TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 1, pp. 305-322, February 2012
This paper is available online at http://tjm.math.ntu.edu.tw

EXPOSED POINTS AND STRONGLY EXPOSED POINTS IN MUSIELAK-ORLICZ SEQUENCE SPACES

Zhongrui Shi and Chunyan Liu

Abstract. In this paper we give criteria of exposed points and strongly exposed points in Musielak-Orlicz sequence spaces endowed with Luxemburg norm.

1. INTRODUCTION

It is well known that both the exposed points and strongly exposed points are basic concepts in the geometric theory of Banach spaces. They have numerous application, such as in separation theory and control theory. Criteria for exposed points and strongly exposed points in all classical Orlicz spaces were given in [1, 2, 3] . In recent years, Zhao and Cui in [4] discussed the problem in Musielak-Orlicz sequence spaces under some restriction. In this paper, by counterexamples, we show that the criteria of extreme points in [15] and exposed points in [4] are not true, and we give the criteria for exposed points and strongly exposed points in arbitrary Musielak-Orlicz sequence spaces equipped with Luxemburg norm by getting rid of the restriction on Musielak-Orlicz function in [4].

Let $[X, \|\cdot\|]$ be a Banach space; $S(X)$ and $B(X)$ be the unit sphere and unit ball of X, respectively; X^* be the dual space of X. For $x \in S(X)$, denote $\text{Grad}(x) = \{f \in S(X^*) : f(x) = 1\}$. A point $x \in S(X)$ is called an extreme point of $B(X)$ if $y, z \in B(X)$ and $y+z = 2x$ imply $y = z$. A point $x \in S(X)$ is called an exposed point of $B(X)$ if there exists $f \in \text{Grad}(x)$ such that $1 = f(x) > f(y)$ for all $y \in B(X)\backslash\{x\}$[5]; moreover, if such f satisfies that $x_n \in B(X)$, $f(x_n) \to f(x)$ imply $x_n \to x (n \to \infty)$, then x is called a strongly exposed point of $B(X)$[6], where f is called an exposed functional of x. It is obvious that an exposed point is also an extreme point.

Received February 2, 2010, accepted October 27, 2010.
Communicated by Bor-Luh Lin.
2010 *Mathematics Subject Classification*: 46B20, 46E30.
Key words and phrases: Musielak-Orlicz sequence spaces, Luxemburg norm, Exposed points, Strongly exposed points.
This work was supported by the National Natural Science Foundation (10971129) and SH(J50101).

Let $\mathbb{N}$ be the set of all natural numbers; $\mathbb{R}$ be the set of all real numbers. By $M = \{M_i\}_{i=1}^{\infty}$ we denote a Musielak-Orlicz function sequence provided that for each $i \in \mathbb{N},\ M_i : (-\infty, +\infty) \to [0, +\infty]$ satisfying

1. $M_i(0) = 0, \lim_{u\to\infty} M_i(u) = \infty$ and $M_i(u_i) < \infty$ for some $u_i > 0$;
2. $M_i(u)$ is even convex and left continuous in $[0, +\infty)$.

$p_i^-(u)$ and $p_i(u)$ denote the left-hand and the right-hand derivatives of $M_i(u)$, respectively. The function sequence $N = \{N_i\}_{i=1}^{\infty}$, where $N_i(v) = \sup_{u>0}\{u|v| - M_i(u)\}$, which has the same property as $M_i(u)$, is called the complementary function sequence of M. $q_i^-(s) = \sup\{t : p_i(t) < s\}$ and $q_i(s) = \sup\{t : p_i(t) \le s\}$ are the left-hand and the right-hand derivatives of $N_i(u)$, respectively[9]. Set

$$\alpha_i = \sup\{u \ge 0 : M_i(u) = 0\}, \quad \beta_i = \sup\{u > 0 : M_i(u) < \infty\},$$
$$\tilde{\alpha}_i = \sup\{u \ge 0 : N_i(u) = 0\}, \quad \tilde{\beta}_i = \sup\{u > 0 : N_i(u) < \infty\}$$
$$SC_{M_i} = \{u \in \mathbb{R} : \forall\, \varepsilon > 0,\ M_i(u) < \frac{M_i(u+\varepsilon) + M_i(u-\varepsilon)}{2}\}.$$

Clearly, SC_{M_i} is the set of all strictly convex points of M_i. An interval $[a, b]$ is called a structurally affine interval of $M_i(u)$ (SAI(M_i) for short) provided that $M_i(u)$ is affine on $[a, b]$ and it is not affine either on $[a - \varepsilon, b]$ or on $[a, b + \varepsilon]$ for all $\varepsilon > 0$[9]. Denote

$$SC_{M_i}^- = \{u \in SC_{M_i} : \exists\, \varepsilon > 0 \text{ s.t. } M_i \text{ is affine on } [u, u+\varepsilon]\},$$
$$SC_{M_i}^+ = \{u \in SC_{M_i} : \exists\, \varepsilon > 0 \text{ s.t. } M_i \text{ is affine on } [u-\varepsilon, u]\}.$$

It is obviously that

$$SC_{M_i} = \mathbb{R} \setminus \bigcup_n (a_n, b_n), \text{where } [a_n, b_n] \in SAI(M_i), n = 1, 2, \cdots.$$

We say that $M = \{M_i\}_{i=1}^{\infty}$ satisfies the δ_2^0-condition $\{M \in \delta_2^0$ for short$\}$ if there exist $a > 0, K > 0, i_0 \in \mathbb{N}$ and $c_i \ge 0 (i > i_0)$ with $\sum_{i>i_0} c_i < \infty$ such that $M_i(2u) \le KM_i(u) + c_i$ holds for all $i > i_0$ and all u with $M_i(u) \le a$. It is known that $h_M = l_M$ if and only if $M \in \delta_2^0$[7].

Let l^0 denote the space of all real sequences $u = \{u(i)\}_{i=1}^{\infty}$. As usual, for $u \in l^0$, we denote $\operatorname{supp} u = \{i \in \mathbb{N} : u(i) \ne 0\}$. For each $u = \{u(i)\}_{i=1}^{\infty} \in l^0$, we define the modular ρ_M of u by $\rho_M(u) = \sum_{i=1}^{\infty} M_i(u(i))$. The linear set $\{u \in l^0 : \rho_M(\lambda u) < \infty$ for some $\lambda > 0\}$ endowed with Luxemburg norm

$$\|u\|_{(M)} = \inf\{\lambda > 0 : \rho_M(\frac{u}{\lambda}) \le 1\}$$

or the Orlicz norm

$$\|u\|_M = \sup\left\{\sum_{i=1}^{\infty} u(i)v(i) : \rho_N(v) \le 1\right\} = \inf_{k>0} \frac{1}{k}(1 + \rho_M(ku))$$

is a Banach space, denoted by $l_{(M)}$ or l_M, and it is called the Musielak-Orlicz sequence space [9, 10, 11]. The subspace $\{u \in l_M : \forall\, \lambda > 0, \exists\, i_\lambda$ such that $\sum_{i>i_\lambda} M_i(\lambda u(i)) < \infty\}$ equipped with the norm $\|\cdot\|_{(M)}$(or $\|\cdot\|_M$), which is also a Banach space, is denoted by $h_{(M)}$(or h_M). Denote $\theta_M(u) = \inf\{\lambda > 0 : \sum_{i>i_\lambda} M_i(\frac{u(i)}{\lambda}) < \infty$ for some $i_\lambda\}$. It is known that $\theta_M(u) = \text{dist}(u, h_{(M)}) = \text{dist}(u, h_M)$[12] and $(h_{(M)})^* = l_N$, $(h_M)^* = l_{(N)}$ [8, 9, 10, 11].

We say that $\varphi \in (l_M)^*$ is a singular functional ($\varphi \in F$ for short) if $\varphi(u) = 0$ for all $u \in h_M$. The dual space of l_M is represented in the form $(l_M)^* = l_N \oplus F$, i.e., each $f \in (l_M)^*$ has the unique representation $f = v + \varphi$, where $\varphi \in F$ and $v \in l_N$, and v is called the regular functional with $\langle u, v\rangle = \sum_{i=1}^{\infty} u(i)v(i)$ for all $u = \{u(i)\}_{i=1}^{\infty} \in l_{(M)}$ [8, 9, 10, 11]. It is well known that $\|f\| = \|v\|_N + \|\varphi\|$ for every $f \in l_{(M)}^*$[7]. For $u \in S(l_{(M)})$(or $S(l_M)$), we denote $\text{RGrad}(u) = \{v \in S(l_N)(\text{or } S(l_{(N)}) : \langle u, v\rangle = 1\}$.

