TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 1, pp. 335-352, February 2012
This paper is available online at http://tjm.math.ntu.edu.tw

LINEAR REGULARITY FOR AN INFINITE SYSTEM FORMED BY *p*-UNIFORMLY SUBSMOOTH SETS IN BANACH SPACES

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Abstract. In this paper, we introduce and study p-uniform subsmoothness of a collection of infinitely many closed sets in a Banach space. Using variational analysis and techniques, we mainly study linear regularity for a collection of infinitely many closed sets satisfying p-uniform subsmoothness. The necessary or/and sufficient conditions on the linear regularity are obtained in this case. In particular, we extend the characterizations of linear regularity for a collection of infinitely many closed convex sets to the nonconvex setting.

1. INTRODUCTION

R. A. Poliquin and R. T. Rockafellar [1] introduced and studied the concept of prox-regularity for functions and sets in the finite-dimensional context. This notion is an extension of convexity and has been extensively studied by many authors (see [2, 3, 4] and references therein). Aussel, Daniilidis and Thibaut [5] introduced and studied the notion of subsmoothness for a closed set which is an extension of prox-regularity and smoothness, and established several interesting and valuable properties for approximate convex functions and submonotone subdifferential mappings therein. Recently, the authors [6] introduced and considered the uniform subsmoothness of infinitely-many closed subsets in Banach spaces, and used it to study the interrelationship among linear regularity, property(G), CHIP and strong CHIP. Motivated by [5] and [6], in this paper, we introduce and consider the p-uniform subsmoothness for a collection of infinitely many closed sets in Banach space setting.

Received September 1, 2010, accepted November 22, 2010.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: 90C31, 90C25, 49J52, 46B20.
Key words and phrases: Linear regularity, Subsmoothness, Clarke subdifferential, Normal cone, Asplund space.
This research was supported by the National Natural Science Foundation of P. R. China (Grant No. 11061039).

The concept of linear regularity is well-known in mathematical programming since it plays an important role in metric regularity/subregularity, error bounds and approximation theory. In particular, it is utilized to establish a linear convergence rate of iterates generated by the cyclic projection algorithm for finding the projection from a point to the intersection of finitely many closed convex sets (see [7] and references therein). In early 1970s, Jameson [8] presented a characterization for the linear regularity of two closed convex cones. In terms of Jameson's property(G), Bauschke, Borwein and Li [9] provided a characterization of the linear regularity for a finite system of closed convex cones. Recently, Ng and Yang [10] extended the results in [9] to a finite collection of closed convex sets in a Banach space. Furthermore Li, Ng and Pong [11] and Zheng and Ng [12] studied the linear regularity for a collection of infinitely many closed convex sets in a Banach space, respectively. In [12], Zheng and Ng introduced the notion of weak* p-sum for infinitely many closed convex sets in dual spaces and generalized Jameson's property(G) to an infinite system of closed convex cones of a Banach space. Zheng and Ng considered the local linear regularity for the nonconvex setting in [13] where the case of finitely many subsmooth sets was studied and several necessary and/or sufficient conditions for the local linear regularity of this case were given. They further in [14] introduced the notion of L-subsmoothness for locally Lipschizian functions and studied metric regularity for this class of functions. Inspired by [6, 12, 13] and [14], in this paper, we mainly study the case of infinitely many closed sets in nonconvex setting, and provide some sufficient and/or necessary conditions for the local linear regularity of a collection formed by infinitely many closed sets satisfying p-uniform subsmoothness.

The paper is organized as follows. In Section 2, we recall some notions in variational analysis and approximate projection theorems established recently in [13], which will be of use in the proof of our main results. In Section 3, we introduce and study a notion of p-uniform subsmoothness. Then, we provide necessary and/or sufficient conditions for the p-local linear regularity of a collection of infinitely many closed sets with the assumption of p-uniform subsmoothness.

2. Preliminaries

Let X be a Banach space with topological dual X^*, and $\langle \cdot, \cdot \rangle$ be the duality pairing between X and X^*. Let B_X and B_{X^*} denote the closed unit balls of X and X^*, respectively. For a nonempty subset A of X, we denote ∂A the boundary of A with respect to the norm topology.

Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $x \in \mathrm{dom}(\phi) := \{y \in X : \phi(y) < +\infty\}$ and $h \in X$. We denote the generalized Rockafellar directional derivative of ϕ at x along the direction h by $\phi^\circ(x; h)$ which is defined by (see [15])

$$\phi^\circ(x;h) := \lim_{\varepsilon\downarrow 0}\limsup_{z\xrightarrow{\phi}x,\, t\downarrow 0}\inf_{w\in h+\varepsilon B_X}\frac{\phi(z+tw)-\phi(z)}{t},$$

where $z \xrightarrow{\phi} x$ means that $z \to x$ and $\phi(z) \to \phi(x)$. When ϕ is locally Lipschitzian around x, $\phi^\circ(x;h)$ reduces to Clarke's directional derivative; that is

$$\phi^\circ(x;h) = \limsup_{z\to x,\, t\downarrow 0}\frac{\phi(z+th)-\phi(z)}{t}.$$

Recall [16] that ϕ is regular at x if ϕ is Lipschitz around x and admits directional derivatives $\phi'(x;h)$ at x for all $h \in X$ with $\phi'(x;h) = \phi^\circ(x;h)$, where $\phi'(x;h)$ is defined by

$$\phi'(x;h) := \lim_{t\to 0^+}\frac{\phi(x+th)-\phi(x)}{t}.$$

The Clarke subdifferential of ϕ at x is defined by

$$\partial_c\phi(x) := \{x^* \in X^* : \langle x^*, h\rangle \le \phi^\circ(x;h)\ \forall h \in X\},$$

and the Fréchet subdifferential of ϕ at x is defined by

$$\hat\partial\phi(x) := \Big\{x^* \in X^* : \liminf_{y\to x}\frac{\phi(y)-\phi(x)-\langle x^*, y-x\rangle}{\|y-x\|} \ge 0\Big\}.$$

Let A be a closed subset of X and $a \in A$. The Clarke normal cone of A at a, denoted by $N_c(A,a)$, is defined by

$$N_c(A,a) := \partial_c\delta_A(a),$$

where δ_A denotes the indicator function of A; that is $\delta_A(y) = 0$ if $y \in A$ and $\delta_A(y) = +\infty$ if $y \notin A$. For $\varepsilon \ge 0$, the set of ε-normal to A at a is defined by

$$\hat N_\varepsilon(A,a) := \{x^* \in X^* : \limsup_{y\xrightarrow{A}a}\frac{\langle x^*, y-a\rangle}{\|y-a\|} \le \varepsilon\},$$

where $y \xrightarrow{A} a$ means $y \to a$ and $y \in A$. When $\varepsilon = 0$, $\hat N_\varepsilon(A,a)$ is a convex cone which is called the Fréchet normal cone of A at a and is denoted by $\hat N(A,a)$. It is known (cf.[17, Corollary 1.96]) that $\hat N(A,u) \cap B_{X^*} = \hat\partial d(\cdot, A)(u)$ for all $u \in A$. Hence $x^* \in \hat N(A,u) \cap B_{X^*}$ if and only if for any $\varepsilon > 0$ there exists $r > 0$ such that

$$\langle x^*, x-u\rangle \le d(x,A) + \varepsilon\|x-u\| \quad \forall x \in B(u,r). \tag{2.1}$$

When A is convex, one has

$$\hat{N}(A,a) = N_c(A,a) = \{x^* \in X^* : \langle x^*, x-a\rangle \le 0\ \forall x \in A\}.$$

Recall that a Banach space X is called an Asplund space if every continuous convex function defined on an open convex subset D of X is Fréchet differentiable at each point of a dense G_δ subset of D. It is well known that X is an Asplund space if and only if every separable subspace of X has a separable dual space (cf.[18]). In particular, every reflexive Banach space is an Asplund space.

