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RESOLVENT AVERAGE ON SECOND-ORDER CONE

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Abstract. Recently Bauschke et al. introduced a very interesting and new notion of proximal average in the context of convex analysis, and studied this subject systemically in [3–7] from various viewpoints. In addition, this new concept was applied to positive semidefinite matrices under the name of resolvent average, and basic properties of the resolvent average are successfully established by themselves from a totally different view and techniques of convex analysis rather than the classical matrix diagonalization [8]. Inspired by their works and the well-known fact that the second-order cone is the other typical example of symmetric cone, we study the resolvent average on second-order cone, and derive corresponding results in a different manner from Bauschke et al. [8].

1. Introduction and Preliminaries

Recently Bauschke et al. [5] proposed the following interesting and new notion of proximal average of convex functions.

Definition 1. Let $f = (f_1, \ldots, f_m)$ and $\lambda = (\lambda_1, \ldots, \lambda_m)$ where all $f_i : \mathbb{R}^n \to (-\infty, +\infty]$ are proper convex lower semicontinuous and λ_i's are positive real numbers with $\lambda_1 + \cdots + \lambda_m = 1$. Let $\mathsf{q} = \frac{1}{2} \| \cdot \|^2$ be the quadratic energy function and μ be a positive number. Then *the λ-weighted proximal average of f with parameter μ* is

$$\begin{aligned} p_\mu(f, \lambda) &= \lambda_1 \star (f_1 + \mu \star \mathsf{q}) \square \cdots \square \lambda_m \star (f_1 + \mu \star \mathsf{q}) - \mu \star \mathsf{q} \\ &= \big(\lambda_1 (f_1 + \mu^{-1}\mathsf{q})^* + \cdots + \lambda_m (f_m + \mu^{-1}\ \mathsf{q})^*\big)^* - \mu^{-1}\mathsf{q} \end{aligned}$$

where * stands for the Fenchel conjugate, and the epi-multiplication and the infimal convolution (or epi-addition) are denoted by $\star$ and $\square$, respectively. Their motivation

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for the new object originally came from fixed point theory [2], and recent systematic studies were accomplished in [3–7] from various viewpoints. The striking feature of the proximal average lies in the fact that it provides a parametric family of convex functions that continuously transform one convex function into another even when the domains of the functions do not intersect, so that it may avoid serious defects of the well-known arithmetic average and epigraphical one while we are averaging convex, lower semicontinuous functions in the framework of convex analysis and optimization problems [5].

On the other hand, this new concept was applied to positive semidefinite matrices under the name of *resolvent average*, and basic properties of the resolvent average are successfully established by themselves from a totally different view and techniques of convex analysis rather than the classical matrix diagonalization [8]. As is well known, there are two typical examples of a symmetric cone which are useful in the theory of optimization: the one is the cone of positive semidefinite matrices and the other is the second-order cone [1, 10]. Inspired by the work of Bauschke et al. [8] and this observation, we study the resolvent average on second-order cone in terms of symmetric cones, and present corresponding results in a different manner. An elementary and useful approach exploiting some intrinsic properites of second-order cone is proposed in this paper.

Now let us take a brief look at standard definitions and terminologies concerned with the second-order cone (in short, SOC). First, recall that SOC is the closed convex cone

$$\mathcal{K} := \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1\right\}.$$

Then the Löwner partial order on $\mathbb{R}^n$ induced from $\mathcal{K}$ is defined by $x \preceq y :\Longleftrightarrow y - x \in \mathcal{K}$, and $x < y :\Longleftrightarrow y - x \in \mathrm{int}\mathcal{K}$. Also note that the Euclidean space $\mathbb{R}^n$ with the Jordan product defined by

$$x \circ y = (\langle x, y\rangle,\ x_1 y_2 + y_1 x_2)$$

is a Euclidean Jordan algebra equipped with the standard inner product $\langle \cdot, \cdot \rangle$ where $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. (For further details concerning a Euclidean Jordan algebra V, readers may refer to [10]). For each $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the *determinant* and the *trace* of x are defined by

$$\det(x) = x_1^2 - \|x_2\|^2, \quad \mathrm{tr}(x) = 2x_1.$$

In addition, x is said to be *invertible* if $\det(x) \neq 0$. In this case, x has a unique inverse y in the sense that $x \circ y = y \circ x = e$ where $e = (1, 0, \cdots, 0)$ is the unit element of the Euclidean Jordan algebra $V = (\mathbb{R}^n, \circ)$. In fact, we have

$$y = x^{-1} = \frac{1}{x_1^2 - \|x_2\|^2}(x_1, -x_2).$$

From the definition of determinant, it immediately holds

$$\det(x+y) = (x_1+y_1)^2 - \|x_2+y_2\|^2 = \det(x) + \det(y) + 2(x_1y_1 - \langle x_2, y_2\rangle).$$

This simple equality is very useful, so it often appears in this paper. We close this section with mentioning a reference [9] by Chen containing many interesting properties of determinant and trace in the context of the second-order cone $\mathcal{K}$.

2. The Resolvent Average of Two Variables

Let $e = (1, 0, \cdots, 0)$ be the unit element of the Euclidean Jordan algebra $V = (\mathbb{R}^n, \circ)$. Let $a,\, b \succeq 0$, (namely, $a,\, b \in \mathcal{K}$) and $0 < \lambda < 1$, $\mu > 0$ be real numbers. Define *the resolvent average* $\mathcal{R}_\mu(\lambda, 1-\lambda; a, b)$ *of* a *and* b *with the parameter* μ to be

$$\mathcal{R}_\mu(\lambda, 1-\lambda; a, b) = \left(\lambda\big(a + \frac{e}{\mu}\big)^{-1} + (1-\lambda)\big(b + \frac{e}{\mu}\big)^{-1}\right)^{-1} - \frac{e}{\mu}.$$

Along the corresponding results regarding positive semidefine matrices due to Bauschuke et al. [8], we first write down some elementary properties of the resolvent average $\mathcal{R}_\mu(\lambda, 1-\lambda; a, b)$ with proofs, for the sake of completeness.