2. Main Results

For the convenience of reading, we present some auxiliary lemmas.

Lemma 1. [13]. *Let* $u \in l_M \setminus \{0\}$. *If* $\sum_{i\in \text{supp} u} N_i(\tilde{\beta}_i) > 1$, *then* $\|u\|_M = \frac{1}{k}(1 + \rho_M(ku))$ *if and only if* $k \in K_M(u)$, *where* $K_M(u) = [k_u^*, k_u^{**}]$ *and*

$$k_u^* = \inf\left\{k > 0 : \rho_N(p(k|u|)) = \sum_{i=1}^{\infty} N_i(p_i(k|u(i)|)) \geq 1\right\},$$
$$k_u^{**} = \sup\{k > 0 : \rho_N(p(k|u|)) \leq 1\}.$$

If $\sum_{i\in \text{supp} u} N_i(\tilde{\beta}_i) \leq 1$, *then* $\|u\|_M = \sum_{i\in \text{supp} u} |u(i)|\tilde{\beta}_i$,

Lemma 2. *Let* $u \in l_{(M)}$, *then* $f = v+\varphi$ *with* $K_N(v) \neq \emptyset$, *where* $v \in l_N, \varphi \in F$, *is a support functional of* u *if and only if*

(1) $\rho_M(u) = 1$,

(2) $\varphi(u) = \|\varphi\|$,

(3) $u(i)v(i) \geq 0$ *and* $p_i^-(|u(i)|) \leq k|v(i)| \leq p_i(|u(i)|)$ *for all* $i \in \mathbb{N}$ *and* $k \in K_N(v)$.

Proof. It can be proceeded in an analogous way as the proof of Theorem 1.76 in [9]. ∎

Lemma 3. [14]. *Suppose that* $M \in \delta_2^0$. *If* $u_n, u \in l_{(M)}, \rho_M(u_n) \to \rho_M(u)$ *and* $u_n(i) \to u(i)$ *as* $n \to \infty$ *for each* $i \in \mathbb{N}$ *then* $\|u_n - u\|_{(M)} \to 0$.

Lemma 4. $u \in S(l_{(M)})$ *is an extreme point of* $B\left(l_{(M)}\right)$ *if and only if*

(*i*) $|u(i)| = \beta_i (i = 1, 2, \cdots)$ *or*

(*ii*) (*a*) $\rho_M(u) = 1$;

(*b*) *if* $u(i) = 0$, *then* $\alpha_i = 0$;

(*c*) $\mu\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} \leq 1$ *and if* $|u(i_0)| \in \mathbb{R} \setminus SC_{M_{i_0}}$ *then* $|u(i_0)| > \alpha_{i_0}$.

Proof. Necessity. We can obtain that Conditions (i) or (ii)(a), (ii)(b) and $\mu\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} \leq 1$ are necessary from the process of the proof of Theorem 1 in [15], where μ is the counting measure.

If $\{i \in \mathbb{N} : |u(i)| \in R \setminus SC_{M_i}\} = \{i_0\}$ and $|u(i_0)| \in (0, \alpha_{i_0})$, due to $\alpha_{i_0} \in SC_{M_{i_0}}$, then there is an $\varepsilon > 0$ such that $|u(i_0)| \pm \varepsilon \in (0, \alpha_{i_0})$. Setting $v = \sum\limits_{i \neq i_0} u(i)e_i + (u(i_0) + \varepsilon \text{sign} u(i_0))e_{i_0}$ and $w = \sum\limits_{i \neq i_0} u(i)e_i + (u(i_0) - \varepsilon \text{sign} u(i_0))e_{i_0}$, where

$$e_i = (0, \cdots, 0, \overset{i}{1}, 0, \cdots),$$

we have $v + w = 2u,\ v \neq w$ and $\rho_M(v) = \rho_M(w) = \rho_M(u) = 1$, a contradiction with that u is an extreme point of $B(l_{(M)})$.

Sufficiency.

Case 1. $|u(i)| = \beta_i (i = 1, 2, \cdots)$. We can get that u is an extreme point by the same arguments of sufficiency of Theorem 1 in [15].

Case 2. $\rho_M(u) = 1$. Let $v, w \in S(l_{(M)}),\ v + w = 2u$. Since $1 = \rho_M(u) = \rho_M(\frac{v+w}{2}) \leq \frac{\rho_M(v) + \rho_M(w)}{2} \leq 1$, we have $\rho_M(v) = \rho_M(w) = 1$. Hence

$$0 = \frac{\rho_M(v) + \rho_M(w)}{2} - \rho_M(u) = \sum_{i=1}^{\infty} \Big(\frac{M_i(v(i)) + M_i(w(i))}{2} - M_i(\frac{v(i) + w(i)}{2}) \Big).$$

By the convexity of $M_i(u)$, we derive that

$$M_i(u(i)) = M_i(\frac{v(i) + w(i)}{2}) = \frac{M_i(v(i)) + M_i(w(i))}{2}.$$

By Condition (ii)(c), we get that $|u(i)| (i \neq i_0)$ is the strictly convex point of $M_i(u)$, then $v(i) = w(i) = u(i) (i \neq i_0)$. Since

$$M_{i_0}(v(i_0)) = 1 - \sum_{i \neq i_0} M_i(v(i)) = 1 - \sum_{i \neq i_0} M_i(u(i)) = M_{i_0}(u(i_0)),$$

$|u(i_0)| > \alpha_{i_0}$ and M_{i_0} is strictly increasing for $u > \alpha_{i_0}$, we see $|v(i_0)| = |u(i_0)|$. Similarly, we have $|w(i_0)| = |u(i_0)|$. Combining this with $v(i_0) + w(i_0) = 2u(i_0)$, we obtain $u(i_0) = v(i_0) = w(i_0)$. Hence $v = w = u$. ■

Remark 1. (Theorem 1 of [15]):
$u \in S(l_{(M)})$ is an extreme point if and only if
(1) $|u(i)| = \beta_i (i = 1, 2, \cdots)$ or $\rho_M(u) = 1$;
(2) $\alpha_i = 0 (i \notin suppu)$;
(3) $\mu\{i : |u(i)|$ is not the strictly convex point of $M_i(u)\} \leq 1$.

Lemma 4 shows that this result is not true.

Next we will discuss the exposed points. First we need to point out that Lemma 3 of [4] is not true.

Remark 2. (Lemma 3 of [4]):
If $u \in S(l_{(M)})$ and $|u(i)| \neq \beta_i$ for some $i \in \mathbb{N}$, then $Grad(u) \ni f = v + \varphi(v \in l_N,\ \varphi \in F)$ implies $K_N(v) \neq \emptyset$.

Let us see the following counterexample:

Example 1. Define

$$M_i(u) = \begin{cases} 0 & |u| \leq 1 \\ \infty & |u| > 1, \end{cases}$$

then $N_i(v) = v\ (i = 1, 2, \cdots)$. Take $u = (\frac{1}{2}, 1, 0, \cdots)$ and $v = (0, 1, 0, \cdots)$, then $\|v\|_N = 1$ and $\langle u, v\rangle = 1$. Since $\rho_M(q(kv)) = 0 < 1$ for any $k > 0$, then $k_v^* = \infty$, i.e., $K_N(v) = \emptyset$.

Lemma 5. *Let $u \in S(l_{(M)})$ be an exposed point of $B\left(l_{(M)}\right)$ with $|u(i)| \neq \beta_i$ for some $i \in \operatorname{supp} u$. If $f = v + \varphi \in S(l^*_{(M)})(v \in l_N, \varphi \in F)$ is an exposed functional of u, then $v \neq 0$ and $K_N(v) \neq \emptyset$.*

Proof. If $v = 0$, then $1 = f(u) = \varphi(u) = \varphi(u - [u]_n)$ and $u \neq u - [u]_n$ for some $n \in \mathbb{N}$. This contradicts with the fact that f is an exposed functional of u, where $[u]_n = (u(1), u(2), \cdots, u(n), 0, 0, \cdots)$.