Recall that a closed set A in X is said to be subsmooth at $a \in A$ if for any $\varepsilon > 0$ there exists $r > 0$ such that

$$\langle x^* - u^*, x - u\rangle \ge -\varepsilon\|x-u\|$$

whenever $x, u \in A \cap B(a,r)$, $x^* \in N_c(A,x) \cap B_{X^*}$ and $u^* \in N_c(A,u) \cap B_{X^*}$.

It follows from [13] that if A is subsmooth at a, then A is Clarke regular at a; that is

$$\text{subsmoothness of } A \text{ at } a \Longrightarrow N_c(A,a) = \hat{N}(A,a). \tag{2.2}$$

The following approximate projection results(recently established in [13]) will be useful in the proofs of our main results.

Lemma 2.1. *Let X be a Banach space (resp., an Asplund space) and A be a closed nonempty subset of X. Let $\gamma \in (0,1)$. Then for any $x \notin A$ there exist $a \in \partial A$ and $a^* \in N_c(A,a)$(resp., $a^* \in \hat{N}(A,a)$) with $\|a^*\| = 1$ such that*

$$\gamma\|x-a\| < \min\Big\{d(x,A), \langle a^*, x-a\rangle\Big\}.$$

In Sections 3, we will need the following inequality.

Lemma 2.2. *Let $p \in [1,+\infty)$. Then there exists $M = M(p) > 0$ such that*

$$(a+b)^p \le M\big(|a|^p + |b|^p\big)\ \forall\, a, b \in \mathbb{R}. \tag{2.3}$$

Taking $M = 2^p$ and by virtue of the trivial inequality $a+b \le 2\max\big\{|a|,|b|\big\}$, the proof can be obtained.

3. p-Local Linear Regularity of p-Uniformly Subsmooth Sets

In this section, we study p-local linear regularity of a collection of infinitely many closed sets in Banach space X. Let I be an arbitrary nonempty index and let $p \in [1,+\infty)$. Recall that $l^p(I)$ is a classic Banach space and its interesting and important properties can be found in Day [19]. We denote

$$l^p_+(I) := \{(t_i)_{i\in I} \in l^p(I) : t_i \ge 0\ \forall i \in I\}.$$

We first recall the notion of p-local linear regularity (cf.[12]).

Definition 3.1. Let $\{A_i : i \in I\}$ be a collection of closed sets in X. Assume that $A := \bigcap_{i\in I} A_i$ is nonempty. Let $p \in [1, +\infty)$. We say that the collection $\{A_i : i \in I\}$ is p-locally linearly regular at $a \in A$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, A) \le \tau\Big(\sum_{i\in I}(d(x, A_i))^p\Big)^{\frac{1}{p}} \quad \forall x \in B(a, \delta). \tag{3.1}$$

Note that (3.1) holds trivially if $\big(\sum_{i\in I}(d(x, A_i))^p\big)^{\frac{1}{p}} = +\infty$, so we are inspired to consider the general case and it is necessary to study the following concept introduced in [12].

Definition 3.2. We say that $d(\cdot, A_i)_{i\in I}$ is of type l^p if $(d(x, A_i))_{i\in I} \in l^p(I)$ for each $x \in X$.

In order to study p-local linear regularity for the collection of closed sets, we introduce a new notion of p-uniform subsmoothness which is inspired by the definition of subsmoothness ([cf. [4, 6, 7, 13 and references therein]).

Definition 3.3. Let $\{A_i : i \in I\}$ be a collection of closed sets in X. Suppose that $A := \bigcap_{i\in I} A_i$ is nonempty. We say that
(i) the collection $\{A_i : i \in I\}$ is p-uniformly subsmooth at $a \in A$, if for any $\varepsilon > 0$ there exist $\delta > 0$ and $(\omega_i)_{i\in I} \in l^p(I)$ with $\sum_{i\in I} |\omega_i|^p \le 1$ such that whenever $i \in I$, $a_i \in A_i \cap B(a, \delta)$ and $a_i^* \in N_c(A_i, a_i) \cap B_{X^*}$, one has

$$\langle a_i^*, x - a_i\rangle \le |\omega_i|\varepsilon\|x - a_i\| \quad \forall\, x \in A_i \cap B(a, \delta); \tag{3.2}$$

(ii) the collection $\{A_i : i \in I\}$ is p-uniformly subsmooth on A, if $\{A_i : i \in I\}$ is p-uniformly subsmooth at each $a \in A$.

It is easy to verify from the definition that the collection $\{A_i : i \in I\}$ is p-uniformly subsmooth on A if each A_i is closed and convex.

The following proposition gives a characterization for the notion of p-uniform subsmoothness.

Proposition 3.1. *Let X be a Banach space and $\{A_i : i \in I\}$ be a collection of closed subsets in X. Suppose that $A := \bigcap_{i\in I} A_i$ is nonempty. Then $\{A_i : i \in I\}$ is p-uniformly subsmooth at $a \in A$ if and only if for any $\varepsilon > 0$ there exist $\delta > 0$ and $(\omega_i)_{i\in I} \in B_{l^p(I)}$ such that whenever $i \in I$, $a_i \in A_i \cap B(a, \delta)$ and $a_i^* \in N_c(A_i, a_i) \cap B_{X^*}$, one has*

$$\langle a_i^*, x - a_i\rangle \le d(x, A_i) + |\omega_i|\varepsilon\|x - a_i\| \quad \forall\, x \in B(a, \delta). \tag{3.3}$$

Proof. Note that $d(x, A_i) = 0$ for all $x \in A_i$; so the sufficiency part follows from that (3.3) implies (3.2). Conversely, suppose that $\{A_i : i \in I\}$ is p-uniformly subsmooth at a. Let any $\varepsilon \in (0, +\infty)$. Then there exist $\delta > 0$ and $(\omega_i)_{i\in I} \in B_{l^p(I)}$ such that whenever $i \in I$, $a_i \in A_i \cap B(a, \delta)$ and $a_i^* \in N_c(A_i, a_i) \cap B_{X^*}$, one has

$$\langle a_i^*, x - a_i\rangle \le \frac{|\omega_i|\varepsilon}{2}\|x - a_i\| \quad \forall x \in B(a, 2\delta) \cap A_i. \tag{3.4}$$