Proposition 1. *Let* $a \succeq c \succeq 0$ *and* $b \succeq d \succeq 0$. *Then*

$$\mathcal{R}_\mu(\lambda, 1-\lambda; a, b) \succeq \mathcal{R}_\mu(\lambda, 1-\lambda; c, d).$$

Moreover, if additionally either $a \succ c$ *or* $b \succ d$, *we have*

$$\mathcal{R}_\mu(\lambda, 1-\lambda; a, b) \succ \mathcal{R}_\mu(\lambda, 1-\lambda; c, d).$$

Proof. For $\mu > 0$, clearly $a + \frac{e}{\mu} \succeq c + \frac{e}{\mu} \succ 0$, and $b + \frac{e}{\mu} \succeq d + \frac{e}{\mu} \succ 0$ so that

$$0 \prec \big(a + \frac{e}{\mu}\big)^{-1} \preceq \big(c + \frac{e}{\mu}\big)^{-1}, \quad \text{and} \quad 0 \prec \big(b + \frac{e}{\mu}\big)^{-1} \preceq \big(d + \frac{e}{\mu}\big)^{-1}.$$

Hence we get

$$(1)\quad 0 \prec \lambda\big(a + \frac{e}{\mu}\big)^{-1} + (1-\lambda)\big(b + \frac{e}{\mu}\big)^{-1} \preceq \lambda\big(c + \frac{e}{\mu}\big)^{-1} + (1-\lambda)\big(d + \frac{e}{\mu}\big)^{-1}.$$

Note that for $x, y \succ 0$, by Chen [9, Proposition 3.3]

$$x \succeq y \;\Leftrightarrow\; x^{-1} \preceq y^{-1}$$
$$x \succ y \;\Leftrightarrow\; x^{-1} \prec y^{-1}.$$

It follows from (1) and the first equivalence that

$$\mathcal{R}_\mu(\lambda, 1-\lambda; a, b) \succeq \mathcal{R}_\mu(\lambda, 1-\lambda; c, d).$$

In addition, by the second equivalence, we obtain analogously that

$$\mathcal{R}_\mu(\lambda, 1-\lambda; a, b) \succ \mathcal{R}_\mu(\lambda, 1-\lambda; c, d)$$

if additionally either $a \succ c$ or $b \succ d$. ∎

As an immediate consequence, we obtain the following.

Corollary 1. *Let* $a, b \succ 0$. *Then*

$$\mathcal{R}_\mu(\lambda, 1-\lambda; a, b) \succ 0.$$

Proof. Taking $c = d = 0$ in Proposition 1 yields the result. ∎

From now on, unless otherwise specified, $a, b \succ 0$ is assumed and the abbreviation $d(x)$ is used for $\det(x)$.

Proposition 2. *For any* $a,\ b \succ 0$*, we have*

(*i*) $(a+b)^{-1} = \dfrac{d(a)}{d(a+b)}a^{-1} + \dfrac{d(b)}{d(a+b)}b^{-1}$

(*ii*) $d(a^{-1}+b^{-1}) = \dfrac{d(a+b)}{d(a)d(b)}$

(*iii*) $(a^{-1}+b^{-1})^{-1} = \dfrac{d(b)}{d(a+b)}a + \dfrac{d(a)}{d(a+b)}b.$

Proof.

$$\text{(i)} \qquad \begin{aligned}(a+b)^{-1} &= \frac{1}{d(a+b)}(a_1+b_1, -a_2-b_2) = \frac{(a_1, -a_2)+(b_1, -b_2)}{d(a+b)} \\ &= \frac{d(a)}{d(a+b)}a^{-1} + \frac{d(b)}{d(a+b)}b^{-1}\end{aligned}$$

$$\text{(ii)} \qquad \begin{aligned}d(a^{-1}+b^{-1}) &= d(a^{-1}) + d(b^{-1}) + 2((a^{-1})_1(b^{-1})_1 - \langle (a^{-1})_2,\ (b^{-1})_2\rangle) \\ &= \frac{1}{d(a)} + \frac{1}{d(b)} + 2\left(\frac{a_1b_1}{d(a)d(b)} - \frac{\langle a_2,\ b_2\rangle}{d(a)d(b)}\right) \\ &= \frac{d(a)+d(b)+2(a_1b_1 - \langle a_2,\ b_2\rangle)}{d(a)d(b)} \\ &= \frac{d(a+b)}{d(a)d(b)}\end{aligned}$$

$$\text{(iii)} \qquad \begin{aligned}(a^{-1}+b^{-1})^{-1} &= \frac{d(a^{-1})}{d(a^{-1}+b^{-1})}a + \frac{d(b^{-1})}{d(a^{-1}+b^{-1})}b \\ &= \frac{d(a)d(b)}{d(a+b)}\frac{1}{d(a)}a + \frac{d(a)d(b)}{d(a+b)}\frac{1}{d(b)}b \\ &= \frac{d(b)}{d(a+b)}a + \frac{d(a)}{d(a+b)}b\end{aligned}$$

where the first equality comes from (i) and the second follows from (ii). ■

Theorem 1. (Self-duality of resolvent average). *For any* $a,\ b \succ 0$*, positive numbers* $0 < \lambda < 1$ *and* $\mu > 0$*, it holds*

$$[\mathcal{R}_{\mu}(\lambda, 1-\lambda; a, b)]^{-1} = \mathcal{R}_{\mu^{-1}}(\lambda, 1-\lambda; a^{-1}, b^{-1}).$$

To prove Theorem 1, we need three nontrivial equalities more as follows:

Proposition 3. *For any* $a,\ b \succ 0$*, positive numbers* $0 < \lambda < 1$ *and* $\mu > 0$*, we get*

$$(i) \qquad \frac{d(b^{-1}+\mu e)}{\lambda(1-\lambda)^2 d\big(\frac{1}{\lambda}(a^{-1}+\mu e)+\frac{1}{1-\lambda}(b^{-1}+\mu e)\big)} = \frac{\mu^2}{d\big(\lambda(a+\frac{e}{\mu})^{-1}+(1-\lambda)(b+\frac{e}{\mu})^{-1}-\mu e\big)}\cdot\frac{\lambda d(a)}{d(a+\frac{e}{\mu})}.$$

$$(ii) \qquad \frac{d(a^{-1}+\mu e)}{\lambda^2(1-\lambda) d\big(\frac{1}{\lambda}(a^{-1}+\mu e)+\frac{1}{1-\lambda}(b^{-1}+\mu e)\big)} = \frac{\mu^2}{d\big(\lambda(a+\frac{e}{\mu})^{-1}+(1-\lambda)(b+\frac{e}{\mu})^{-1}-\mu e\big)}\cdot\frac{(1-\lambda) d(b)}{d(b+\frac{e}{\mu})}.$$