If $K_N(v) = \emptyset$,i.e, $k_v^* = \infty$, then

$$\begin{aligned} \|v\|_N &= \lim_{k\to\infty} \frac{1}{k}(1 + \rho_N(kv)) = \lim_{k\to\infty} \sum_{i\in\operatorname{supp} v} \frac{N_i(kv(i))}{k} \\ &= \lim_{k\to\infty} \sum_{i\in\operatorname{supp} v} \frac{N_i(kv(i))|v(i)|}{k|v(i)|} = \sum_{i\in\operatorname{supp} v} |v(i)|\beta_i. \end{aligned}$$

Since

$$1 = f(u) = \langle u, v\rangle + \varphi(u) = \sum_{i=1}^{\infty} v(i)u(i) + \varphi(u) \leq \|v\|_N + \|\varphi\| = \|f\| = 1,$$

we have $\langle v, u\rangle = \|v\|_N$. We claim that $\operatorname{supp} v = \operatorname{supp} u$. Otherwise, suppose for some $j \in \operatorname{supp} v \setminus \operatorname{supp} u$, then

$$\|v\|_N = \langle v, u\rangle = \langle v - v(j)e_j, u\rangle \leq \|v - v(j)e_j\|_N.$$

Since $\beta_i > 0(\forall i \in \mathbb{N},)$ it reaches a contradiction:

$$\|v\|_N = \sum_{i\in\mathbb{N}} |v(i)|\beta_i > \sum_{i\in\mathbb{N}\setminus\{j\}} |v(i)|\beta_i = \|v - v(j)e_j\|_N.$$

Suppose for some $j \in \operatorname{supp} u \setminus \operatorname{supp} v$, then $u \neq u - u(j)e_j$ and

$$1 = f(u) = \langle v, u\rangle + \varphi(u) = \langle v, u - u(j)e_j\rangle + \varphi(u - u(j)e_j) = f(u - u(j)e_j),$$

a contradiction with the fact that f is an exposed functional of u. So, $\operatorname{supp} u = \operatorname{supp} v$. Therefore, we obtain that

$$\|v\|_N = \langle v, u\rangle = \sum_{i=1}^{\infty} u(i)v(i) \leq \sum_{i=1}^{\infty} |u(i)\|v(i)| < \sum_{i=1}^{\infty} |v(i)|\beta_i = \|v\|_N.$$

This contradiction shows that $K_N(v) \neq \emptyset$. ■

Before we prove the following lemmas, similarly to smooth points and strongly smooth points, we introduce the regular smooth points and strongly regular smooth points of $B(l_M)$. That is, $u \in S(l_M)$ is said to be a regular smooth point of $B(l_M)$ if $RGrad(u) = \{v\}$,i.e, u has and only has one regular supporting functional. Moreover, a regular smooth point u is called a strongly regular smooth point of $B(l_M)$ if for $v_n \in B(l_{(N)})$, $\langle v_n, u\rangle \to 1$ implies $v_n \to v \quad (n \to \infty)$.

Lemma 6. *Let $u \in S(l_M)$. Then u is a regular smooth point of $B\,(l\,_M)$ if and only if*

(*I*) *if $u(i) = 0$ then $\tilde{\alpha}_i = 0$;*

(*II*) *if $\sum_{i\in\operatorname{supp} u} N_i(\tilde{\beta}_i) \leq 1$, then $\sum_{i\in\operatorname{supp} u} N_i(\tilde{\beta}_i) = 1$ or $\operatorname{supp} u = \mathbb{N}$.*

(*III*) *if $\sum_{i\in\operatorname{supp} u} N_i(\tilde{\beta}_i) > 1$, then $\rho_N(p^-(k|u|)) = 1$ or $\rho_N(p(k|u|)) = 1$ or $\mu\{i : p_i^-(k|u(i)|) < p_i(k|u(i)|)\} \leq 1$ where $k \in K_M(u)$.*

Proof. The proof is similar to Theorem 1 in [16]. ■

Lemma 7. *Let $u \in S(l_M)$. Then u is a strongly regular smooth point if and only if $N \in \delta_2^0$ and*

(*I*) *if $u(i) = 0$ then $\tilde{\alpha}_i = 0$,*

(*II*) *if $\sum_{i\in \text{supp}\, u} N_i(\tilde{\beta}_i) \leq 1$ then $\sum_{i\in \text{supp}\, u} N_i(\tilde{\beta}_i) = 1$ and $\mu(\text{supp}\, u) < \infty$,*

(*III*) *if $\sum_{i\in \text{supp}\, u} N_i(\tilde{\beta}_i) > 1$, then*

(a) $\rho_N(p^-(k|u|)) = 1$ *or*

(b) $\theta_M(ku) < 1$ *but either* $\rho_N(p(k|u|)) = 1$ *or* $\mu\{i : p_i^-(k|u(i)|) < p_i(k|u(i)|)\} \leq 1$ *where* $k \in K_M(u)$.

Proof. Necessity. First we show that $N \in \delta_2^0$.

Let $v \in S(l_N)$ be the unique element of RGrad(u). Suppose $N \notin \delta_2^0$. If $\theta_N(v) > 0$, we take $z = 0$; if $\theta_N(v) = 0$, take $z \in S(l_{(N)})$ with $\theta_N(z) \neq 0$ (see Theorem 5 in [17]). Then there exists $\varphi \in S(F)$ such that $\varphi(v - z) = \varphi(-z) \neq 0$.

From $\rho_N(z) \leq 1$, take a increasing sequence $\{m_n\}$ such that $\sum_{i=m_n+1} N_i(z(i)) < \frac{1}{n}$. Setting $v_n = \sum_{i=1}^{m_n} v(i)e_i + \sum_{i=m_n+1}^{\infty} z(i)e_i$, we have $\rho_N(v_n) < 1 + \frac{1}{n}$ and $1 \leftarrow 1 + \frac{1}{n} \geq \langle v_n, u\rangle = \sum_{i=1}^{m_n} v(i)u(i) + \sum_{i=m_n+1}^{\infty} z(i)u(i) \to \langle v, u\rangle = 1 (n \to \infty)$. But $\|v - v_n\|_{(N)} \geq \varphi(v - v_n) = \varphi(v - z - [v]_{m_n} + [z]_{m_n}) = \varphi(v - z) \neq 0 \ \ (\forall n \in \mathbb{N})$, from $\|\frac{v_n}{1+\frac{1}{n}} - v_n\|_{(N)} \to 0$, which contradicts with the fact that u is a strongly regular smooth point. Hence $N \in \delta_2^0$.

Since u is also a regular smooth point, the condition (I) holds.

When $\sum_{i\in \text{supp}\, u} N_i(\tilde{\beta}_i) \leq 1$, then $\sum_{i\in \text{supp}\, u} N_i(\tilde{\beta}_i) = 1$ or supp $u = \mathbb{N}$ applying Lemma 6. Noticing that $N \in \delta_2^0$, we see that there are at most finite $i \in$ supp u with $\tilde{\beta}_i < \infty$, so $\mu(\text{supp}\, u) < \infty$ and $\sum_{i\in \text{supp}\, u} N_i(\tilde{\beta}_i) = 1$.

When $\sum_{i\in \text{supp}\, u} N_i(\tilde{\beta}_i) > 1$, if suppose that (III) dose not hold, then $\rho_N(p^-(k|u|)) < 1$ and $\theta(ku) = 1$, where $1 = \|u\|_M = \frac{1}{k}(1 + \rho_M(ku))$.

Let v is the unique element of RGrad(u). Since $\rho_N(v) = 1$ and $p_i^-(k|u(i)|) \leq |v(i)| \leq p_i(k|u(i)|) (i \in \mathbb{N})$, there exists an $i_0 \in \mathbb{N}$ satisfying $p_{i_0}^-(k|u(i_0)|) < |v(i_0)|$. Set $c = N_{i_0}(v(i_0)) - N_{i_0}(p_{i_0}^-(k|u(i_0)|))$, then $0 < c < 1$.