Fix $i \in I$, and let $x \in B(a, \delta)$, $a_i \in A_i \cap B(a, \delta)$ and $a_i^* \in N_c(A_i, a_i) \cap B_{X^*}$. Noting that $d(x, A_i) \le \|x - a\| < \delta$, one can take a sequence $\{u_n\} \subset A_i \cap B(x, \delta)$ such that $\|x - u_n\| \to d(x, A_i)$. Since $\|u_n - a\| \le \|u_n - x\| + \|x - a\| < 2\delta$, it follows from (3.4) that

$$\begin{aligned}\langle a_i^*, x - a_i\rangle &= \langle a_i^*, x - u_n\rangle + \langle a_i^*, u_n - a_i\rangle \\ &\le \|x - u_n\| + \frac{|\omega_i|\varepsilon}{2}\|u_n - a_i\| \\ &\le \|x - u_n\| + \frac{|\omega_i|\varepsilon}{2}(\|u_n - x\| + \|x - a_i)\|\end{aligned}$$

Taking limits as $n \to \infty$, one has

$$\langle a_i^*, x - a_i\rangle \le d(x, A_i) + \frac{|\omega_i|\varepsilon}{2}(d(x, A_i) + \|x - a_i\|) \le d(x, A_i) + |\omega_i|\varepsilon\|x - a_i\|.$$

This shows that the necessity part holds. The proof is completed. ■

Let $\{C_i : i \in J\}$ be a family of subsets of X containing the origin. The set $\sum\limits_{i\in J} C_i$ is defined by

$$\sum_{i\in J} C_i := \begin{cases} \big\{\sum_{i\in J_0} a_i : a_i \in C_i, \emptyset \ne J_0 \subset J \text{ being finite}\big\} & \text{if } J \ne \emptyset \\ \{0\} & \text{if } J = \emptyset\end{cases}$$

Proposition 3.2. *Let X be a Banach space, $\{A_i : i \in I\}$ be a collection of closed subsets in X, $a \in A := \bigcap_{i\in I} A_i$ and $p, q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\{A_i : i \in I\}$ is p-uniformly subsmooth at a and that $d(\cdot, A_i)_{i\in I}$ is of type l^p. Then for any $\mu = (\mu_i)_{i\in I} \in l_+^q(I)$ with $\|\mu\| \le 1$, one has*

$$\overline{\sum_{i\in I} \mu_i(N_c(A_i, a) \cap B_{X^*})}^{w^*} \subset \hat{\partial}\Big(\big(\sum_{i\in I} d(\cdot, A_i)^p\big)^{\frac{1}{p}}\Big)(a).$$

Proof. Let x^* be an arbitrary point in $\overline{\sum_{i\in I} \mu_i(N_c(A_i, a) \cap B_{X^*})}^{w^*}$ and take a generalized sequence $\{x_k^*\} \subset \sum_{i\in I} \mu_i(N_c(A_i, a)\cap B_{X^*})$ such that $x_k^* \xrightarrow{w^*} x^*$. Then, for each k, there exist a finite subset $I_k \subset I$, $x_k^*(j) \in N_c(A_j, a) \cap B_{X^*}(j \in I_k)$ such that

$$x_k^* = \sum_{j\in I_k} \mu_j x_k^*(j).$$

Since $\{A_i : i \in I\}$ is p-uniformly subsmooth, for each $\varepsilon > 0$ there exist $\delta > 0$ and $(\omega_i)_{i\in I} \in B_{l^p(I)}$ such that (3.3) holds. Thus, for any $x \in B(a,\delta)$, by (3.3) and Hölder inequality, one has

$$\begin{aligned}\langle x_k^*, x-a\rangle &\le \sum_{j\in I_k} \mu_j \langle x_k^*(j), x-a\rangle \le \sum_{j\in I_k} \mu_j \big(d(x,A_j) + |\omega_j|\varepsilon\|x-a\|\big)\\ &\le \Big(\sum_{i\in I} \mu_i^q\Big)^{\frac{1}{q}} \Big(\big(\sum_{i\in I} d(x,A_i)^p\big)^{\frac{1}{p}} + \varepsilon\|x-a\| \Big(\sum_{i\in I} |\omega_i|^p\Big)^{\frac{1}{p}}\Big)\\ &\le \Big(\big(\sum_{i\in I} d(x,A_i)^p\Big)^{\frac{1}{p}} + \varepsilon\|x-a\|\end{aligned}$$

(thanks to $\|\mu\| \le 1$ and $(\omega_i)_{i\in I} \in B_{l^p(I)}$). By passing to the limits, one has

$$\langle x^*, x-a\rangle \le \Big(\sum_{i\in I} d(x,A_i)^p\Big)^{\frac{1}{p}} + \varepsilon\|x-a\|$$

for all $x \in B(a,\delta)$. This implies that $x^* \in \hat{\partial}\Big(\big(\sum_{i\in I} d(\cdot,A_i)^p\big)^{\frac{1}{p}}\Big)(a)$. The proof is completed. ■

Using the results presented in Section 2, we will provide necessary or/and sufficient conditions for p-local linear regularity under the assumption of p-uniform subsmoothness. First, we need to establish the following lemmas which are of some independent interests and inspired by [12, Lemma 3.1 and Lemma 3.2].

Lemma 3.1. *Let $\{A_i : i \in I\}$ be a collection of closed sets of a Banach space X such that $A := \bigcap_{i\in I} A_i$ is nonempty. Let $a \in A$ and $p, q \in (1,+\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\{A_i : i \in I\}$ is p-uniformly subsmooth at a and that $d(\cdot,A_i)_{i\in I}$ is of type l^p. Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$\phi(x) := \Big(\sum_{i\in I} d(x,A_i)^p\Big)^{\frac{1}{p}} \quad \forall x \in X.$$

Then

$$\phi'(a;h) = \Big(\sum_{i\in I} d^\circ_{A_i}(a;h)^p\Big)^{\frac{1}{p}} \quad \forall h \in X.$$

Proof. We first show that $d(\cdot,A_i)$ is regular at a for each $i \in I$; that is

$$d^\circ_{A_i}(a;h) = d'_{A_i}(a;h) \quad \forall h \in X, \tag{3.5}$$

where $d^\circ_{A_i}(a;h)$ and $d'_{A_i}(a;h)$ denote the Clarke's directional derivative and the directional derivative of $d(\cdot,A_i)$ at a along the direction h, respectively.

Let $h \in X$ and $\varepsilon \in (0,+\infty)$. Since $(A_i)_{i\in I}$ is p-uniformly subsmooth at a, for each $\varepsilon > 0$ there exist $\delta_1 \in (0,+\infty)$ and $(\omega_i)_{i\in I} \in B_{l^p(I)}$ such that whenever $i \in I$, $a_i \in A_i \cap B(a,\delta_1)$ and $a_i^* \in N_c(A_i,a_i) \cap B_{X^*}$, one has

$$\langle a_i^*, z-a_i\rangle \le d(z,A_i) + |\omega_i|\varepsilon\|a-a_i\| \quad \forall\, z \in B(a,\delta_1). \tag{3.6}$$

Fix $i \in I$ and take $t > 0$ sufficiently small such that $a+th \in B(a,\delta_1)$. Noting that $\partial_c d(\cdot,A_i)(a) \subset N_c(A_i,a)\cap B_{X^*}$(cf. [15, Proposition 2.4.2]), it follows from (3.6) and [15, Proposition 2.1.1] that

$$d^\circ_{A_i}(a;h) \le \frac{d(a+th,A_i)}{t} + \varepsilon\|h\|,\ \forall\, t>0 \text{ small enough} \tag{3.7}$$

(thanks to $d(x,A_i)=0$). Taking limits as $\varepsilon \to 0^+$, one has

$$d^\circ_{A_i}(a;h) \le \liminf_{t\to 0^+} \frac{d(a+th,A_i)}{t}. \tag{3.8}$$

On the other hand, from the definition of Clarke's directional derivative, one has

$$\limsup_{t\to 0^+} \frac{d(a+th,A_i)}{t} \le d^\circ_{A_i}(a;h).$$

This and (3.8) imply that $d^\circ_{A_i}(a;h) = d'_{A_i}(a;h)$.