Proof. (i) First, we observe that

$$\begin{aligned}
&d\left(\frac{1}{\lambda}(a^{-1}+\mu e)+\frac{1}{1-\lambda}(b^{-1}+\mu e)\right)\\
&= d\left((\lambda a)^{-1}+((1-\lambda)b)^{-1}+\frac{\mu}{\lambda(1-\lambda)}e\right)\\
&= \frac{d(\lambda a+(1-\lambda)b)}{d(\lambda a)d((1-\lambda)b)}+\frac{\mu^2}{\lambda^2(1-\lambda)^2}+\frac{2\mu}{\lambda(1-\lambda)}\left(\frac{a_1}{\lambda d(a)}+\frac{b_1}{(1-\lambda)d(b)}\right)\\
&= \frac{d(\lambda a+(1-\lambda)b)+\mu^2 d(a)d(b)+2\mu(\lambda b_1 d(a)+(1-\lambda)a_1 d(b))}{\lambda^2(1-\lambda)^2 d(a)d(b)}.
\end{aligned} \tag{2}$$

where the second equality is due to Proposition 2(ii). By (2), we see that

$$\begin{aligned}
LHS \text{ of (i)} &= \frac{d\big(b^{-1}+(\frac{e}{\mu})^{-1}\big)}{\lambda(1-\lambda)^2}\\
&\quad\cdot\frac{\lambda^2(1-\lambda)^2 d(a)d(b)}{d(\lambda a+(1-\lambda)b)+\mu^2 d(a)d(b)+2\mu(\lambda b_1 d(a)+(1-\lambda)a_1 d(b))}\\
&= \frac{\lambda\mu^2 d(a)d(b+\frac{e}{\mu})}{d(\lambda a+(1-\lambda)b)+\mu^2 d(a)d(b)+2\mu(\lambda b_1 d(a)+(1-\lambda)a_1 d(b))}\\
&= \frac{\lambda\mu^2 d(a)d(b+\frac{e}{\mu})}{M_1}
\end{aligned} \tag{3}$$

where the second equality follows from Proposition 2(ii) and

$$M_1 = d(\lambda a + (1-\lambda)b) + \mu^2 d(a)d(b) + 2\mu(\lambda b_1 d(a) + (1-\lambda)a_1 d(b)).$$

Secondly,

$$\begin{aligned}
& d\Big(\lambda\big(a+\frac{e}{\mu}\big)^{-1} + (1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1} - \mu e\Big) \\
&= d\Big(\big(\frac{1}{\lambda}(a+\frac{e}{\mu})\big)^{-1} + \big(\frac{1}{1-\lambda}(b+\frac{e}{\mu})\big)^{-1} - \mu e\Big) \\
&= \frac{d\big(\frac{1}{\lambda}(a+\frac{e}{\mu}) + \frac{1}{1-\lambda}(b+\frac{e}{\mu})\big)}{d\big(\frac{1}{\lambda}(a+\frac{e}{\mu})\big)d\big(\frac{1}{1-\lambda}(b+\frac{e}{\mu})\big)} + \mu^2 \\
& \quad -2\mu\Big(\frac{\lambda}{d(a+\frac{e}{\mu})}\big(a_1+\frac{1}{\mu}\big) + \frac{1-\lambda}{d(b+\frac{e}{\mu})}\big(b_1+\frac{1}{\mu}\big)\Big) \\
&= \frac{d\big((1-\lambda)(a+\frac{e}{\mu}) + \lambda(b+\frac{e}{\mu})\big) + \mu^2 d\big(a+\frac{e}{\mu}\big)d\big(b+\frac{e}{\mu}\big)}{d\big(a+\frac{e}{\mu}\big)d\big(b+\frac{e}{\mu}\big)} \\
& \quad -2\mu\frac{\big(\lambda(a_1+\frac{1}{\mu})d(b+\frac{e}{\mu}) + (1-\lambda)(b_1+\frac{1}{\mu})d(a+\frac{e}{\mu})\big)}{d\big(a+\frac{e}{\mu}\big)d\big(b+\frac{e}{\mu}\big)}.
\end{aligned} \tag{4}$$

From (4), we have

$$RHS \text{ of (i)} = \frac{\lambda\mu^2 d(a)d(b+\frac{e}{\mu})}{M_2} \tag{5}$$

where

$$\begin{aligned}
M_2 &= d\big((1-\lambda)a + \lambda b + \frac{e}{\mu}\big) + \mu^2 d\big(a+\frac{e}{\mu}\big)d\big(b+\frac{e}{\mu}\big) \\
& \quad - 2\mu\big(\lambda(a_1+\frac{1}{\mu})d(b+\frac{e}{\mu}) + (1-\lambda)(b_1+\frac{1}{\mu})d(a+\frac{e}{\mu})\big).
\end{aligned}$$

By (3) and (5), it suffices to check that $M_1 = M_2$. Put $d(a) = \alpha,\ d(b) = \beta$ and $\gamma = a_1 b_1 - \langle a_2, b_2\rangle$. Then

$$M_1 = \lambda^2\alpha + (1-\lambda)^2\beta + 2\lambda(1-\lambda)\gamma + 2\mu(\lambda b_1\alpha + (1-\lambda)a_1\beta) + \alpha\beta\mu^2 \tag{6}$$

$$\begin{aligned}
M_2 &= (1-\lambda)^2\alpha + \lambda^2\beta + 2\lambda(1-\lambda)\gamma + \frac{1}{\mu^2} + \frac{2((1-\lambda)a_1 + \lambda b_1)}{\mu} \\
& \quad +\mu^2\left(\alpha + \frac{1}{\mu^2} + \frac{2a_1}{\mu}\right)\left(\beta + \frac{1}{\mu^2} + \frac{2b_1}{\mu}\right) \\
& \quad -2\mu\Big(\lambda\big(a_1+\frac{1}{\mu}\big)\big(\beta+\frac{1}{\mu^2}+\frac{2b_1}{\mu}\big) + (1-\lambda)\big(b_1+\frac{1}{\mu}\big)\big(\alpha+\frac{1}{\mu^2}+\frac{2a_1}{\mu}\big)\Big).
\end{aligned} \tag{7}$$

After deleting $2\lambda(1-\lambda)\gamma$, multiplying both sides by μ^2 and rearrangement, we have only to verify that

$$\begin{aligned}
&(\lambda^2\alpha + (1-\lambda)^2\beta)\mu^2 + 2(\lambda b_1\alpha + (1-\lambda)a_1\beta)\mu^3 + \alpha\beta\mu^4 \\
&\qquad + 2\big(\lambda(a_1\mu+1)(\beta\mu^2+1+2b_1\mu) + (1-\lambda)(b_1\mu+1)(\alpha\mu^2+1+2a_1\mu)\big) \\
&= ((1-\lambda)^2\alpha + \lambda^2\beta)\mu^2 + 1 + 2\mu((1-\lambda)a_1 + \lambda b_1) \\
&\qquad + (\alpha\mu^2+1+2a_1\mu)(\beta\mu^2+1+2b_1\mu).
\end{aligned}$$

This may look a little bit complicated, however, this is nothing but elementary expansion by decreasing order with respect to μ and cancellation process of both sides. This completes the proof of (i).