Since $1 = \theta_M(ku) = \lim\limits_{n\to\infty} \|ku - [ku]_n\|_M$(see Lemma 1 of [12]), there exists a sequence $\{w_n\} \subset S(l_{(N)})$ such that $\|ku - [ku]_n\|_M \geq \sum_{i=n+1}^{\infty} w_n(i)ku(i) = \langle w_n, ku - [ku]_n\rangle \geq \|ku - [ku]_n\|_M - \frac{1}{n}$, i.e., $\langle ku - [ku]_n, w_n\rangle \to 1 (n \to \infty)$. Without loss of generality, we may assume that $w_n = \sum_{i=n+1}^{\infty} w_n(i)e_i$. For any $n > i_0$, setting

$$v_n = \sum_{i\neq i_0, i=1}^{n} v(i)e_i + p_{i_0}^-(k|u(i_0)|)\text{sign}u(i_0)e_{i_0} + \sum_{i=n+1}^{\infty} cw_n(i)e_i,$$

we have

$$
\begin{aligned}
\rho_N(v_n) &= \sum_{i\neq i_0, i=1}^{n} N_i(v(i)) + N_{i_0}(p_{i_0}^{-}(k|u(i_0)|)) + \sum_{i=n+1}^{\infty} N_i(cw_n(i)) \\
&\leq \sum_{i=1}^{n} N_i(v(i)) - c + c\sum_{i=n+1}^{\infty} N_i(w_n(i)) \\
&= \sum_{i=1}^{n} N_i(v(i)) - c\,(1-\rho_N(w_n)) \leq \rho_N(v) = 1
\end{aligned}
$$

and as $n > i_0$,

$$
\begin{aligned}
\langle v_n, ku\rangle &= \sum_{i\neq i_0, i=1}^{n} v(i)ku(i) + p_{i_0}^{-}(k|u(i_0)|)k|u(i_0)| + \sum_{i=n+1}^{\infty} cw_n(i)ku(i) \\
&= \sum_{i\neq i_0, i=1}^{n} [N_i(v(i)) + M_i(ku(i))] + N_{i_0}(p_{i_0}^{-}(k|u(i_0)|)) \\
&\quad + M_{i_0}(k|u(i_0)|) + c\,\langle w_n, ku - [ku]_n\rangle \\
&= \sum_{i=1}^{n}[N_i(v(i)) + M_i(ku(i))] + c\,(\langle w_n, ku-[ku]_n\rangle - 1) \\
&\to \rho_N(v) + \rho_M(ku) = 1 + \rho_M(ku) = k \qquad (n\to\infty),
\end{aligned}
$$

i.e., $\langle v_n, u\rangle \to 1$. But

$$\|v_n - v\|_{(N)} \geq \|(|v(i_0)| - p_{i_0}^{-}(k|u(i_0)|))e_{i_0}\|_{(N)} > 0\,(n > i_0),$$

a contradiction with the fact that u is a strongly regular smooth point.

Sufficiency.

Case 1. $\sum_{i\in \text{supp}\, u} N_i(\tilde{\beta}_i) \leq 1$.

Then $\sum_{i\in \text{supp} u} N_i(\tilde{\beta}_i) = 1$ and $v = \sum_{i\in\text{supp}\,u} \tilde{\beta}_i \text{sign} u(i) e_i$ is the unique element of RGrad(u). Let $v_n \in S(l_{(N)})$ satisfying $\langle v_n, u\rangle \to 1\,(n\to\infty)$, then

$$\sum_{i\in\text{supp} u} u(i)\big(\tilde{\beta}_i \text{sign} u(i) - v_n(i)\big) = \sum_{i\in\text{supp}\,u} |u(i)|\big(\tilde{\beta}_i - v_n(i)\text{sign} u(i)\big) \to 0 \qquad (n\to\infty).$$

Hence $v_n(i) \to \tilde{\beta}_i \text{sign} u(i)$ $(\forall i \in \text{supp}\,u)$ as $n\to\infty$. Combining with $\mu(\text{supp}\,u) < \infty$, we get

$$1 \geq \rho_N(v_n) \geq \sum_{i\in\text{supp}\,u} N_i(v_n(i)) \to \sum_{i\in\text{supp}\,u} N_i(\tilde{\beta}_i) = 1.$$

Therefore $\lim_{n\to\infty} \rho_N(v_n) = \rho_N(v) = 1$ and $\sum_{i\notin \text{supp}\,u} N_i(v_n(i)) \to 0\,(n\to\infty)$. Since $\tilde{\alpha}_i = 0\,(i \notin \text{supp}\,u)$, $v_n(i) \to 0\,(i\notin \text{supp}\,u)$ as $n\to\infty$. By Lemma 3, $\|v_n - v\|_{(N)} \to 0\,(n\to\infty)$.

Case 2. $\sum_{i\in\operatorname{supp} u} N_i(\tilde{\beta}_i) > 1$.

Then $1 = \|u\|_M = \frac{1}{k}(1+\rho_M(ku))$, where $k \in K_M(u)$. Let $v_n \in S(l_{(N)})$, $\langle v_n, u\rangle \to 1\,(n\to\infty)$. From

$$\begin{aligned} 1 \leftarrow \langle v_n, u\rangle &= \frac{1}{k}\sum_{i=1}^{\infty} v_n(i)ku(i) \\ &\le \frac{1}{k}\sum_{i=1}^{\infty}[N_i(v_n(i)) + M_i(ku(i))] \le \frac{1}{k}(\rho_N(v_n) + \rho_M(ku)) \\ &\le \frac{1}{k}(1+\rho_M(ku)) = \|u\|_M = 1\,(n\to\infty), \end{aligned}$$

we get

$$\lim_{n\to\infty}\rho_N(v_n) = 1, \tag{2.1}$$

$$\sum_{i=1}^{\infty}[N_i(v_n(i)) + M_i(ku(i)) - v_n(i)ku(i)] \to 0 \quad (n\to\infty). \tag{2.2}$$

Let v be the unique element of RGrad(u). In order to prove $\|v_n - v\|_{(N)} \to 0$, applying Lemma 3, we only need to verify that $\lim_{n\to\infty} v_n(i) = v(i)$ for all $i \in \mathbb{N}$.

Subcase 2.1 $\rho_N(p^-(k|u|)) = 1$.

In this case $v = \{p_i^-(k|u(i)|)\operatorname{sign} u(i)\}_{i=1}^{\infty}$ is the unique element of RGrad(u). Now, we will prove

$$\lim_{n\to\infty} v_n(i) = p_i^-(k|u(i)|)\operatorname{sign} u(i)\ (\forall i\in\mathbb{N}).$$

First, $\underline{\lim}_{n\to\infty} |v_n(i)| \ge p_i^-(k|u(i)|)$ for every $i \in \mathbb{N}$. Otherwise, suppose for some $i_0 \in \mathbb{N}$ and a $\delta > 0$ such that $\underline{\lim}_{n\to\infty} |v_n(i_0)| < p_{i_0}^-(k|u(i_0)|) - \delta$. We may assume $|v_n(i_0)| \le p_{i_0}^-(k|u(i_0)|) - \delta$ for every n. Consider function $f(x) = N_{i_0}(x) + M_{i_0}(ku(i_0)) - ku(i_0)x$. Since f is continuous on the bounded closed set $D = \{x \in \mathbb{R} : |x| \le p_{i_0}^-(k|u(i_0)|) - \delta\}$ and $f(x) > 0\,(x \in D)$, there exists $\varepsilon_0 > 0$ such that $f(x) \ge \varepsilon_0$ for all $x \in D$. This leads to a contradiction:

$$0 \leftarrow N_{i_0}(v_n(i_0)) + M_{i_0}(ku(i_0)) - ku(i_0)v_n(i_0) \ge \varepsilon_0 > 0\,(n\to\infty).$$

Second, suppose $\overline{\lim}_{n\to\infty} |v_n(j_0)| > p_{j_0}^-(k|u(j_0)|)$ for some $j_0 \in \mathbb{N}$, then it reaches a contradiction:

$$\begin{aligned}1 = \lim_{n\to\infty}\rho_N(v_n) &= \overline{\lim_{n\to\infty}}\left(\sum_{i\neq j_0} N_i(v_n(i)) + N_{j_0}(v_n(j_0))\right)\\ &\geq \underline{\lim_{n\to\infty}}\sum_{i\neq j_0} N_i(v_n(i)) + \overline{\lim_{n\to\infty}} N_{j_0}(v_n(j_0))\\ &> \sum_{i\neq j_0} N_i\left(p_i^-(k|u(i)|)\right) + N_{j_0}(p_{j_0}^-(k|u(j_0)|)) = \rho_N(v) = 1.\end{aligned}$$

Summarily, $\lim_{n\to\infty}|v_n(i)| = p_i^-(k|u(i)|)\,(i = 1, 2, \cdots)$. If $p_i^-(k|u(i)|) = 0$, then $\lim_{n\to\infty} v_n(i) = 0 = v(i)$; if $p_i^-(k|u(i)|) \neq 0$, then by (2.2), $v_n(i)u(i) > 0$ for large n and $\lim_{n\to\infty} v_n(i) = p_i^-(k|u(i)|)\text{sign}u(i)$. Therefore $\lim_{n\to\infty} v_n(i) = v(i)$.