Next, we show that for each $h \in X$, one has

$$\phi'(a;h) = \Big(\sum_{i\in I}\big(d^\circ_{A_i}(a;h)\big)^p\Big)^{\frac{1}{p}}. \tag{3.9}$$

Let $h \in X$ and $\varepsilon \in (0,\frac{1}{2})$. By Proposition 3.1, there exist $\delta_2 > 0$ and $(\omega_i)_{i\in I} \in B_{l^p(I)}$ such that whenever $i \in I$, $a_i \in A_i \cap B(a,\delta_2)$ and $a_i^* \in N_c(A_i,a_i)\cap B_{X^*}$, one has

$$\langle a_i^*, y-a_i\rangle \le d(y,A_i) + \frac{|\omega_i|}{2}\varepsilon\|x-a_i\| \quad \forall\, y \in B(a,\delta_2). \tag{3.10}$$

Take $\delta_3 \in (0,\frac{\delta_2}{2})$ such that $\delta_3\|h\| \in (0,\frac{\delta_2}{2})$. Let $t \in (0,\delta_3]$. We denote $a+th$ by z_t. Fix $i \in I$ and we consider that $z_t \in B(a,\delta_3\|h\|)\setminus A_i$. Then $d(z_t,A_i) \le \|z_t-a\| < \delta_3\|h\|$. Let $\gamma \in (\max\{\frac{d(z_t,A_i)}{\delta_3\|h\|},\varepsilon\},1)$. By Lemma 2.1, there exist $z \in \partial A_i$ and $z^* \in N_c(A_i,z)$ with $\|z^*\|=1$ such that

$$\gamma\|z_t-z\| < \min\Big\{\langle z^*, z_t-z\rangle, d(z_t,A_i)\Big\}. \tag{3.11}$$

Thus,

$$\|z-a\| \le \|z-z_t\| + \|z_t-a\| < \frac{d(z_t,A_i)}{\gamma} + \delta_3\|h\| < \delta_2.$$

Note that

$$z_t = \frac{t}{\delta_3}(a+\delta_3 h) + \frac{\delta_3 - t}{\delta_3}a \text{ and } a+\delta_3 h \in B(a,\delta_2).$$

By (3.10), one has

$$\begin{aligned}
\gamma\|z_t - z\| &< \langle z^*, z_t - z\rangle = \frac{t}{\delta_3}\langle z^*, (a+\delta_3 h) - z\rangle + \frac{\delta_3 - t}{\delta_3}\langle z^*, a - z\rangle \\
&\le \frac{t}{\delta_3} d(a+\delta_3 h, A_i) + \frac{|\omega_i|}{2}\varepsilon\Big(\frac{t}{\delta_3}\|a+\delta_3 h - z\| + \frac{\delta_3 - t}{\delta_3}\|a - z\|\Big) \\
&\le \frac{t}{\delta_3} d(a+\delta_3 h, A_i) + |\omega_i|\varepsilon\frac{t}{\delta_3}\frac{\delta_3 - t}{\delta_3}\delta_3\|h\| + \frac{|\omega_i|}{2}\varepsilon\|z_t - z\| \\
&\le \frac{t}{\delta_3} d(a+\delta_3 h, A_i) + |\omega_i| t\varepsilon\|h\| + \varepsilon\|z_t - z\|.
\end{aligned}$$

This implies that

$$(\gamma - \varepsilon) d(z_t, A_i) \le \frac{t}{\delta_3} d(x+\delta_3 h, A_i) + |\omega_i| t\varepsilon\|h\|$$

Taking limits as $\gamma \to 1^-$, one has

$$\begin{aligned}
\frac{d(a+th, A_i)}{t} &\le \frac{1}{1-\varepsilon}\frac{d(a+\delta_3 h, A_i)}{\delta_3} + \frac{\varepsilon}{1-\varepsilon}|\omega_i|\|h\| \\
&\le \frac{2}{\delta_3} d(a+\delta_3 h, A_i) + 2|\omega_i|\varepsilon\|h\|.
\end{aligned}$$

By using Lemma 2.2, there exists $M = M(p) > 0$ such that for any $t \in (0, \delta_3]$, one has

$$\begin{aligned}
\sum_{i\in I}\Big(\frac{d(x+th, A_i)}{t}\Big)^p &\le M\left((\frac{2}{\delta_3})^p \sum_{i\in I}\Big(d(a+\delta_3 h, A_i)\Big)^p + (2\varepsilon\|h\|)^p \sum_{i\in I}|\omega_i|^p\right) \\
&< +\infty
\end{aligned}$$

since $(d(\cdot, A_i))_{i\in I}$ is of type l^p and $\sum_{i\in I}|\omega_i|^p \le 1$. This and (3.5) imply that

$$\lim_{t\to 0^+}\sum_{i\in I}\left(\frac{d(a+th, A_i)}{t}\right)^p = \sum_{i\in I}\Big(d^{\circ}_{A_i}(a;h)\Big)^p.$$

Hence

$$\phi'(a;h) = \lim_{t\to 0^+}\frac{\phi(a+th)}{t} = \Big(\sum_{i\in I}\big(d^{\circ}_{A_i}(x;h)\big)^p\Big)^{\frac{1}{p}}$$

(thanks to $\phi(a) = 0$). The proof is completed. ∎

Let P and Q be metric spaces. Recall that a set-valued mapping $F : P \to 2^Q$ is lower semicontinuous if, for any $x_0 \in P, y_0 \in F(x_0)$ and any neighborhood V of y_0, there exists a neighborhood U of x_0 such that $V \cap F(x) \neq \emptyset$ for each $x \in U$. It is clear that $F : P \to 2^Q$ is lower semicontinuous if and only if, for each $y \in Q$, the real-valued function $x \mapsto d(y, F(x))$ is upper semicontinuous(see [20]).