(ii) Replacing a, b and λ in (i) by b, a and $1-\lambda$, respectively, yields the result. ■

Proposition 4. *For any* $a,\ b \succ 0$, *positive numbers* $0<\lambda<1$ *and* $\mu>0$, *we have*

$$\begin{aligned}
&\frac{1}{d\big(\lambda(a+\frac{e}{\mu})^{-1} + (1-\lambda)(b+\frac{e}{\mu})^{-1} - \mu e\big)} \cdot \left(\frac{\lambda}{d(a+\frac{e}{\mu})} + \frac{1-\lambda}{d(b+\frac{e}{\mu})}\right) \\
&\qquad - \frac{d\big((\lambda(a+\frac{e}{\mu})^{-1} + (1-\lambda)(b+\frac{e}{\mu})^{-1}\big)}{d\big(\lambda(a+\frac{e}{\mu})^{-1} + (1-\lambda)(b+\frac{e}{\mu})^{-1} - \mu e\big)} \\
&= \frac{\frac{1}{\lambda(1-\lambda)^2} d(b^{-1}+\mu e) + \frac{1}{\lambda^2(1-\lambda)} d(a^{-1}+\mu e)}{d\big(\frac{1}{\lambda}(a^{-1}+\mu e) + \frac{1}{1-\lambda}(b^{-1}+\mu e)\big)} - 1.
\end{aligned}$$

Proof. By (4), we know that

$$\begin{aligned}
d\left(\lambda\big(a+\frac{e}{\mu}\big)^{-1} + (1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1} - \mu e\right) &= \frac{M_2}{d(a+\frac{e}{\mu})d(b+\frac{e}{\mu})} \\
d\left(\lambda\big(a+\frac{e}{\mu}\big)^{-1} + (1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1}\right) &= \frac{d\big((1-\lambda)a + \lambda b + \frac{e}{\mu}\big)}{d(a+\frac{e}{\mu})d(b+\frac{e}{\mu})}.
\end{aligned}$$

Hence

$$LHS = \frac{\lambda d(b+\frac{e}{\mu}) + (1-\lambda)d(a+\frac{e}{\mu}) - d\big((1-\lambda)a + \lambda b + \frac{e}{\mu}\big)}{M_2}. \tag{8}$$

On the other hand, it follows from (2) that

$$d\left(\frac{1}{\lambda}(a^{-1}+\mu e) + \frac{1}{1-\lambda}(b^{-1}+\mu e)\right) = \frac{M_1}{\lambda^2(1-\lambda)^2 d(a)d(b)}.$$

Moreover, we see

$$\frac{1}{\lambda(1-\lambda)^2}d(b^{-1}+\mu e) = \frac{d(b+\frac{e}{\mu})\mu^2}{\lambda(1-\lambda)^2 d(b)}$$
$$\frac{1}{\lambda^2(1-\lambda)}d(a^{-1}+\mu e) = \frac{d(a+\frac{e}{\mu})\mu^2}{\lambda^2(1-\lambda) d(a)}.$$

Thus

$$RHS = \frac{\big(\lambda d(a)d(b+\frac{e}{\mu}) + (1-\lambda)d(b)d(a+\frac{e}{\mu})\big)\mu^2 - M_1}{M_1}. \tag{9}$$

As seen in (8) and (9), it is sufficient to show that

$$\begin{aligned} &\left(\lambda d\big(b+\frac{e}{\mu}\big) + (1-\lambda)d\big(a+\frac{e}{\mu}\big) - d\big((1-\lambda)a+\lambda b+\frac{e}{\mu}\big)\right)M_1 \\ &= \left(\big(\lambda d(a)d\big(b+\frac{e}{\mu}\big) + (1-\lambda)d(b)d\big(a+\frac{e}{\mu}\big)\big)\mu^2 - M_1\right)M_2. \end{aligned} \tag{10}$$

Using (6) and (7) with an elementary calculation, we obtain that

$$LHS \text{ of } (10) = \lambda(1-\lambda)(\alpha+\beta-2\gamma)(\delta+2\lambda(1-\lambda)\gamma+2\mu(\lambda b_1\alpha+(1-\lambda)a_1\beta)+\alpha\beta\mu^2)$$

where $\delta = \lambda^2\alpha + (1-\lambda)^2\beta$. Hence

$$\begin{aligned} &\mu^2 \cdot (LHS \text{ of } (10)) \\ &= \lambda(1-\lambda)\left((\alpha+\beta)\delta\mu^2 + 2(\alpha+\beta)\big(\lambda b_1\alpha + (1-\lambda)a_1\beta\big)\mu^3 + (\alpha+\beta)\alpha\beta\mu^4\right) \\ &-2\lambda(1-\lambda)\left(\delta\mu^2 + 2\lambda(1-\lambda)\big(\gamma-\frac{\alpha+\beta}{2}\big)\mu^2 + 2\big(\lambda b_1\alpha+(1-\lambda)a_1\beta\big)\mu^3 + \alpha\beta\mu^4\right)\gamma. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mu^2 \cdot (RHS \text{ of } (10)) \\ &= \Big(\lambda(\alpha\beta\mu^2 + \alpha + 2b_1\alpha\mu) + (1-\lambda)(\alpha\beta\mu^2 + \beta + 2a_1\beta\mu) \\ &\qquad - \delta - 2\lambda(1-\lambda)\gamma - \alpha\beta\mu^2 - 2\big(\lambda b_1\alpha + (1-\lambda)a_1\beta\big)\mu\Big) \cdot (\mu^2 M_2) \\ &= \big(\lambda\alpha + (1-\lambda)\beta - \delta - 2\lambda(1-\lambda)\gamma\big) \cdot (\mu^2 M_2) \\ &= \big(\lambda\alpha + (1-\lambda)\beta - \delta - 2\lambda(1-\lambda)\gamma\big) \cdot \end{aligned}$$