Subcase 2.2. $\theta_M(ku) < 1$ and $\rho_N(p^-(k|u|)) < \rho_N(p(k|u|)) = 1$.

In this case, $v = \{p_i(k|u(i)|)\text{sign}u(i)\}_{i=1}^\infty$ is the unique element of RGrad(u). Take $\eta > 0$ with $\theta_M(ku) < 1 - \eta < 1$. We claim

$$\lim_{m\to\infty}\sup_n \sum_{i=m+1}^\infty N_i(v_n(i)) = 0. \tag{2.3}$$

Otherwise, there would be an $\varepsilon_0 > 0$ and $m_j, n_j \to \infty\,(j\to\infty)$ such that $\sum_{i=m_j+1}^\infty N_i(v_{n_j}(i)) \geq \varepsilon_0$. Combining with (2.2), it reaches a contradiction:

$$\begin{aligned}0 \leftarrow &\sum_{i=m_j+1}^\infty \left[N_i\left(v_{n_j}(i)\right) + M_i\left(ku(i)\right) - v_{n_j}(i)ku(i)\right]\\ \geq &\sum_{i=m_j+1}^\infty \Bigg[N_i\left(v_{n_j}(i)\right) + M_i\left(ku(i)\right)\\ &-(1-\eta)\left(N_i\left(v_{n_j}(i)\right) + M_i\left(\frac{1}{1-\eta}ku(i)\right)\right)\Bigg]\\ \geq &\sum_{i=m_j+1}^\infty \left[\eta N_i\left(v_{n_j}(i)\right) - (1-\eta)M_i\left(\frac{1}{1-\eta}ku(i)\right)\right] \to \eta\varepsilon_0 > 0 \quad (j\to\infty).\end{aligned}$$

Similar to the proof of $\underline{\lim}_{n\to\infty}|v_n(i)| \geq p_i^-(k|u(i)|)(\forall i\in\mathbb{N})$ in Subcase 2.1, we can get $\overline{\lim}_{n\to\infty}|v_n(i)| \leq p_i\left(k|u(i)|\right)\ (\forall\, i\in\mathbb{N})$.

If $\underline{\lim}_{n\to\infty}|v_n(i_0)| < p_{i_0}\left(k|u(i_0)|\right)$ for some $i_0\in\mathbb{N}$, $|u(i_0)| > \alpha_{i_0}$ and $k|u(i_0)| \leq \beta_{i_0}$, then there exists an $\varepsilon_0 > 0$ such that

$$N_{i_0}\left(\underline{\lim_{n\to\infty}}|v_n(i_0)|\right) < N_{i_0}\left(p_{i_0}(k|u(i_0)|)\right) - \varepsilon_0.$$

By (2.3), choosing $i_1 > i_0$ such that $\sum_{i=i_1+1}^{\infty} N_i(v_n(i)) < \frac{\varepsilon_0}{2}$ for all n, it reaches a contradiction:

$$\begin{aligned}
1 &= \lim_{n\to\infty} \rho_N(v_n) = \underline{\lim}_{n\to\infty} \left(\sum_{i\neq i_0} N_i(v_n(i)) + N_{i_0}(v_n(i_0)) \right) \\
&\leq \overline{\lim}_{n\to\infty} \left(\sum_{\substack{i=1\\ i\neq i_0}}^{i_1} N_i(v_n(i)) + \sum_{i=i_1+1}^{\infty} N_i(v_n(i)) \right) + \underline{\lim}_{n\to\infty} N_{i_0}(v_n(i_0)) \\
&\leq \sum_{\substack{i=1\\ i\neq i_0}}^{i_1} \overline{\lim}_{n\to\infty} N_i(v_n(i)) + \frac{\varepsilon_0}{2} + N_{i_0}\left(\underline{\lim}_{n\to\infty} |v_n(i_0)| \right) \\
&\leq \sum_{\substack{i=1\\ i\neq i_0}}^{i_1} N_i\left(p_i\left(k|u(i)|\right)\right) + \frac{\varepsilon_0}{2} + N_{i_0}\left(p_{i_0}(k|u(i_0)|)\right) - \varepsilon_0 \\
&\leq \sum_{i=1}^{i_1} N_i\left(p_i\left(k|u(i)|\right)\right) - \frac{\varepsilon_0}{2} \leq \rho_N(p(k|u|)) - \frac{\varepsilon_0}{2} = 1 - \frac{\varepsilon_0}{2}.
\end{aligned}$$

So, $\lim_{n\to\infty} |v_n(i)| = p_i(k|u(i)|)\,(i = 1, 2, \cdots)$, hence $\lim\limits_{n\to\infty} v_n(i) = p_i(k|u(i)|)$ sign$u(i)(i = 1, 2, \cdots)$ by the same argument of Subcase 2.1.

Subcase 2.3. $\theta_M(ku) < 1,\ \rho_N(p^-(k|u|)) < 1 < \rho_N(p(k|u|))$ and there exists an unique $i_0 \in \mathbb{N}$ satisfying $p_{i_0}^-(k|u(i_0)|) < p_{i_0}(k|u(i_0)|)$.

In this case, $v = \sum_{i\neq i_0} p_i^-(k|u(i)|)\mathrm{sign}u(i)e_i + N_{i_0}^{-1}\left(1 - \sum_{i\neq i_0} N_i(p_i^-(k|u(i)|))\right)$ $\mathrm{sign}u(i_0)e_{i_0}$ is the unique element of RGrad(u). Repeating the proof of the Subcases 2.1, 2.2, we can prove that $\lim_{n\to\infty} v_n(i) = v(i)$ for any $i \neq i_0$. It follows from (2.3) that $\lim_{n\to\infty} \sum_{i\neq i_0} N_i(v_n(i)) = \sum_{i\neq i_0} N_i(v(i))$. Recalling $\lim_{n\to\infty} \rho_N(v_n) = \rho_N(v) = 1$, we have $\lim_{n\to\infty} N_{i_0}(v_n(i_0)) = N_{i_0}(v(i_0))$. By the continuity of $N_{i_0}^{-1}$ at $N_{i_0}(v(i_0))$, $\lim_{n\to\infty} |v_n(i_0)| = |v(i_0)|$. Again in virtue of (2.2) $\lim\limits_{n\to\infty} v_n(i_0) = v(i_0)$. So, $\lim_{n\to\infty} v_n(i) = v(i)\,(i = 1, 2, \cdots)$. ∎

Remark 3. (Theorem 1 of [4]):

Suppose $u \in S(l_{(M)})$ and $|u(i_0)| \neq \beta_{i_0}$ for some $i_0 \in \mathbb{N}$. Then u is an exposed point if and only if

(I) (1) $\rho_M(u) = 1$; (2) $\mu\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} \leq 1$; (3) if $u(i) = 0$ then $\alpha_i = 0$,

(II) $\rho_N(p^-(|u|)) < \infty$,

(II) if $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \{i_0\}$, then $\{i \neq i_0 : |u(i)| \in SC_{M_i}^- \cup SC_{M_i}^+, p_i^-(|u(i)|) = p_i(|u(i)|)\} = \emptyset$,

(IV) if $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \backslash SC_{M_i}\} = \emptyset$, then $\{i \in \mathbb{N} : |u(i)| \in SC^-_{M_i}, p^-_i(|u(i)|) = p_i(|u(i)|)\} = \emptyset$ or $\{i \in \mathbb{N} : |u(i)| \in SC^+_{M_i}, p^-_i(|u(i)|) = p_i(|u(i)|)\} = \emptyset$.

From the following theorem, we see that this result is not true. We shall establish a new criterion for exposed points of $B(l_{(M)})$ and get rid of the limitation in [4].

