varepsilon)e(x) = H(x, y, u) + \varepsilon e(x) + [intC(x)]^c.$$

Take any $\varepsilon > 0$ and let ε be fixed. Clearly,

$$H(x, y, u) + \varepsilon e(x) + [intC(x)]^c \subseteq H(x, y, u) + [intC(x)]^c + B^+(0, \varepsilon)e(x).$$

Thus, we only need to show that

$$H(x, y, u) + [intC(x)]^c + B^+(0, \varepsilon)e(x) \subseteq H(x, y, u) + \varepsilon e(x) + [intC(x)]^c.$$

In fact, if $z \in H(x, y, u) + [intC(x)]^c + B^+(0, \varepsilon)e(x)$, then there exists some $\gamma \in B^+(0, \varepsilon)$ such that

$$z \in H(x, y, u) + [intC(x)]^c + \gamma e(x).$$

Noting that $\gamma \le \varepsilon$ and $e(x) \in intC(x)$, we have $(\gamma-\varepsilon)e(x)+intC(x) \supseteq intC(x)$, and so

$$\begin{aligned} & H(x, y, u) + [intC(x)]^c + \gamma e(x) \\ = \ & H(x, y, u) + [intC(x)]^c + \varepsilon e(x) + (\gamma - \varepsilon)e(x) \\ = \ & H(x, y, u) + [(\gamma - \varepsilon)e(x) + intC(x)]^c + \varepsilon e(x) \\ \subseteq \ & H(x, y, u) + [intC(x)]^c + \varepsilon e(x). \end{aligned}$$

It follows that $z \in H(x, y, u) + [intC(x)]^c + \varepsilon e(x)$ and so

$$H(x, y, u) + [intC(x)]^c + B^+(0, \varepsilon)e(x) \subseteq H(x, y, u) + \varepsilon e(x) + [intC(x)]^c.$$

This implies that

$$H(x,y,u)+[intC(x)]^c+B^+(0,\varepsilon)e(x)=H(x,y,u)+\varepsilon e(x)+[intC(x)]^c$$

and hence,

$$0\in G(x,y,u)+B^+(0,\varepsilon)e(x)\Longleftrightarrow H(x,y,u)+\varepsilon e(x)\nsubseteq -intC(x).$$

Now it follows that $\Omega(\varepsilon)=\Omega_{(WVEP)}(\varepsilon)$ for all $\varepsilon>0$, and Definitions 3.1 and 3.2 coincide with Definitions 4.3 and 4.4, respectively.

Let $W(x)=[-intC(x)]^c=Z\backslash(-intC(x))$ for all $x\in X$. If H is $u.s.c.$ and compact-valued and W is closed, then, by Lemma 2.2, G is closed. From Theorems 3.1-3.5, we have the following results concerned with LP well-posedness for (WVEP).

Theorem 4.6. *Let E be complete. Assume that K is continuous and compact-valued, T and H are $u.s.c.$ and compact-valued and W is closed. Then $($WVEP$)$ is LP well-posed if and only if*

$$\Omega_{(WVEP)}(\varepsilon)\neq\emptyset,\ \forall\,\varepsilon>0,\ \ \textit{and}\ \ \mathrm{diam}(\Omega_{(WVEP)}(\varepsilon))\to 0\ \textit{as}\ \varepsilon\to 0.$$

Theorem 4.7. *Let E be complete. Assume that K is continuous and compact-valued, T and H are $u.s.c.$ and compact-valued and W is closed. Then $($WVEP$)$ is LP well-posed in the generalized sense if and only if*

$$\Omega_{(WVEP)}(\varepsilon)\neq\emptyset,\ \forall\,\varepsilon>0,\ \ \textit{and}\ \ \mu(\Omega_{(WVEP)}(\varepsilon))\to 0\ \textit{as}\ \varepsilon\to 0.$$

Theorem 4.8. *(WVEP) is well-posed in the generalized sense if and only if $S_{(WVEP)}$ is nonempty and compact and $e(\Omega_{(WVEP)}(\varepsilon),S_{(WVEP)})\to 0$ as $\varepsilon\to 0$.*

Theorem 4.9. *Let E be finite-dimensional. Assume that*

(i) *K is continuous and closed-valued;*

(ii) *T and H are $u.s.c.$ and compact-valued;*

(iii) *W is closed;*

(iv) *there exists $\varepsilon>0$ such that $\Omega_{(WVEP)}(\varepsilon)$ is nonempty and bounded.*

Then $($WVEP$)$ is LP well-posed if and only if $($WVEP$)$ has a unique solution.

Theorem 4.10. *Let E be finite-dimensional. Assume that*

(i) *K is continuous and closed-valued;*

(ii) *T and H are $u.s.c.$ and compact-valued;*

(iii) *W is closed;*

(iv) *there exists $\varepsilon>0$ such that $\Omega_{(WVEP)}(\varepsilon)$ is nonempty and bounded.*

Then $(WVEP)$ *is LP well-posed in the generalized sense if and only if* $(WVEP)$ *has a nonempty solution set* $S_{(WVEP)}$.