$$
\begin{aligned}
&\Big(\alpha\beta\mu^4 + \big(2\lambda b_1\alpha + 2(1-\lambda)a_1\beta\big)\mu^3 + \delta\mu^2 + 2\lambda(1-\lambda)\gamma\mu^2\Big)\\
&= \big(\lambda\alpha + (1-\lambda)\beta - \delta\big)\cdot\big(\alpha\beta\mu^4 + \big(2\lambda b_1\alpha + 2(1-\lambda)a_1\beta\big)\mu^3 + \delta\mu^2\big)\\
&- 2\lambda(1-\lambda)\Big(\alpha\beta\mu^4 + \big(2\lambda b_1\alpha + 2(1-\lambda)a_1\beta\big)\mu^3 + \big(2\delta - \lambda\alpha - (1-\lambda)\beta\big)\mu^2\\
&\qquad + 2\lambda(1-\lambda)\gamma\mu^2\Big)\gamma
\end{aligned}
$$

where the third equality comes from (7). In addition, it is clear that the second parts of $\mu^2\cdot(LHS \text{ of } (10))$ and $\mu^2\cdot(RHS \text{ of } (10))$ coincide. Thus to get a conclusion, it should be checked that

$$
\begin{aligned}
&\lambda(1-\lambda)\Big((\alpha+\beta)\delta\mu^2 + 2(\alpha+\beta)\big(\lambda b_1\alpha + (1-\lambda)a_1\beta\big)\mu^3 + (\alpha+\beta)\alpha\beta\mu^4\Big)\\
&= \big(\lambda\alpha + (1-\lambda)\beta - \delta\big)\cdot\big(\alpha\beta\mu^4 + \big(2\lambda b_1\alpha + 2(1-\lambda)a_1\beta\big)\mu^3 + \delta\mu^2\big).
\end{aligned}
$$

However, this is direct from the equality

$$
\lambda\alpha + (1-\lambda)\beta - \delta = \lambda\alpha + (1-\lambda)\beta - \lambda^2\alpha - (1-\lambda)^2\beta = \lambda(1-\lambda)(\alpha+\beta).
$$

This completes our proof. ■

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. By Proposition 2(iii), we see that

$$
\begin{aligned}
RHS &= \big(\lambda(a^{-1}+\mu e)^{-1} + (1-\lambda)(b^{-1}+\mu e)^{-1}\big)^{-1} - \mu e\\
&= \Big(\big(\frac{1}{\lambda}(a^{-1}+\mu e)\big)^{-1} + \big(\frac{1}{1-\lambda}(b^{-1}+\mu e)\big)^{-1}\Big)^{-1} - \mu e\\
&= \frac{d\big(\frac{1}{1-\lambda}(b^{-1}+\mu e)\big)}{d\big(\frac{1}{\lambda}(a^{-1}+\mu e) + \frac{1}{1-\lambda}(b^{-1}+\mu e)\big)}\cdot\frac{1}{\lambda}(a^{-1}+\mu e)\\
&\quad + \frac{d\big(\frac{1}{\lambda}(a^{-1}+\mu e)\big)}{d\big(\frac{1}{\lambda}(a^{-1}+\mu e) + \frac{1}{1-\lambda}(b^{-1}+\mu e)\big)}\cdot\frac{1}{1-\lambda}(b^{-1}+\mu e) - \mu e\\
&= \frac{d(b^{-1}+\mu e)}{\lambda(1-\lambda)^2 d\big(\frac{1}{\lambda}(a^{-1}+\mu e) + \frac{1}{1-\lambda}(b^{-1}+\mu e)\big)}a^{-1}\\
&\quad + \frac{d(a^{-1}+\mu e)}{\lambda^2(1-\lambda) d\big(\frac{1}{\lambda}(a^{-1}+\mu e) + \frac{1}{1-\lambda}(b^{-1}+\mu e)\big)}b^{-1}\\
&\quad + \left(\frac{\frac{1}{\lambda(1-\lambda)^2}d(b^{-1}+\mu e) + \frac{1}{\lambda^2(1-\lambda)}d(a^{-1}+\mu e)}{d\big(\frac{1}{\lambda}(a^{-1}+\mu e) + \frac{1}{1-\lambda}(b^{-1}+\mu e)\big)} - 1\right)(\mu e).
\end{aligned}
\tag{11}
$$

On the other hand, by Proposition 2(i), we have

$$
\begin{aligned}
\lambda\big(a+\tfrac{e}{\mu}\big)^{-1} + (1-\lambda)\big(b+\tfrac{e}{\mu}\big)^{-1} &= \frac{\lambda d(a)}{d(a+\frac{e}{\mu})}a^{-1} + \frac{(1-\lambda)d(b)}{d(b+\frac{e}{\mu})}b^{-1}\\
&\quad + \left(\frac{\lambda}{d(a+\frac{e}{\mu})} + \frac{1-\lambda}{d(b+\frac{e}{\mu})}\right)\left(\frac{e}{\mu}\right)
\end{aligned}
\tag{12}
$$

Now it follows from (12) and Proposition 2(iii) that

$$
\begin{aligned}
LHS &= \Big(\big(\lambda\big(a+\frac{e}{\mu}\big)^{-1}+(1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1}\big)^{-1}+\big(-\mu e\big)^{-1}\Big)^{-1} \\
&= \frac{d(-\mu e)}{d\big(\lambda\big(a+\frac{e}{\mu}\big)^{-1}+(1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1}-\mu e\big)} \\
&\quad \Big(\lambda\big(a+\frac{e}{\mu}\big)^{-1}+(1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1}\Big) \\
&\quad +\frac{d\big(\lambda\big(a+\frac{e}{\mu}\big)^{-1}+(1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1}\big)}{d\big(\lambda\big(a+\frac{e}{\mu}\big)^{-1}+(1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1}-\mu e\big)}(-\mu e) \\
&= \frac{\mu^2}{M_3}\Big(\frac{\lambda d(a)}{d(a+\frac{e}{\mu})}\Big)a^{-1}+\frac{\mu^2}{M_3}\Big(\frac{(1-\lambda)d(b)}{d(b+\frac{e}{\mu})}\Big)b^{-1}+M_4(\mu e)
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
M_3 &= d\Big(\lambda\big(a+\frac{e}{\mu}\big)^{-1}+(1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1}-\mu e\Big), \text{ and} \\
M_4 &= \text{the } LHS \text{ of Proposition 4.}
\end{aligned}
$$

By means of Propositions 3 and 4, we see that all of the coefficients for $a^{-1},\ b^{-1}$ and μe of (11) and (13) are exactly the same, which completes the proof. ∎

We denote the well known *arithmetic average* and *harmonic average of* a *and* b by

$$
\begin{aligned}
\mathcal{A}(\lambda, 1-\lambda; a, b) &= \lambda a+(1-\lambda)b \\
\mathcal{H}(\lambda, 1-\lambda; a, b) &= \big(\lambda a^{-1}+(1-\lambda)b^{-1}\big)^{-1}.
\end{aligned}
$$

Then as a direct consequence of Theorem 1, the following result is obtained.