Theorem 1. *$u \in S(l_{(M)})$ is an exposed point of $B\left(l_{(M)}\right)$ if and only if*

(I) $|u(i)| = \beta_i (i = 1, 2, \cdots)$ *or*

(II) (*i*) $\rho_M(u) = 1$,

(*ii*) *if* $u(i) = 0$ *then* $\alpha_i = 0$,

(*iii*) (*1*) $|u(i)| = \beta_i (\forall i \in \operatorname{supp} u)$ *or*

(*2*) (*a*) $\rho_N(p_-(|u|)) < \infty$,

(*b*) *if* $|u(i)| = \alpha_i > 0$, *then* $M_i(u)$ *is not smooth at* α_i,

(*c*) *if* $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \emptyset$ *then either*

$$\{i \in \mathbb{N} : |u(i)| \in SC^-_{M_i} \text{ and } p^-_i(|u(i)|) = p_i(|u(i)|)\} = \emptyset$$

or

$$\{i \in \mathbb{N} : |u(i)| \in SC^+_{M_i} \text{ and } p^-_i(|u(i)|) = p_i(|u(i)|)\} = \emptyset,$$

(*d*) *if* $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \{i_0\}$, *then* $|u(i_0)| > \alpha_{i_0}$,

$$\{i \in \mathbb{N} \setminus \{i_0\} : |u(i)| \in SC^-_{M_i} \cup SC^+_{M_i} \text{ and } p^-_i(|u(i)|) = p_i(|u(i)|)\} = \emptyset.$$

Proof. Necessity. Since u is also an extreme point, the condition (I), (II)(i), (II)(ii) and $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \backslash SC_{M_i}\} = \emptyset$ or $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \backslash SC_{M_i}\} = \{i_0\}$ and $|u(i_0)| > \alpha_{i_0}$ are necessary.

While $|u(i)| < \beta_i$ for some $i \in \operatorname{supp} u$.

Suppose that $\rho_N(p^-(|u|)) = \infty$. For an exposed functional of u $f = v + \varphi$, then $v \neq 0$ and $K_N(v) \neq \emptyset$ by Lemma 5. Take $k \in K_N(v)$, then $p^-_i(|u(i)|) \leq k(|v(i)|) \leq p_i(|u(i)|)$ for all $i \in \mathbb{N}$ by Lemma 2. Hence, $\infty = \rho_N(p^-(|u|)) \leq \rho_N(kv) \leq k - 1$, a contradiction. Hence (a) is necessary.

Suppose that $|u(j)| = \alpha_j > 0$ and $M_j(u)$ is smooth at α_j. Define $u' = \sum_{i \neq j} u(i) e_i$. Then $\rho_M(u') = \rho_M(u) = 1$ and $u' \neq u$. Let $f = v + \varphi$ be an exposed functional of u and $k \in K_N(v)$, then $p^-_j(|u(j)|) \leq k|v(j)| \leq p_j(|u(j)|) = p_j(\alpha_j) = 0$. Notice $k \geq 1$, we have

$$1 = f(u) = \langle v, u \rangle + \varphi(u) = \left\langle v, u' \right\rangle + \varphi(u' + u(j) e_j) = \left\langle v, u' \right\rangle + \varphi(u') = f(u'),$$

which contradicts the fact that u is an exposed point. Hence (b) is necessary.

While $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \emptyset$. Suppose

$$\{i \in \mathbb{N} : |u(i)| \in SC^-_{M_i}, p^-_i(|u(i)|) = p_i(|u(i)|)\} \neq \emptyset$$
$$\{i \in \mathbb{N} : |u(i)| \in SC^+_{M_i}, p^-_i(|u(i)|) = p_i(|u(i)|)\} \neq \emptyset.$$

Without loss of generality, we assume that $|u(1)| = a_1, |u(2)| = b_2$, where $[a_i, b_i] \in SAI(M_i)(i = 1, 2)$ and $p^-_1(a_1) = p_1(a_1)$, $p^-_2(b_2) = p_2(b_2)$. Take $\varepsilon_1, \varepsilon_2 > 0$ such that $|u(1)| + \varepsilon_1 \in (a_1, b_1), |u(2)| - \varepsilon_2 \in (a_2, b_2)$ and $p_1(|u(1)|)\varepsilon_1 = p_2(|u(2)|)\varepsilon_2$. Then $M_1(u(1)) + M_2(u(2)) = M_1(|u(1)| + \varepsilon_1) + M_2(|u(2)| - \varepsilon_2)$. Setting $u' = (u(1) + \varepsilon_1 \text{sign} u(1), u(2) - \varepsilon_2 \text{sign} u(2), u(3), \cdots)$, we have $u' \neq u$ and $\rho_M(u') = \rho_M(u) = 1$. Let $f = v + \varphi$ be an exposed functional of u. In virtue of Lemma 2, $\varphi(u) = \|\varphi\|$ and $u(i)v(i) \geq 0, p^-_i(|u(i)|) \leq k|v(i)| \leq p_i(|u(i)|)$ for all $i \in \mathbb{N}$, where $k \in K_N(v)$. By the definition of $u', p^-_i(|u'(i)|) \leq k|v(i)| \leq p_i(|u'(i)|), u'(i)v(i) \geq 0(\forall i \in \mathbb{N})$ and $\varphi(u') = \varphi(u + \varepsilon_1 \text{sign} u(1)e_1 - \varepsilon_2 \text{sign} u(2)e_2) = \varphi(u) = \|\varphi\|$. Again by Lemma 2, we get $f(u') = 1$, a contradiction with the fact that u is an exposed point. Hence (c) is necessary.

Finally while $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \{i_0\}$. Suppose

$$\{i \in \mathbb{N} : |u(i)| \in SC^-_{M_i} \cup SC^+_{M_i} \text{ and } p^-_i(|u(i)|) = p_i(|u(i)|)\} \neq \emptyset.$$

By repeating the same arguments as above, we also get a contradiction with the fact that u is an exposed point. Hence (d) is necessary.

Sufficiency. We consider in two cases.

Case 1. $|u(i)| = \beta_i(\forall\, i \in \text{supp} u)$.

For any $i \in \text{supp} u$, since $N_i(y)$ is continuous at 0, there exists $v(i) > 0$ such that $N_i(v(i)) < \frac{1}{2^i}$. Set $v = \{v(i)\text{sign} u(i)\}_{i=1}^\infty$, then supp v = supp u, $v \in l_N$ and $\|v\|_N = \sum_{i \in \text{supp} u} |v(i)|\beta_i$. Hence $\frac{v}{\|v\|_N} \in$ Grad(u). Since supp $u = \mathbb{N}$ or $\rho_M(u) = \sum_{i \in \text{supp} u} M_i(\beta_i) = 1$ and $\alpha_i = 0$ $(i \notin \text{supp} u = \text{supp} v)$, by the Lemma 6, $u = \{\beta_i \text{sign} u(i)\}_{i=1}^\infty$ is the unique element of RGrad$(\frac{v}{\|v\|_N})$. By the definition of regular smooth point, u is an exposed point of $B\left(l_{(M)}\right)$.

Case 2. $|u(i)| < \beta_i$ for some $i \in \text{supp} u$ and $\rho_M(u) = 1$.

Denote $J = \{i \in \mathbb{N} : p^-_i(|u(i)|) < p_i(|u(i)|)\}$. When $\rho_N(p^-(|u|)) < \infty$, for each $i \in J$ we choose $\varepsilon_i > 0$ such that $p^-_i(|u(i)|) + \varepsilon_i < p_i(|u(i)|)$ and $\sum_{i \notin J} N_i(p^-_i(|u(i)|)) + \sum_{i \in J} N_i(p^-_i(|u(i)|) + \varepsilon_i) < \infty$. Set $w = \{w(i)\text{sign} u(i)\}_{i=1}^\infty$ and $v = \frac{w}{\|w\|_N}$, where

$$w(i) = \begin{cases} p^-_i(|u(i)|) & i \notin J \\ p^-_i(|u(i)|) + \varepsilon_i & i \in J \end{cases}.$$

Then

$$1 \geq \langle v, u \rangle = \frac{1}{\|w\|_N} \sum_{i=1}^{\infty} w(i)|u(i)| = \frac{1}{\|w\|_N} \sum_{i=1}^{\infty} (N_i(w(i)) + M_i(u(i)))$$
$$= \frac{1}{\|w\|_N} (\rho_N(\|w\|_N |v|) + 1) \geq \|v\|_N = 1.$$

Hence, $v \in Grad(u)$ and $\|w\|_N \in K_N(v)$.