Remark 4.5. In Theorems 4.6 and 4.7, (i) if E is finite-dimensional, then the condition "K is continuous and compact-valued" can be weaken by "K is continuous and closed-valued"; (ii) if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed but not necessarily compact, and the condition "H is $u.s.c.$ and compact-valued" can be weakened by "$(x, y) \multimap H(x, y, u)$ is $u.s.c.$ and compact-valued".

Remark 4.6. (a) By Remark 4.5 (ii), we know that Theorems 4.7 and 4.10 generalize Theorems 3.1 and 3.3 of Li and Li [24], respectively; (b) Theorem 4.8 ia a generalization of Theorem 3.3 in Huang et al. [13].

4.3. LP well-posedness for strong vector equilibrium problems

Let $C : X \multimap Z$ be a multivalued mapping such that for any $x \in X$, $C(x)$ is a proper, closed and convex cone in Z. Let $e : X \to Z$ be a continuous mapping such that for any $x \in X$, $e(x) \in C(x)$. Let $H : X \times Y \times X \multimap Z$ be a multivalued mapping.

We consider the following strong vector equilibrium problems.

(SVEP): Find $x \in X_0$ such that $x \in K(x)$ and there exists $y \in T(x)$ satisfying

$$H(x, y, u) \subseteq C(x), \ \forall\, u \in K(x).$$

Denote by $S_{(SVEP)}$ the solution set of (SVEP).

Definition 4.5. A sequence $\{x_n\} \subseteq X$ is said to be an LP approximating solution sequence for (SVEP) if there exist a sequence $\{\varepsilon_n\}$ of real positive numbers with $\varepsilon_n \to 0$ and a sequence $\{y_n\}$ with $y_n \in T(x_n)$ such that, for each $n \in N$, $d(x_n, X_0) \leq \varepsilon_n$ and $d(x_n, K(x_n)) \leq \varepsilon_n$ with

$$H(x_n, y_n, u) + \varepsilon_n e(x_n) \subseteq C(x_n), \ \forall\, u \in K(x_n).$$

Definition 4.6. (SVEP) is said to be LP well-posed if (SVEP) has a unique solution x, and every LP approximating solution sequence for (SVEP) converges to x, and (SVEP) is said to be LP well-posed in the generalized sense if (SVEP) has a nonempty solution set $S_{(SVEP)}$, and every LP approximating solution sequence for (SVEP) has some subsequence which converges to some point of $S_{(SVEP)}$.

Remark 4.7. If $X_0 = X$ and for every $(x, y, u) \in X \times Y \times X$, $intC(x) \neq \emptyset$, $e(x) \in intC(x)$ and $H(x, y, u) = -H(x, u)$, then (SVEP) reduces to (GVQEP 2) of Li et al. [25]. Thus, LP approximating solution sequence for (SVEP) reduces to LP approximating solution sequence for (GVQEP 2) of Li et al. [25] and LP well-posedness in the generalized sense of (SVEP) reduces to LP well-posedness of (GVQEP 2) of Li et al. [25].

Define the approximating solution set for (SVEP) by

$$\Omega_{(SVEP)}(\varepsilon) = \{x \in X : d(x, X_0) \leq \varepsilon,\ d(x, K(x)) \leq \varepsilon \text{ and } \exists\, y \in T(x) \text{ such that } H(x, y, u) + \varepsilon e(x) \subseteq C(x),\ \forall\, u \in K(x)\}, \quad \forall\, \varepsilon > 0.$$

Let $G(x, y, u) = [H(x, y, u) - (C(x))^c]^c$ for all $(x, y, u) \in X \times Y \times X$. Then, (GQVIP) reduces to (SVEP), and so $S = S_{(SVEP)}$, $\Omega(\varepsilon) = \Omega_{(SVEP)}(\varepsilon)$ for all $\varepsilon > 0$, and Definitions 3.1 and 3.2 reduce to Definitions 4.5 and 4.6, respectively. Indeed, $S = S_{(SVEP)}$ is trival. Next, we show that, for each $\varepsilon > 0$,

$$[H(x, y, u) - (C(x))^c]^c + B^+(0, \varepsilon)e(x) = [H(x, y, u) - (C(x))^c]^c + \varepsilon e(x).$$

Take any $\varepsilon > 0$ and let ε be fixed. Clearly,

$$[H(x, y, u) - (C(x))^c]^c + B^+(0, \varepsilon)e(x) \supseteq [H(x, y, u) - (C(x))^c]^c + \varepsilon e(x).$$

Thus, we only need to show that

$$[H(x, y, u) - (C(x))^c]^c + B^+(0, \varepsilon)e(x) \subseteq [H(x, y, u) - (C(x))^c]^c + \varepsilon e(x).$$