Theorem 2. (Arithmetic-resolvent-harmonic average inequalities). *For any* $a,\ b\succ 0$, *positive numbers* $0<\lambda<1$ *and* $\mu>0$, *we have*

$$
\mathcal{H}(\lambda, 1-\lambda; a, b)\preceq \mathcal{R}_\mu(\lambda, 1-\lambda; a, b)\preceq \mathcal{A}(\lambda, 1-\lambda; a, b).
$$

Proof. By the operator convexity of the inversion $x\mapsto x^{-1}$ for $x\succ 0$, we first get

$$
\begin{aligned}
\Big(\lambda\big(a+\frac{e}{\mu}\big)^{-1}+(1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1}\Big)^{-1}-\frac{e}{\mu} &\preceq \lambda\big(a+\frac{e}{\mu}\big)+(1-\lambda)\big(b+\frac{e}{\mu}\big)-\frac{e}{\mu} \\
&= \lambda a+(1-\lambda)b.
\end{aligned} \tag{14}
$$

From Corollary 1, the reverse order property of the inversion and replacing $a,\ b$ and μ in (14) by a^{-1}, b^{-1} and $1/\mu$ respectively, we have

$$\big(\lambda a^{-1}+(1-\lambda)b^{-1}\big)^{-1} \preceq \Big(\big(\lambda\big(a^{-1}+\mu e\big)^{-1}+(1-\lambda)\big(b^{-1}+\mu e\big)^{-1}\big)^{-1}-\mu e\Big)^{-1}$$
$$= \Big(\lambda\big(a+\frac{e}{\mu}\big)^{-1}+(1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1}\Big)^{-1}-\frac{e}{\mu}$$

where the equality follows from Theorem 1. This completes the proof. ∎

Theorem 3. (Limits). *For any* $a, b \succ 0$, *positive numbers* $0<\lambda<1$ *and* $\mu>0$, *we have*

(i) $\lim_{\mu\to 0^+} \mathcal{R}_\mu(\lambda, 1-\lambda; a, b) = \mathcal{A}(\lambda, 1-\lambda; a, b)$

(ii) $\lim_{\mu\to +\infty} \mathcal{R}_\mu(\lambda, 1-\lambda; a, b) = \mathcal{H}(\lambda, 1-\lambda; a, b)$

Proof. (i) Replacing a^{-1}, b^{-1} and μ by a, b and $\frac{1}{\mu}$, respectively, in (11) yields that

$$\begin{aligned}\mathcal{R}_\mu(\lambda, 1-\lambda; a, b) &= \Big(\lambda\big(a+\frac{e}{\mu}\big)^{-1}+(1-\lambda)\big(b+\frac{e}{\mu}\big)^{-1}\Big)^{-1}-\frac{e}{\mu}\\ &= \frac{d(b+\dfrac{e}{\mu})}{\lambda(1-\lambda)^2 d\big(\frac{1}{\lambda}(a+\frac{e}{\mu})+\frac{1}{1-\lambda}(b+\frac{e}{\mu})\big)}\,a\\ &\quad+\frac{d(a+\dfrac{e}{\mu})}{\lambda^2(1-\lambda) d\big(\frac{1}{\lambda}(a+\dfrac{e}{\mu})+\dfrac{1}{1-\lambda}(b+\dfrac{e}{\mu})\big)}\,b\\ &\quad+\left(\frac{\dfrac{1}{\lambda(1-\lambda)^2}d(b+\dfrac{e}{\mu})+\dfrac{1}{\lambda^2(1-\lambda)}d(a+\dfrac{e}{\mu})}{d\big(\frac{1}{\lambda}(a+\frac{e}{\mu})+\frac{1}{1-\lambda}(b+\frac{e}{\mu})\big)}-1\right)\frac{e}{\mu}.\end{aligned} \tag{15}$$

It is easy to check that

$$\begin{aligned} &d\big(a+\frac{e}{\mu}\big) = d(a)+\frac{1}{\mu^2}+\frac{2a_1}{\mu}, d\big(b+\frac{e}{\mu}\big) = d(b)+\frac{1}{\mu^2}+\frac{2b_1}{\mu} \quad \text{and}\\ &d\left(\frac{1}{\lambda}\big(a+\frac{e}{\mu}\big)+\frac{1}{1-\lambda}\big(b+\frac{e}{\mu}\big)\right)\\ &= d\big(\frac{a}{\lambda}+\frac{b}{1-\lambda}\big)+\frac{1}{\lambda^2(1-\lambda)^2}\cdot\frac{1}{\mu^2}+\frac{2((1-\lambda)a_1+\lambda b_1)}{\lambda^2(1-\lambda)^2}\cdot\frac{1}{\mu}.\end{aligned} \tag{16}$$

From the above three equalities (16), we can readily obtain that

$$\begin{aligned} &\lim_{\mu\to 0^+}\frac{d(b+\frac{e}{\mu})}{\lambda(1-\lambda)^2 d\big(\frac{1}{\lambda}(a+\frac{e}{\mu})+\frac{1}{1-\lambda}(b+\frac{e}{\mu})\big)} = \lambda,\\ &\lim_{\mu\to 0^+}\frac{d(a+\frac{e}{\mu})}{\lambda^2(1-\lambda) d\big(\frac{1}{\lambda}(a+\frac{e}{\mu})+\frac{1}{1-\lambda}(b+\frac{e}{\mu})\big)} = 1-\lambda \quad \text{and}\\ &\lim_{\mu\to 0^+}\left(\frac{\frac{1}{\lambda(1-\lambda)^2}d(b+\frac{e}{\mu})+\frac{1}{\lambda^2(1-\lambda)}d(a+\frac{e}{\mu})}{d\big(\frac{1}{\lambda}(a+\frac{e}{\mu})+\frac{1}{1-\lambda}(b+\frac{e}{\mu})\big)}-1\right)\cdot\frac{1}{\mu} = 0,\end{aligned} \tag{17}$$

which entails

$$\lim_{\mu\to 0^+} \mathcal{R}_\mu(\lambda, 1-\lambda; a, b) = \lambda a + (1-\lambda)b = \mathcal{A}(\lambda, 1-\lambda; a, b)$$

by (15).