Subcase 2.1. $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \emptyset$ and $\{i \in \mathbb{N} : |u(i)| \in SC^-_{M_i}, p^-_i(|u(i)|) = p_i(|u(i)|)\} = \emptyset$.

In this case we have $q_i(\|w\|_N |v(i)|) = \sup\{t : p_i(t) \leq \|w\|_N |v(i)|\} = |u(i)|$ for all $i \in \mathbb{N}$. So, $\rho_M(q(\|w\|_N |v|)) = \rho_M(u) = 1$. By Conditions (II)(ii) and (II)(iii)(2)(b), we can get $\alpha_i = 0$ when $v(i) = 0$. By Lemma 6, RGrad(v) has the unique element u. i.e., u is an exposed point.

Subcase 2.2. $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \emptyset$ and $\{i \in \mathbb{N} : |u(i)| \in SC^+_{M_i}, p^-_i(|u(i)|) = p_i(|u(i)|)\} = \emptyset$.

From $q^-_i(\|w\|_N |v(i)|) = \sup\{t : p_i(t) < \|w\|_N |v(i)|\} = |u(i)| (i \in \mathbb{N})$, we have $\rho_M(q^-(\|w\|_N |v|)) = \rho_M(u) = 1$. By the Condition (II)(ii) , $\alpha_i = 0$ when $v(i) = 0$. According to Lemma 6, we can get that u is the unique element of RGrad(v). Thus, u is an exposed point.

Subcase 2.3. $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \{i_0\}$ and $\{i \neq i_0 : |u(i)| \in SC^-_{M_i} \cup SC^+_{M_i}, p^-_i(|u(i)|) = p_i(|u(i)|)\} = \emptyset$.

Denote $|u(i_0)| \in (a_{i_0}, b_{i_0})$, where $[a_{i_0}, b_{i_0}] \in SAI(M_{i_0})$. By the definition of v, $q^-_i(\|w\|_N |v(i)|) = q_i(\|w\|_N |v(i)|) = |u(i)|$ for all $i \in \mathbb{N} \setminus \{i_0\}$ and $q^-_{i_0}(\|w\|_N |v(i_0)|)$ $= a_{i_0} < b_{i_0} = q_{i_0}(\|w\|_N |v(i_0)|)$. Combining (II)(ii) and (II)(iii)(2)(d): $|u(i_0)| > \alpha_{i_0}$, we have $\alpha_i = 0$ for all $i \notin \mathrm{supp} v$. In virtue of Lemma 6, u is the unique element of RGrad(v) . Hence, u is an exposed point of $B\left(l_{(M)}\right)$. ■

Lemma 8. *If $M \notin \delta^0_2$, then $B\left(l_{(M)}\right)$ does not have any strongly exposed point.*

Proof. Let $u \in S(l_{(M)})$, then $\theta(u) \leq 1$.

While $\theta(u) = 1$.

For any $\varepsilon > 0, j \in \mathbb{N}$, by the definition of θ, $\sum_{i=j}^{\infty} M_i(\frac{u(i)}{1-\varepsilon}) = \infty$. Take $0 = n_0 < n_1 < n_2 < \cdots$, such that

$$\sum_{i=n_{k-1}+1}^{n_k} M_i \left(\frac{u(i)}{1 - \frac{1}{k}} \right) > 1 \quad (k = 1, 2, \cdots).$$

Set $u^k = u - [u]^{n_k}_{n_{k-1}}$, where $[u]^{n_k}_{n_{k-1}} = \sum_{i=n_{k-1}+1}^{n_k} u(i)e_i$, then $u^k \in B\left(l_{(M)}\right)$. For $f = v + \varphi \in \mathrm{Grad}(u)$ $(v \in l_N, \varphi \in F)$,

$$1 \geq f(u^k) = \left\langle u - [u]_{n_{k-1}}^{n_k},\, v \right\rangle + \varphi\left(u - [u]_{n_{k-1}}^{n_k}\right)$$
$$\geq \sum_{i=1}^{n_{k-1}} u(i)v(i) + \varphi(u) \to \langle u, v\rangle + \varphi(u) = f(u) = 1 \quad (k \to \infty)$$

and

$$\left\| u - u^k \right\|_{(M)} = \left\| [u]_{n_{k-1}}^{n_k} \right\|_{(M)} \geq 1 - \frac{1}{k} \to 1 \quad (k \to \infty).$$

This shows that u is not a strongly exposed point.

While $\theta(u) < 1$. By Lemma 1.7 of [7] we have $Grad(u) \subset S(l_N)$. For $v \in Grad(u)$, since $M \notin \delta_2^0$, by Lemma 7, v is not a strongly regular smooth point of $B(l_N)$, i.e., u is not a strongly exposed point. ∎

Finally, we establish the criterion for strongly exposed point of $B(l_{(M)})$.

Theorem 2. $u \in S(l_{(M)})$ *is a strongly exposed point of* $B\left(l_{(M)}\right)$ *if and only if* $M \in \delta_2^0$ *and*

(*I*) $\rho_M(u) = 1$,

(*II*) *if* $u(i) = 0$ *then* $\alpha_i = 0$,

(*III*) (*i*) *if* $|u(i)| = \beta_i$ *for all* $i \in \operatorname{supp} u$, *then* $\mu(\operatorname{supp} u) < \infty$;

(*ii*) *if* $|u(i)| < \beta_i$ *for some* $i \in \operatorname{supp} u$, *then*

(*1*) $\rho_N(p^-(|u|)) < \infty$,

(*2*) *if* $|u(i)| = \alpha_i > 0$ *then* M_i *is not smooth at* α_i,

(*3*) *if* $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \emptyset$, *then either* $\{i \in \mathbb{N} : |u(i)| \in SC_{M_i}^+, p_i^-(|u(i)|) = p_i(|u(i)|)\} = \emptyset$ *or* $\{i \in \mathbb{N} : |u(i)| \in SC_{M_i}^- p_i^-(|u(i)|) = p_i(|u(i)|)\} = \emptyset$ *and* $\theta_N(p^-(|u|)) < 1$, *where* $p^-(|u|) = \{p_i^-(|u(i)|)\}_{i=1}^{\infty}$,

(*4*) *if* $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \{i_0\}$, *then* $|u(i_0)| > \alpha_{i_0}$, $\left\{i \in \mathbb{N} : |u(i)| \in SC_{M_i}^- \bigcup SC_{M_i}^+, p_i^-(|u(i)|) = p_i(|u(i)|)\right\} = \emptyset$ *and* $\theta_N(p^-(|u|)) < 1$.

Proof. Necessity. By Lemma 8, it follows that $M \in \delta_2^0$ is necessary.

First we show that $|u(i)| = \beta_i\,(i \in \operatorname{supp} u)$ imply $\mu(\operatorname{supp} u) < \infty$. Otherwise, $\mu(\operatorname{supp} u) = \infty$. Then for each $j \in \mathbb{N}, \sum_{i>j}^{\infty} M_i(\lambda u(i)) = \infty\,(\lambda > 1)$. Hence $u \notin h_{(M)}$. By $M \in \delta_2^0$, $h_{(M)} = l_{(M)}$, it reaches a contradiction $u \in h_{(M)} = l_{(M)}$.

Since u is also an exposed point, by Theorem 1, it is enough to verify:

1. $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \emptyset$ and $\{i \in \mathbb{N} : |u(i)| \in SC_{M_i}^+, p_i^-(|u(i)|) = p_i(|u(i)|)\} \neq \emptyset$ imply $\theta_N(p^-(|u|)) < 1$;

2. $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \{i_0\}$ implies $\theta_N(p^-(|u|)) < 1$.