Proposition 3.3. *Let X be a Banach space, I be a metric space and let $\{A_i : i \in I\}$ be a collection of closed sets of X. Suppose that $A := \bigcap_{i\in I} A_i$ is nonempty, $i \mapsto A_i$ is lower semicontinuous and that $\{A_i : i \in I\}$ is p-uniformly subsmooth at $a \in A$. Then, for each $h \in X$, $i \mapsto d^\circ_{A_i}(a; h)$ is upper semicontinuous.*

Proof. Since $\{A_i : i \in I\}$ is p-uniformly subsmooth at a, (3.5) holds; that is

$$d^\circ_{A_i}(a; h) = d'_{A_i}(a; h) \quad \forall h \in X.$$

Let $h \in X$. It suffices to show that for any $i_k \to i_0 \in I$, one has

$$\limsup_{k\to\infty} d'_{A_{i_k}}(a; h) \leq d'_{A_{i_0}}(a; h).$$

Let $\varepsilon \in (0, \frac{1}{2})$. By Proposition 3.1, there exist $\delta_2 \in (0, \varepsilon)$ and $(\omega_i)_{i\in I} \in B_{l^p(I)}$ such that (3.10) holds whenever $i \in I$, $a_i \in A_i \cap B(a, \delta_2)$ and $a_i^* \in N_c(A_i, a_i) \cap B_{X^*}$. Take $\delta_3 \in (0, \frac{\delta_2}{2})$ such that $\delta_3\|h\| \in (0, \frac{\delta_2}{2})$. Let $t \in (0, \delta_3]$. Fix $i_k \in I$ and consider $a + th \in B(a, \delta_3\|h\|)\backslash A_{i_k}$. By the computation in the proof of Lemma 3.1, one has

$$\frac{d(a + th, A_{i_k})}{t} \leq \frac{1}{1-\varepsilon}\frac{d(a + \delta_3 h, A_{i_k})}{\delta_3} + \frac{\varepsilon}{1-\varepsilon}|\omega_{i_k}|\|h\|$$

Noting that $i \mapsto A_i$ is lower semicontinuous, by [20, Corollary 1.4.17], $i \mapsto d(a + \delta_3 h, A_i)$ is upper semicontinuous. Then, for any k large enough, one has

$$d(a + \delta_3 h, A_{i_k}) \leq d(a + \delta_3 h, A_{i_0}) + \delta_3^2.$$

This implies that for any k large enough,

$$\frac{d(a + th, A_{i_k})}{t} \leq \frac{1}{1-\varepsilon}\Big(\frac{d(a + \delta_3 h, A_{i_0})}{\delta_3} + \delta_3\Big) + \frac{\varepsilon}{1-\varepsilon}\|h\|$$

Taking limits as $\varepsilon \to 0^+$, we have

$$\limsup_{k\to\infty} d'_{A_{i_k}}(a; h) \leq d'_{A_{i_0}}(a; h).$$

The proof is completed. ∎

Next, we give several definitions with respect to weak*- summable family. Readers are invited to see [7] for more details.

Definition 3.4. Let $\{x_i^* : i \in I\}$ be a family of elements and $\{A_i : i \in I\}$ be a collection of subsets in X^*. We say that

(i) $\{x_i^* : i \in I\}$ is weak*-summable if there exists $x^* \in X^*$ such that for all $h \in X$, one has
$$\langle x^*, h\rangle = \sum_{i\in I}\langle x_i^*, h\rangle.$$
We denote it by $x^* = \sum_{i\in I}^* x_i^*$.

(ii) $\{A_i : i \in I\}$ is weak*-summable if $\{x_i^* : i \in I\}$ is weak*-summable whenever $\{x_i^* : i \in I\} \subset X^*$ with $x_i^* \in A_i (\forall i \in I)$.
We denote by $\sum_{i\in I}^* A_i$ the set $\Big\{ \sum_{i\in I}^* x_i^* : x_i^* \in A_i, i \in I\Big\}$.

If $(t_iA_i)_{i\in I}$ is weak*-summable for each $(t_i)_{i\in I} \in l_+^p(I)$ with $\sum_{i\in I} t_i^p = 1$, we define $\sum_{i\in I}^p(A_i)$ as

$$\sum\nolimits_{i\in I}^p(A_i) := \bigcup_{(t_i)_{i\in I}\in l_+^p(I),\ \sum\limits_{i\in I} t_i^p=1} \sum\nolimits_{i\in I}^* t_iA_i. \tag{3.12}$$

The following lemma provides a characterization for Clarke's subdifferential of ϕ defined in Lemma 3.1. We will give its proof which goes along the way as [12, Lemma 3.2] with a minor modification for the sake of completeness.

Lemma 3.2. *Let $\{A_i : i \in I\}$ be a collection of closed sets of a Banach space X such that $A := \bigcap_{i\in I} A_i$ is nonempty. Let $a \in A$ and $p, q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\{A_i : i \in I\}$ is p-uniformly subsmooth at a and that $d(\cdot, A_i)_{i\in I}$ is of type l^p. Let ϕ be as in Lemma 3.1. Suppose that ϕ is regular at a. Then*

$$\partial_c\phi(a) = \sum\nolimits_{i\in I}^q\big(\hat{N}(A_i, a) \cap B_{X^*}\big). \tag{3.13}$$

Proof. Since ϕ is regular at a, ϕ is locally Lipschitzian around a and

$$\phi^\circ(a; h) = \phi'(a; h)\ \ \forall h \in X.$$

We denote $d(x, A_i)$ by $f_i(x)$ for each $i \in I$. We claim that

$$\partial_c\phi(a) = \sum\nolimits_{i\in I}^q\big(\partial_c f_i(a)\big). \tag{3.14}$$

We will divide it into three steps to prove (3.14):

Step 1. We show that $\sum_{i\in I}^q(\partial_c f_i(a))$ is well defined.

Applying the proof of Lemma 3.1 and by virtue of [15, Proposition 2.1.2], one has

$$f_i^\circ(a;h) \geq 0 \ \text{ and } \ f_i^\circ(a;h) = \max\Big\{\langle x^*, h\rangle : x^* \in \partial_c f_i(a)\Big\}.$$

Hence, for any subset $J \subset I$, any $(t_i)_{i\in I} \in l_+^q(I)$ with $\sum_{i\in I} t_i^q = 1$, any $x_i^* \in \partial_c f_i(a)(i \in I)$, and any $h \in X$, by Hölder inequality,

$$\sum_{i\in J} t_i\langle x_i^*, h\rangle \leq \sum_{i\in J} t_i f_i^\circ(a;h) \leq \Big(\sum_{i\in J}[f_i^\circ(a;h)]^p\Big)^{\frac{1}{p}} \leq \phi^\circ(a;h), \tag{3.15}$$

(the last inequality is from Lemma 3.1).

Step 2. We prove that $\sum_{i\in I}^q(\partial_c f_i(a))$ is convex and weak* closed.