(ii) Similarly, we get from (16)

$$\begin{aligned}
&\lim_{\mu\to+\infty} \frac{d(b+\frac{e}{\mu})}{\lambda(1-\lambda)^2 d\big(\frac{1}{\lambda}(a+\frac{e}{\mu})+\frac{1}{1-\lambda}(b+\frac{e}{\mu})\big)} = \frac{d(b)}{\lambda(1-\lambda)^2 d\big(\frac{a}{\lambda}+\frac{b}{1-\lambda}\big)},\\
(18)\quad &\lim_{\mu\to+\infty} \frac{d(a+\frac{e}{\mu})}{\lambda^2(1-\lambda) d\big(\frac{1}{\lambda}(a+\frac{e}{\mu})+\frac{1}{1-\lambda}(b+\frac{e}{\mu})\big)} = \frac{d(a)}{\lambda^2(1-\lambda) d\big(\frac{a}{\lambda}+\frac{b}{1-\lambda}\big)} \quad \text{and}\\
&\lim_{\mu\to+\infty} \left(\frac{\frac{1}{\lambda(1-\lambda)^2} d(b+\frac{e}{\mu}) + \frac{1}{\lambda^2(1-\lambda)} d(a+\frac{e}{\mu})}{d\big(\frac{1}{\lambda}(a+\frac{e}{\mu})+\frac{1}{1-\lambda}(b+\frac{e}{\mu})\big)} - 1 \right) \cdot \frac{1}{\mu} = 0.
\end{aligned}$$

Thus, by (15)

$$(19)\quad \lim_{\mu\to+\infty} \mathcal{R}_\mu(\lambda, 1-\lambda; a, b) = \frac{d(b)}{\lambda(1-\lambda)^2 d\big(\frac{a}{\lambda}+\frac{b}{1-\lambda}\big)}\ a + \frac{d(a)}{\lambda^2(1-\lambda) d\big(\frac{a}{\lambda}+\frac{b}{1-\lambda}\big)}\ b.$$

On the other hand, we see that

$$\begin{aligned}
\mathcal{H}(\lambda, 1-\lambda; a, b) &= \big(\lambda a^{-1} + (1-\lambda) b^{-1}\big)^{-1} = \left(\big(\frac{a}{\lambda}\big)^{-1} + \big(\frac{b}{1-\lambda}\big)^{-1} \right)^{-1}\\
&= \frac{d(\frac{b}{1-\lambda})}{d\big(\frac{a}{\lambda}+\frac{b}{1-\lambda}\big)} \cdot \frac{a}{\lambda} + \frac{d(\frac{a}{\lambda})}{d\big(\frac{a}{\lambda}+\frac{b}{1-\lambda}\big)} \cdot \frac{b}{1-\lambda}\\
(20)\qquad\qquad &\\
&= \frac{d(b)}{\lambda(1-\lambda)^2 d\big(\frac{a}{\lambda}+\frac{b}{1-\lambda}\big)}\ a + \frac{d(a)}{\lambda^2(1-\lambda) d\big(\frac{a}{\lambda}+\frac{b}{1-\lambda}\big)}\ b\\
&= \lim_{\mu\to+\infty} \mathcal{R}_\mu(\lambda, 1-\lambda; a, b),
\end{aligned}$$

where the third equality is due to Proposition 2(iii) and the last comes from (19). This completes our proof. ■

3. The Resolvent Average of m-Variables

Based on the previous results up to now, we deal with a general m-variables $a_1, \cdots, a_m$ case. Let $\mu > 0$ and $\lambda_i > 0$ with $\sum_{i=1}^m \lambda_i = 1$. Analogously to the two variables case, we define *the resolvent average of nonnegative m-variables* $a_1, \cdots, a_m$ with the parameter μ by

$$\mathcal{R}_\mu(\lambda_1, ..., \lambda_m; a_1, ..., a_m) = \left(\lambda_1\big(a_1 + \frac{e}{\mu}\big)^{-1} + \cdots + \lambda_m\big(a_m + \frac{e}{\mu}\big)^{-1} \right)^{-1} - \frac{e}{\mu}.$$

Following the same argument of Proposition 1, we immediately get two basic properties:

Proposition 5. *Let $a_i \succeq b_i \succeq 0$ for $i = 1, \ldots, m$. Then*

$$\mathcal{R}_\mu(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m) \succeq \mathcal{R}_\mu(\lambda_1, \ldots, \lambda_m; b_1, \ldots, b_m).$$

Moreover, if additionally $a_i \succ b_i$ for some i, we have

$$\mathcal{R}_\mu(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m) \succ \mathcal{R}_\mu(\lambda_1, \ldots, \lambda_m; b_1, \ldots, b_m).$$

Corollary 2. *Let $a_i \succ 0$ for $i = 1, \ldots, m$. Then*

$$\mathcal{R}_\mu(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m) \succ 0.$$

Though it is easily checked, the following recursion formula plays an important role as a ladder to the general case from two variables one.

Proposition 6. (Recursion). *We get*

$$\begin{aligned}
&\mathcal{R}_\mu(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m)\\
&= \mathcal{R}_\mu\big(1-\lambda_m, \lambda_m; \mathcal{R}_\mu(\frac{\lambda_1}{1-\lambda_m}, \ldots, \frac{\lambda_{m-1}}{1-\lambda_m}; a_1, \ldots, a_{m-1}),\, a_m\big).
\end{aligned}$$

Proof. By definition,

$$\begin{aligned}
&\mathcal{R}_\mu\big(1-\lambda_m, \lambda_m; \mathcal{R}_\mu(\frac{\lambda_1}{1-\lambda_m}, \ldots, \frac{\lambda_{m-1}}{1-\lambda_m}; a_1, \ldots, a_{m-1}),\, a_m\big)\\
&= \bigg((1-\lambda_m)\big(\mathcal{R}_\mu(\frac{\lambda_1}{1-\lambda_m}, \ldots, \frac{\lambda_{m-1}}{1-\lambda_m}; a_1, \ldots, a_{m-1}) + \frac{e}{\mu}\big)^{-1}\\
&\qquad + \lambda_m\big(a_m + \frac{e}{\mu}\big)^{-1}\bigg)^{-1} - \frac{e}{\mu}\\
&= \bigg((1-\lambda_m)\bigg(\frac{\lambda_1}{1-\lambda_m}\big(a_1 + \frac{e}{\mu}\big)^{-1} + \cdots + \frac{\lambda_{m-1}}{1-\lambda_m}\big(a_{m-1}\\
&\qquad + \frac{e}{\mu}\big)^{-1}\bigg) + \lambda_m\big(a_m + \frac{e}{\mu}\big)^{-1}\bigg)^{-1} - \frac{e}{\mu}\\
&= \bigg(\lambda_1\big(a_1 + \frac{e}{\mu}\big)^{-1} + \cdots + \lambda_m\big(a_m + \frac{e}{\mu}\big)^{-1}\bigg)^{-1} - \frac{e}{\mu}\\
&= \mathcal{R}_\mu(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m).
\end{aligned}$$