Let $|u(i)| < \beta_i$ for some $i \in \operatorname{supp} u$. Let v be any exposed functional of u. By Lemma 5, $K_N(v) \neq \emptyset$. For $k \in K_N(v)$, we have

$$p_i^-(|u(i)|) \leq k|v(i)| \leq p_i(|u(i)|) \qquad (\forall i \in \mathbb{N}). \tag{2.4}$$

Suppose $\theta_N(p^-(|u|)) = 1$ but either $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \{i_0\}$ or $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \emptyset$ and $\{i \in \mathbb{N} : |u(i)| \in SC^+_{M_i}, p_i^-(|u(i)|) = p_i(|u(i)|)\} \neq \emptyset$. From (2.4), we have $\theta_N(kv) = 1$ and $q_i^-(k|v(i)|) \leq |u(i)| (i \in \mathbb{N})$. Without loss of generality, we may assume that $|u(i_0)| \in (a_{i_0}, b_{i_0}]$ and $p^-_{i_0}(b_{i_0}) = p_{i_0}(b_{i_0})$ when $|u(i_0)| = b_{i_0}$, where $[\,a_{i_0}, b_{i_0}\,] \in \mathrm{SAI}(M_{i_0})$. Then $q^-_{i_0}(k|v(i_0)|) = a_{i_0} < |u(i_0)|$. Since $M_{i_0}(u(i_0)) > M_{i_0}(\alpha_{i_0}) = 0$, $\rho_M(q^-(k|v|)) < \rho_M(u) = 1$. By Lemma 7, v is not a strongly regular smooth point, i.e., u is not a strongly exposed point.

Sufficiency.

Case 1. $|u(i)| = \beta_i$ $(\forall\, i \in \operatorname{supp} u)$ and $\mu(\operatorname{supp} u) < \infty$.

Let v be a supporting functional of u with $\operatorname{supp} v = \operatorname{supp} u$ (we can structure v as the case 1 of sufficiency in Theorem 1). Then v is a strongly regular smooth point of $B(l_N)$ by Lemma 7. Hence, u is a strongly exposed point of $B\left(l_{(M)}\right)$.

Case 2. $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \emptyset$ and $\{i \in \mathbb{N} : |u(i)| \in SC^+_{M_i}, p_i^-(|u(i)|) = p_i(|u(i)|)\} = \emptyset$.

Let v be a supporting functional of u as the case 2 of sufficiency in Theorem 1. Then $\rho_M(q^-(\|w\|_N|v|)) = \rho_M(u) = 1$. By the condition (II), $\alpha_i = 0$ when $v(i) = 0$. In virtue of Conditions (I) and (III)(a) of Lemma 7, v is a strongly regular smooth point of $B(l_N)$, i.e., u is a strongly exposed point of $B(l_{(M)})$.

Case 3. $\theta_N(p^-(|u|)) < 1$. Then, there are $\tau > 0$ and $i_1 \in \mathbb{N}$ such that $\sum\limits_{i>i_1} N_i((1+\tau)p_i^-(|u(i)|)) < \infty$.

For $j \in J$, where $J = \{i \in \mathbb{N} : p_i^-(|u(i)|) < p_i(|u(i)|)\}$ and take $\varepsilon_i > 0$ satisfying $p_i^-(|u(i)|) + \varepsilon_i < p_i(|u(i)|)$ such that $\sum\limits_{i \notin J} N_i(p_i^-(|u(i)|)) + \sum\limits_{i \in J} N_i(p_i^-(|u(i)|) + \varepsilon_i) < \infty$ and $\sum\limits_{i \notin J, i > i_1} N_i((1+\tau)p_i^-(|u(i)|)) + \sum\limits_{i \in J, i > i_1} N_i((1+\tau)(p_i^-(|u(i)|) + \varepsilon_i)) < \infty$. Set $w = \{w(i) \operatorname{sign} u(i)\}_{i=1}^\infty$ and $v = \frac{w}{\|w\|_N}$, where

$$w(i) = \begin{cases} p_i^-(|u(i)|) & i \notin J \\ p_i^-(|u(i)|) + \varepsilon_i & i \in J \end{cases}$$

Then $\langle v, u \rangle = 1$, $\|w\|_N \in K_N(v)$ and $\theta_N(v) < 1$.

Subcase 3.1. $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \emptyset$ and $\{i \in \mathbb{N} : |u(i)| \in SC^-_{M_i}, p_i^-(|u(i)|) = p_i(|u(i)|)\} = \emptyset$. Then $\rho_M(q(\|w\|_N|v|)) = \rho_M(u) = 1$. By

Conditions (II) and (III)(ii)(2), $\alpha_i = 0(i \notin \operatorname{supp} v)$. In virtue of Conditions (I) and (III)(b) of Lemma 7, v is a strongly regular smooth point of $B(l_N)$. i.e., u is a strongly exposed point of $B\left(l_{(M)}\right)$.

Subcase 3.2. $\{i \in \mathbb{N} : |u(i)| \in \mathbb{R} \setminus SC_{M_i}\} = \{i_0\}$ and $\{i : |u(i)| \in SC^-_{M_i} \bigcup SC^+_{M_i}, p^-_i(|u(i)|) = p_i(|u(i)|)\} = \emptyset$. Then $q^-_i(\|w\|_N|v(i)|) = q_i(\|w\|_N|v(i)|) = |u(i)|$ for $i \in \mathbb{N} \setminus \{i_0\}$ and $q^-_{i_0}(\|w\|_N|v(i_0)|) = a_{i_0} < b_{i_0} = q_{i_0}(\|w\|_N|v(i_0)|)$. Again by Conditions (II) and (III)(ii)(4), $\alpha_i = 0$ if $v(i) = 0$. Hence v is a strongly regular smooth point due to Conditions (I) and (III)(b) of Lemma 7, i.e., u is a strongly exposed point. ■

References

1. B. Wang, Exposed points of Orlicz spaces, *Journal of Baoji Teacher College* (*Scince*), **12(2)** (1989), 43-49.

2. M. Li, B. Wang and T. Wang, Strongly exposed points of Orlicz sequence spaces, *Acta Mathematica Sinica*, **42(4)** (1999), 645-648.

3. T. Wang, D. Ji and Z. Shi, The criteria of strongly exposed points in Orlicz spaces, *Comment. Math. Univ. Carolin.*, **35(4)** (1994), 721-724.

4. L. Zhao and Y. Cui, Exposed points in Musielak-Orlicz sequence space endowed with the Luxemburg norm, *Natur. Sci. J. Harbin Normal Univ.*, **12(4)** (2005), 3-6.

5. S. Straszewicz, *Über* exponierte punkte abgeschlossener punktmengen, *Fund. Math.*, **24** (1935), 139-143.

6. J. Lindenstranss, On oprators which attain their norm, *Israel J. Math.*, **1** (1963), 139-148.

7. H. Hudzik and Y. Ye, Support functional and smoothness in Musielak-Orlicz sequence spaces endowed with the Luxemburg norm, *Comment. Math. Univ. Carolin.*, **31(4)** (1990), 661-684.

8. M. A. Krasnoselskii and Ya. B. Rutickii, *Convex function and Orlicz spaces*, Groningen, 1961.

9. S. Chen, *Geometry of Orlicz spaces*, Dissertationes Mathematicae, 356, Warszawa, 1996.

10. J. Musielak, Orlicz spaces and modular spaces, in: *Lecture Notes in Math.*, Vol. 1034, Springer-Verlag, 1983.

11. M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces*, Marcel Dekker Inc., 1991.

12. Z. Shi and R. Zhang, On smoothness of Orlicz-Musielak sequence spaces, *J. Mathematics*, **19(4)** (1999), 354-360.

13. L. Cao and T. Wang, Some notes about $K(x) \neq \phi$ in Musielak-Orlicz spaces, *Natur. Sci. J. Harbin Normal Univ.*, **16(4)** (2000), 1-4.

14. M. Zuo, Y. Cui and H. Hudzik, On the points of local uniform rotundity and weak local uniform rotundity in Musielak-Orlicz sequence spaces equipped with the Orlicz norm, *Nonlinear Analysis*, **71** (2009), 4906-4915.

15. X. Liu and T. Wang, Extreme points and strongly extreme points of Musielak-Orlicz sequence spaces, *Acta Mathematica Sinica* (*English Series*), **21(2)** (2005), 267-268.

16. T. Wang and S. Bian, Smooth points and strongly (very) smooth points of Musielak-Orlicz spaces with Orlicz norm, *Acta Analysis Functionals Applicata*, **1(1)** (1999), 61-68.

17. A. Kaminska, Rotundity of Orlicz-Musielak Sequence spaces, *Bull. Polish. Acad. Sci, Math.*, **29** (1981), 137-144.

Zhongrui Shi and Chunyan Liu
Department of Mathematics
Shanghai University
Shanghai 200444
P. R. China
E-mail: zshi@shu.edu.cn
chyliu@shu.edu.cn