It is not hard to verify that $\sum_{i\in I}^q(\partial_c f_i(a))$ is convex. It remains to show that it is weak* closed. To do this, let $x^* \in \overline{\sum_{i\in I}^q(\partial_c f_i(a))}^{w^*}$. Then there exist a direct set Λ and nets $(t_i(k))_{k\in\Lambda}$, $(x_i^*(k))_{k\in\Lambda}(i \in I)$ such that $t_i(k) \geq 0, \sum_{i\in I}(t_i(k))^q = 1$, $x_i^*(k) \in \partial_c f_i(a)$, and

$$\sum\nolimits_{i\in I}^* t_i(k)x_i^*(k) \xrightarrow{w^*} x^*. \tag{3.16}$$

Define $g_k := (t_i(k))_{i\in I}(\forall\, k \in \Lambda)$. Noting that $\{g_k\}_{k\in\Lambda}$ is a net in the unit ball of $l^q(I)$, without loss of generality(considering subnet if necessary), we can assume that g_k weak*-converges to some $(\lambda_i)_{i\in I} \in l_+^q(I)$ and $\sum_{i\in I}\lambda_i^q = 1$. Let $I^+ := \{i \in I : \lambda_i > 0\}$. Then I^+ is at most countable. Noting that $\lim\limits_k t_i(k) = \lambda_i = 0$ for each $i \in I\backslash I^+$, it follows from (3.15) that

$$\sum\nolimits_{i\in I\backslash I^+}^* t_i(k)x_i^*(k) \xrightarrow{w^*} 0.$$

This and (3.16) imply that

$$\sum\nolimits_{i\in I^+}^* t_i(k)x_i^*(k) \xrightarrow{w^*} x^*. \tag{3.17}$$

Without loss of generality we can assume I^+ to be the set $\mathbf{N}$ of natural numbers. Noting that $\partial_c f_i(a)$ is weak* compact and $\{x_i^*(k)\}_{k\in\Lambda} \subset \partial_c f_i(a)$ for each $i \in \mathbf{N}$, there exists a subnet $\Lambda_1 \subset \Lambda$ such that $\{x_1^*(k)\}_{k\in\Lambda_1}$ weak*-converges to some $a_1^* \in \partial_c f_1(a)$. Thus there exists a subnet $\Lambda_2 \subset \Lambda_1$ such that $\{x_2^*(k)\}_{k\in\Lambda_2}$ weak*-converges to some $a_2^* \in \partial_c f_2(a),\cdots$. By this way, there must exist a subnet $\Lambda_{n+1} \subset \Lambda_n$ such that $\{x_{n+1}^*(k)\}_{k\in\Lambda_{n+1}}$ weak*-converges to some $a_{n+1}^* \in \partial_c f_{n+1}(a),\cdots$, and so on. We claim that

$$x^* = \sum\nolimits_{i\in\mathbf{N}}^* \lambda_i a_i^*. \tag{3.18}$$

To see this, let $h \in X$ and $\varepsilon > 0$. By (3.15), there exists $n_0 \in \mathbf{N}$ such that

$$\max\bigg\{\Big(\sum_{i=n_0}^\infty[f_i^\circ(a;h)]^p\Big)^{\frac{1}{p}}, \Big(\sum_{i=n_0}^\infty[f_i^\circ(a;-h)]^p\Big)^{\frac{1}{p}}\bigg\} < \varepsilon.$$

Then, for any $n \geq n_0$, any $(t_i)_{i\in\mathbf{N}} \in l_+^q(\mathbf{N})$ with $\sum_{i\in\mathbf{N}} t_i^q = 1$ and any $x_i^* \in \partial_c f_i(a)(i \in \mathbf{N})$, by (3.15), one has

$$\left|\sum_{i=n+1}^{\infty} t_i\langle x_i^*, h\rangle\right| \leq \max\left\{\Big(\sum_{i=n_0}^{\infty}[f_i^\circ(a;h)]^p\Big)^{\frac{1}{p}}, \Big(\sum_{i=n_0}^{\infty}[f_i^\circ(a;-h)]^p\Big)^{\frac{1}{p}}\right\} < \varepsilon.$$

Noting that $\{t_i(k)\}_{k\in\Lambda_n}$ converges to λ_i and $\{x_i^*(k)\}_{k\in\Lambda_i}$ weak*-converges to a_i^* for $1 \leq i \leq n$, it follows from (3.17) that

$$\left|\sum_{i=1}^{n}\langle\lambda_i a_i^*, h\rangle - \langle x^*, h\rangle\right| \leq \varepsilon \ \forall n \geq n_0.$$

This shows that (3.18) holds. Hence $x^* = \sum_{i\in\mathbf{N}}^* \lambda_i a_i^* \in \sum_{i\in I}^q(\partial_c f_i(a))$. This implies that $\sum_{i\in I}^q(\partial_c f_i(a))$ is weak* closed.

Step 3. We prove that $\big(\sum_{i\in I}[f_i^\circ(a;\cdot)]^p\big)^{\frac{1}{p}}$ is the support function of the weak* closed set $\sum_{i\in I}^q(\partial_c f_i(a))$.

Granting this, it follows from Step 2, Lemma 3.1 and [15, Proposition 2.1.4] that $\sum_{i\in I}^q(\partial_c f_i(a)) = \partial_c\phi(a)$ since $\phi^\circ(a;\cdot)$ is the support function of the weak* closed convex set $\partial_c\phi(a)$.

Let $h \in X$. By (3.15), one has

$$\sup\Big\{\langle x^*, h\rangle : x^* \in \sum\nolimits_{i\in I}^q(\partial_c f_i(a))\Big\} \leq \Big(\sum_{i\in I}[f_i^\circ(a;h)]^p\Big)^{\frac{1}{p}} \tag{3.19}$$

On the other hand, since f_i is Lipschitz, it follows from [15, Proposition 2.1.2] that for each $i \in I$, there exists $z_i^* \in \partial_c f_i(a)$ such that

$$\langle z_i^*, h\rangle = f_i^\circ(a;h).$$

Noting that $l^q(I)$ is reflexive and $\big(f_i^\circ(a;h)\big)_{i\in I} \in l^p(I) = \big(l^q(I)\big)^*$, by James's Theorem(cf.[21]), there exists $(t_i)_{i\in I} \in l_+^q(I)$ with $\sum_{i\in I} t_i^q = 1$ such that

$$\Big(\sum_{i\in I}[f_i^\circ(a;h)]^p\Big)^{\frac{1}{p}} = \sum_{i\in I} t_i f_i^\circ(a;h) = \Big\langle \sum\nolimits_{i\in I}^* t_i z_i^*, h\Big\rangle.$$

Thus $\big(\sum_{i\in I}[f_i^\circ(a;\cdot)]^p\big)^{\frac{1}{p}}$ is the support function of $\sum_{i\in I}^q(\partial_c f_i(a))$.

Next, we prove that

$$\partial_c f_i(a) = \hat{N}(A_i, a) \cap B_{X^*} \ \ \forall i \in I \tag{3.20}$$

Fix $i \in I$. Since A_i is subsmooth at a, it follows from (2.2) that $N_c(A_i, a) = \hat{N}(A_i, a)$. Hence

$$\partial_c f_i(a) \subset N_c(A_i, a) \cap B_{X^*} = \hat{N}(A_i, a) \cap B_{X^*} = \hat{\partial} f_i(a) \subset \partial_c f_i(a).$$

This implies that

$$\partial_c f_i(a) = \hat{N}(A_i, a) \cap B_{X^*}.$$

Hence (3.13) holds by (3.14) and (3.20). The proof is completed. ■

For convenience to state the main results in this section, we need some notations. Let $\{K_i : i \in I\}$ be a collection of weak* closed subsets of X^* and $p \in [1, +\infty)$. We define the weak* p-sum of $(K_i)_{i\in I}$ by

$$p - \sum\nolimits_{i\in I}^{*} K_i := \left\{ \sum\nolimits_{i\in I}^{*} x_i^* : x_i^* \in K_i (\forall i \in I), \sum_{i\in I} \|x_i^*\|^p < +\infty \right\},$$

provided that for any $(x_i^*)_{i\in I}$ with $x_i^* \in K_i$ $(i \in I)$ and $\sum_{i\in I} \|x_i^*\|^p < +\infty$ there exists $x^* \in X^*$ such that $x^* = \sum_{i\in I}^{*} x_i^*$.