Indeed, if $z \in [H(x, y, u) - (C(x))^c]^c + B^+(0, \varepsilon)e(x)$, then there exists some $\gamma \in B^+(0, \varepsilon)$ such that

$$z \in [H(x, y, u) - (C(x))^c]^c + \gamma e(x).$$

Noting that $\gamma \leq \varepsilon$ and $e(x) \in C(x)$, we have $(\gamma - \varepsilon)e(x) - [C(x)]^c \supseteq -[C(x)]^c$, and so

$$\begin{aligned}
& [H(x, y, u) - (C(x))^c]^c + \gamma e(x) \\
= \ & [H(x, y, u) - (C(x))]^c + \varepsilon e(x) + (\gamma - \varepsilon)e(x) \\
= \ & [H(x, y, u) - (C(x))^c + (\gamma - \varepsilon)e(x)]^c + \varepsilon e(x) \\
\subseteq \ & [H(x, y, u) - (C(x))^c]^c + \varepsilon e(x).
\end{aligned}$$

It follows that

$$z \in [H(x, y, u) - (C(x))^c]^c + \varepsilon e(x).$$

This implies that

$$[H(x, y, u) - (C(x))^c]^c + B^+(0, \varepsilon)e(x) \subseteq [H(x, y, u) - (C(x))^c]^c + \varepsilon e(x).$$

Thus,

$$[H(x, y, u) - (C(x))^c]^c + B^+(0, \varepsilon)e(x) = [H(x, y, u) - (C(x))^c]^c + \varepsilon e(x).$$

Therefore,

$$0 \in G(x, y, u) + B^+(0, \varepsilon)e(x) \Longleftrightarrow H(x, y, u) + \varepsilon e(x) \subseteq C(x).$$

From this, it is easy to know that $\Omega(\varepsilon) = \Omega_{(SVEP)}(\varepsilon)$ for all $\varepsilon > 0$, and Definitions 3.1 and 3.2 coincide with Definitions 4.5 and 4.6, respectively.

If H is $l.s.c.$ and C is closed, then, by Lemma 2.2, it is easy to know that G is closed. Thus, by Theorems 3.1-3.5, we can obtain the following results of LP well-posedness for (SVEP).

Theorem 4.11. *Let E be complete. Assume that K is continuous and compact-valued, T is $u.s.c.$ and compact-valued, H is $l.s.c.$ and C is closed. Then (SVEP) is LP well-posed if and only if*

$$\Omega_{(SVEP)}(\varepsilon) \neq \emptyset, \ \forall \varepsilon > 0, \quad \textit{and} \quad \text{diam}(\Omega_{(SVEP)}(\varepsilon)) \to 0 \textit{ as } \varepsilon \to 0.$$

Theorem 4.12. *Let E be complete. Assume that K is continuous and compact-valued, T is $u.s.c.$ and compact-valued, H is $l.s.c.$ and C is closed. Then (SVEP) is LP well-posed in the generalized sense if and only if*

$$\Omega_{(SVEP)}(\varepsilon) \neq \emptyset, \ \forall \varepsilon > 0, \quad \textit{and} \quad \mu(\Omega_{(SVEP)}(\varepsilon)) \to 0 \textit{ as } \varepsilon \to 0.$$

Theorem 4.13. $($*SVEP*$)$ *is well-posed in the generalized sense if and only if $S_{(SVEP)}$ is nonempty and compact and $e(\Omega_{(SVEP)}(\varepsilon), S_{(SVEP)}) \to 0$ as $\varepsilon \to 0$.*

Theorem 4.14. *Let E be finite-dimensional. Assume that*

(i) *K is continuous and closed-valued;*

(ii) *T is $u.s.c.$ and compact-valued;*

(iii) *H is $l.s.c.$;*

(iv) *C is closed;*

(v) *there exists $\varepsilon > 0$ such that $\Omega_{(SVEP)}(\varepsilon)$ is nonempty and bounded.*

Then (SVEP) is LP well-posed if and only if (SVEP) has a unique solution.

Theorem 4.15. *Let E be finite-dimensional. Assume that*

(i) *K is continuous and closed-valued;*

(ii) *T is $u.s.c.$ and compact-valued;*

(iii) *H is $l.s.c.$;*

(iv) *C is closed;*

(v) *there exists $\varepsilon > 0$ such that $\Omega_{(SVEP)}(\varepsilon)$ is nonempty and bounded.*

Then (*SVEP*) *is LP well-posed in the generalized sense if and only if* (*SVEP*) *has a nonempty solution set* $S_{(SVEP)}$.

Remark 4.8. In Theorems 4.11 and 4.12, (i) if E is finite-dimensional, then the condition "K is continuous and compact-valued" can be weaken by "K is continuous and closed-valued"; (ii) if K is a constant mapping, i.e., $K(x) \equiv \tilde{X}$ ($\tilde{X}$ is a subset of X) for all $x \in X$, then $\tilde{X}$ is only need to be assumed to be closed but not necessarily compact, and the condition "H is $l.s.c.$" can be weakened by "$(x, y) \multimap H(x, y, u)$ is $l.s.c.$".

Remark 4.9. If $X_0 = X$ and for every $(x, y, u) \in X \times Y \times X$, $H(x, y, u) = -H(x, u)$, then (SVEP) reduces to (GVQEP 2) of Li et al. [25], and so, by Theorems 4.11-4.15, we can get some results of LP well-posedness for (GVQEP 2) and LP well-posedness in the generalized sense for (GVQEP 2). However, Li et al. [25] only studied the LP well-posedness in the generalized sense for (GVQEP 2).

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