Let $p \in (1, +\infty)$ and $\tau > 0$. We call that $(K_i)_{i\in I}$ has property $(G, \tau)_p$ if

$$\big(p - \sum\nolimits_{i\in I}^{*} K_i\big) \cap B_{X^*} \subset \tau \sum\nolimits_{i\in I}^{p} (K_i \cap B_{X^*}).$$

If each K_i is a cone, it is easy to verify that

$$(G, \tau)_p \Longleftrightarrow \big(p - \sum\nolimits_{i\in I}^{*} K_i\big) \cap B_{X^*} \subset (0, \tau] \sum\nolimits_{i\in I}^{p} (K_i \cap B_{X^*}). \tag{3.21}$$

Under the suitable assumptions, some necessary or/and sufficient conditions for p-locally linear regularity can be obtained through the following theorems.

Theorem 3.1. *Let $\{A_i : i \in I\}$ be a collection of closed sets of a Banach space X such that $A := \bigcap_{i\in I} A_i$ is nonempty. Let $a \in A$ and $p, q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\{A_i : i \in I\}$ is p-uniformly subsmooth on A and that $d(\cdot, A_i)_{i\in I}$ is of type l^p. We consider the following statements :*

- (i) *there exist τ_1, $\delta_1 > 0$ such that $N_c(A, x) = q\text{-}\sum_{i\in I}^{*} N_c(A_i, x)$ and the collection $(N_c(A_i, x))_{i\in I}$ has property $(G, \tau_1)_q$* for all $x \in \partial A \cap B(a, \delta_1)$*;*
- (ii) *the collection $\{A_i : i \in I\}$ is p-locally linearly regular at $a \in A$, that is, there exist $\tau, r > 0$ such that*

$$d(x, A) \le \tau \big(\sum_{i\in I} d(x, A_i)^p \big)^{\frac{1}{p}} \quad \forall x \in B(a, r);$$

- (iii) *there exist τ, $\delta > 0$ such that $\hat{N}(A, x) = q\text{-}\sum_{i\in I}^{*} \hat{N}(A_i, x)$ and the collection $(\hat{N}(A_i, x))_{i\in I}$ has property $(G, \tau)_q$* for all $x \in \partial A \cap B(a, \delta)$.

Then, (i) *implies* (ii). *Furthermore, we suppose that $\big(\sum_{i\in I} d(\cdot, A_i)^p\big)^{\frac{1}{p}}$ is regular at all $x \in \partial A$ close to a. Then* (ii) *implies* (iii).

Proof. (i)⇒(ii): Let $\varepsilon \in (0, +\infty)$ such that $\tau_1\varepsilon < 1$. By Proposition 3.1, there exist $\delta_2 \in (0, \delta_1)$ and $(\omega_i)_{i\in I} \in B_{l^p(I)}$ such that whenever $i \in I$, $a_i \in A_i \cap B(a, \delta_2)$ and $a_i^* \in N_c(A_i, a_i) \cap B_{X^*}$, one has

$$\langle a_i^*, x - a_i\rangle \leq d(x, A_i) + |\omega_i|\varepsilon\|x - a_i\| \quad \forall x \in B(a, \delta_2). \tag{3.22}$$

Let $r := \frac{\delta_2}{2}$ and $x \in B(a, r)\backslash A$. Then $d(x, A) \leq \|x - a\| < r$. Let any $\gamma \in (\max\{\frac{d(x,A)}{r}, \tau_1\varepsilon\}, 1)$. By Lemma 2.1, there exist $z \in \partial A$ and $z^* \in N_c(A, z)$ with $\|z^*\| = 1$ such that

$$\gamma\|x - z\| < \min\Big\{d(x, A), \langle z^*, x - z\rangle\Big\}. \tag{3.23}$$

Noting that

$$\|z - a\| \leq \|z - x\| + \|x - a\| < \frac{d(x, A)}{\gamma} + r < 2r = \delta_2,$$

there exist $(t_i)_{i\in I} \in l_+^q(I)$ with $\sum_{i\in I} t_i^q \leq 1$ and $x_i^* \in N_c(A_i, z) \cap B_{X^*}(i \in I)$ such that $z^* = \tau_1 \sum_{i\in I}^* t_i x_i^*$. It follows from (3.22), (3.23), Hölder inequality and Minkowski inequality that

$$\begin{aligned}
\gamma\|x - z\| &< \tau_1 \sum_{i\in I} t_i\langle x_i^*, x - z\rangle \leq \tau_1 \sum_{i\in I} t_i\big(d(x, A_i) + |\omega_i|\varepsilon\|x - z\|\big) \\
&\leq \tau_1\Big(\sum_{i\in I} t_i^q\Big)^{\frac{1}{q}}\Big(\sum_{i\in I}\big(d(x, A_i) + |\omega_i|\varepsilon\|x - z\|\big)^p\Big)^{\frac{1}{p}} \\
&\leq \tau_1\Big(\sum_{i\in I}\big(d(x, A_i)\big)^p\Big)^{\frac{1}{p}} + \tau_1\Big(\sum_{i\in I}|\omega_i|^p\Big)^{\frac{1}{p}}\|x - z\|\varepsilon \\
&\leq \tau_1\Big(\sum_{i\in I}\big(d(x, A_i)\big)^p\Big)^{\frac{1}{p}} + \tau_1\varepsilon\|x - z\|.
\end{aligned}$$

This and $d(x, A) \leq \|x - z\|$ imply that

$$d(x, A) \leq \frac{\tau_1}{\gamma - \tau_1\varepsilon}\big(\sum_{i\in I}(d(x, A_i))^p\big)^{\frac{1}{p}}.$$

Taking limits as $\gamma \to 1^-$, one has

$$d(x, A) \leq \frac{\tau_1}{1 - \tau_1\varepsilon}\left(\sum_{i\in I}\Big(d(x, A_i)\Big)^p\right)^{\frac{1}{p}}.$$

This shows that (ii) holds with $\tau := \frac{\tau_1}{1-\tau_1\varepsilon}$

(ii)⇒ (iii): Let $\phi(x) := \big(\sum_{i\in I} d(x, A_i)^p\big)^{\frac{1}{p}}$. Choose $\sigma \in (0, r)$ such that ϕ is regular at each $x \in \partial A \cap B(a, \sigma)$. Take $\delta = \frac{\sigma}{2}$. Let $x \in B(a, \delta) \cap \partial A$ and $x^* \in \hat{N}(A, x) \cap B_{X^*} = \hat{\partial} d(\cdot, A)(x)$. Then for any $\varepsilon > 0$ there exists $r_1 \in (0, \delta - \|x - a\|)$ such that

$$\langle x^*, z - x\rangle \le d(z, A) + \tau\varepsilon\|z - x\| \quad \forall\, z \in B(x, r_1). \tag{3.24}$$

Noting that $B(x, r_1) \subset B(a, \delta) \subset B(a, \sigma)$, by (ii), one has

$$\langle x^*, z - x\rangle \le \tau\phi(z) + \tau\varepsilon\|z - x\| \quad \forall\, z \in B(x, r_1).$$

This implies that $x^* \in \tau\hat{\partial}\phi(x) \subset \tau\partial_c\phi(x)$(thanks to $\phi(x) = 0$). Hence

$$\hat{N}(A, x) \cap B_{X^*} \subset \tau\partial_c\phi(x). \tag{3.25}$$

Next, we show that

$$\begin{array}{l} \partial_c\phi(x) = \sum^q_{i\in I}\big(\hat{N}(A_i, x) \cap B_{X^*}\big) \\ \hat{N}(A, x) = q - \sum^*_{i\in I}\hat{N}(A_i, x) \end{array} \quad \forall\, x \in \partial A \cap B(a, \delta). \tag{3.26}$$

Granting this, it follows from (3.21) and (3.25) that (iii) holds. Let $x \in \partial A \cap B(a, \delta)$. It follows from Lemmas 3.1 and 3.2 that

$$\partial_c\phi(x) = \sum\nolimits^q_{i\in I}\big(\hat{N}(A_i, x) \cap B_{X^*}\big). \tag{3.27}$$

Thus, we only need to show that $\hat{N}(A, x) = q\text{-}\sum^*_{i\in I}\hat{N}(A_i, x)$.

To do this, let $x^* \in \hat{N}(A, x)\backslash\{0\}$. Then $\frac{x^*}{\|x^*\|} \in \hat{N}(A, x) \cap B_{X^*}$. It follows from (3.25) and (3.27) that there exist $(t_i)_{i\in I} \in l^q_+(I)$ with $\sum_{i\in I} t_i^q = 1$ and $x_i^* \in \hat{N}(A_i, x) \cap B_{X^*}(i \in I)$ such that

$$x^* = \sum\nolimits^*_{i\in I}\tau t_i \cdot \|x^*\| \cdot x_i^*.$$

Note that

$$\sum_{i\in I}\Big\|\tau t_i \cdot \|x^*\| \cdot x_i^*\Big\|^q \le \tau^q\|x^*\|^q\sum_{i\in I} t_i^q < +\infty.$$

This implies that $x^* \in q\text{-}\sum^*_{i\in I}\hat{N}(A_i, x)$ and consequently $\hat{N}(A, x) = q\text{-}\sum^*_{i\in I}\hat{N}(A_i, x)$ since the trivial inclusion $\hat{N}(A, x) \supset q\text{-}\sum^*_{i\in I}\hat{N}(A_i, x)$ holds. ■

Theorem 3.2. *Suppose that X is an Asplund space. Let $\{A_i : i \in I\}$ be a collection of closed sets in X such that $A := \bigcap_{i\in I} A_i$ is nonempty. Let $a \in A$ and $p, q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\{A_i : i \in I\}$ is p-uniformly subsmooth on A , $d(\cdot, A_i)_{i\in I}$ is of type l^p and that $\big(\sum_{i\in I} d(\cdot, A_i)^p\big)^{\frac{1}{p}}$ is regular at all $x \in \partial A$ close to a. Then the following statements are equivalent:*

(i) *there exist* $\tau_1, \delta_1 > 0$ *such that* $\hat{N}(A, x) = q\text{-}\sum^*_{i\in I} \hat{N}(A_i, x)$ *and the collection* $(\hat{N}(A_i, x))_{i\in I}$ *has property* $(G, \tau_1)_q$ for all $x \in \partial A \cap B(a, \delta_1)$*;*

(ii) *the collection* $\{A_i : i \in I\}$ *is* p*-locally linearly regular at* $a \in A$*, that is, there exist* $\tau, \delta > 0$ *such that*

$$d(x, A) \le \tau\Big(\sum_{i\in I} d(x, A_i)^p\Big)^{\frac{1}{p}} \quad \forall\, x \in B(a, \delta).$$

Combining the proof of Theorem 3.1 with the Asplund space version of Lemma 2.1, one can obtain the proof Theorem 3.2 which will be omitted.

References

1. R. A. Poliquin and R. T. Rockafellar, Prox-regular functions in variational analysis, *Trans. Amer. Math. Soc.*, **348** (1996), 1805-1838.
2. F. Bernard and L. Thilbaut, Prox-regularity of functions and sets in Banach spaces, *Set-Valued Anal.*, **12** (2004), 25-47.
3. R. T. Rockafellar and R. J. Wets, *Variational Analysis*, Springer, 1998.
4. R. Poliquin, R. T. Rockafellar and L. Thibault, Local differentiability of distance functions, *Trans. Amer. Math. Soc.*, **352** (2000), 5231-5249.
5. D. Aussel, A. Daniilidis and L. Thibault, Subsmooth sets: Functional characterizations and related concepts, *Trans. Amer. Math. Soc.*, **357** (2005), 1275-1301.
6. X. Y. Zheng, Z. Wei and J.-C. Yao, Uniform subsmoothness and linear regularity for a collection of infinitely many closed sets, *Nonlinear Anal.*, **73** (2010), 413-430.
7. H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.*, **38** (1996), 367-426.
8. G. Jameson, The duality of pair of wedges, *Proc. Lond. Math. Soc.*, **24** (1972), 531-547.
9. H. Bauschke, J. M. Borwein and W. Li, Strong conical hull intersection property, bounded linear regularity, Jameson's property(G), and error bounds in convex optimization, *Math. Program. Ser. A*, **86** (1999), 135-160.
10. K. F. Ng and W. H. Yang, Regularities and their relations to error bounds, *Math. Program., Ser. A*, **99** (2004), 521-538.
11. C. Li, K. F. Ng and T. K. Pong, The SECQ, linear regularity, and the strong CHIP for an infinite system of closed convex sets in normed linear spaces, *SIAM J. Optim.*, **18** (2007), 643-665.
12. X. Y. Zheng and K. F. Ng, Metric regularity and constraint qualification for convex inequalities on Banach spaces, *SIAM. J. Optim.*, **15** (2003), 757-772.
13. X. Y. Zheng and K. F. Ng, Linear regularity for a collection of subsmooth sets in Banach spaces, *SIAM J. Optim.*, **19** (2008), 62-76.

14. X. Y. Zheng and K. F. Ng, Calmness for L-subsmooth multifunctions in Banach spaces, *SIAM. J. Optim.*, **19** (2009), 1648-1673.

15. F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

16. F. H. Clarke, Yu. S. Ledyaev, R. J. Stern and P. R. Wolenski, *Nonsmooth analysis and control theory*, Springer, 1998.

17. B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation I/II*, Springer-Verlag, Berlin, Heidelberg, 2006.

18. R. R. Phelps, *Convex functions, Monotone operators and Differentiability*, Lecture Notes in Math. 1364, Springer, New York, 1989.

19. M. M. Day, *Normed Linear Spaces*, Springer-Verlag, Berlin, 1962.

20. J. P. Aubin and Frankowska, *Set-Valued Analysis*, Birkhäuser Boston, Boston, MA, 1990.

21. J. Diestel, *Geometry of Banach space-selected topics Lect. Notes in Math.*, Nr. 485, Springer-Verlag, 1975.

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