■

Theorem 4. (Self-duality of resolvent average). *For any $a_1, \ldots, a_m \succ 0$, $\lambda_i > 0$ with $\sum_{i=1}^m \lambda_i = 1$ and $\mu > 0$, it holds*

$$[\mathcal{R}_\mu(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m)]^{-1} = \mathcal{R}_{\mu^{-1}}(\lambda_1, \ldots, \lambda_m; a_1^{-1}, \ldots, a_m^{-1}). \tag{21}$$

Proof. Theorem 1 implies that (21) holds for $m = 2$. Suppose that the assertion is true for $m-1$. Employing Proposition 6 with Corollary 2, we have

$$
\begin{aligned}
&[\mathcal{R}_\mu(\lambda_1,\ldots,\lambda_m;a_1,\ldots,a_m)]^{-1}\\
&=\mathcal{R}_{\mu^{-1}}\big(1-\lambda_m,\lambda_m;[\mathcal{R}_\mu(\frac{\lambda_1}{1-\lambda_m},\ldots,\frac{\lambda_{m-1}}{1-\lambda_m};a_1,\ldots,a_{m-1})]^{-1},\ a_m^{-1}\big)\\
&=\mathcal{R}_{\mu^{-1}}\big(1-\lambda_m,\lambda_m;\mathcal{R}_{\mu^{-1}}(\frac{\lambda_1}{1-\lambda_m},\ldots,\frac{\lambda_{m-1}}{1-\lambda_m};a_1^{-1},\ldots,a_{m-1}^{-1}),\ a_m^{-1}\big)\\
&=\mathcal{R}_{\mu^{-1}}(\lambda_1,\ldots,\lambda_m;a_1^{-1},\ldots,a_m^{-1}). \qquad \blacksquare
\end{aligned}
$$

The *arithmetic average* and *harmonic average of* $a_1,\ldots,a_m$ is denoted by

$$
\begin{aligned}
\mathcal{A}(\lambda_1,\cdots,\lambda_m;a_1,\cdots,a_m) &= \lambda_1 a+\cdots+\lambda_m a_m\\
\mathcal{H}(\lambda_1,\cdots,\lambda_m;a_1,\cdots,a_m) &= \big(\lambda_1 a_1^{-1}+\cdots+\lambda_m a_m^{-1}\big)^{-1}.
\end{aligned}
$$

Theorem 5. (Arithmetic-resolvent-harmonic average inequalities). *For any* $a_1,\ldots,a_m\succ 0$, $\lambda_i>0$ *with* $\sum_{i=1}^m\lambda_i=1$ *and* $\mu>0$, *we obtain*

$$
\begin{aligned}
&\mathcal{H}(\lambda_1,\cdots,\lambda_m;a_1,\cdots,a_m)\\
&\preceq \mathcal{R}_\mu(\lambda_1,\ldots,\lambda_m;a_1,\ldots,a_m)\preceq \mathcal{A}(\lambda_1,\cdots,\lambda_m;a_1,\cdots,a_m).
\end{aligned}
$$

Proof. Exactly the same argument as in the proof of Theorem 2 yields the result. $\blacksquare$

To extend Theorem 3 regarding the limiting properties of the resolvent average for two variables to the general m-variables case, we should look at closer the continuity of the function

$$(0,+\infty)\times\mathrm{int}\mathcal{K}\times\mathrm{int}\mathcal{K}\ni(\mu,\ a,\ b)\mapsto\mathcal{R}_\mu(\lambda,\ 1-\lambda;a,\ b)\in\mathrm{int}\mathcal{K}\ \text{ for each } 0<\lambda<1.$$

To this end, for $\mu=0$, let us define

$$\mathcal{R}_0(\lambda,1-\lambda;a,b)=\mathcal{A}(\lambda,1-\lambda;a,b).$$

Then the continuity of the following extended function holds.

Proposition 7. (Continuity at **0**). *Let* $\lambda>0$. *Define* $\mathfrak{F}:[0,+\infty)\times\mathrm{int}\mathcal{K}\times\mathrm{int}\mathcal{K}\to\mathrm{int}\mathcal{K}$ *to be*

$$\mathfrak{F}(\mu,\ a,\ b)=\mathcal{R}_\mu(\lambda,\ 1-\lambda;a,\ b)$$

where $\mathfrak{F}(0,\ a,\ b)=\mathcal{R}_0(\lambda,\ 1-\lambda;a,\ b)=\mathcal{A}(\lambda,1-\lambda;a,b)$. *Then* $\mathfrak{F}$ *is continuous.*

Proof. It suffices to show that $\mathfrak{F}$ is continuous at $(0,a,b)$. Let $\{(\mu_k,a_k,b_k)\}_{k=1}^\infty$ be a convergent sequence to $(0,a,b)$ within $[0,+\infty)\times\mathrm{int}\mathcal{K}\times\mathrm{int}\mathcal{K}$. The sequence $\{(\mu_k,a_k,b_k)\}$ may be divided into two subsequences, that is, the one subsequence

$\{(\mu_{n_k}, a_{n_k}, b_{n_k})\}$ with $\mu_{n_k} = 0$ and the other subsequence $\{(\mu_{n_i}, a_{n_i}, b_{n_i})\}$ with $\mu_{n_i} > 0$. Note that $\mathfrak{F}(\mu_{n_k}, a_{n_k}, b_{n_k}) = \mathcal{R}_{\mu_{n_k}}(\lambda, 1-\lambda; a_{n_k}, b_{n_k}) = \mathcal{A}(\lambda, 1-\lambda; a_{n_k}, b_{n_k})$ converges to $\mathcal{A}(\lambda, 1-\lambda; a, b) = \mathfrak{F}(0, a, b)$ because the function $(x, y) \mapsto \mathcal{A}(\lambda, 1-\lambda; x, y)$ is clearly continuous on int$\mathcal{K} \times$ int$\mathcal{K}$ and $(a_{n_k}, b_{n_k}) \to (a, b)$. Moreover, to check that $\mathfrak{F}(\mu_{n_i}, a_{n_i}, b_{n_i}) = \mathcal{R}_{\mu_{n_i}}(\lambda, 1-\lambda; a_{n_i}, b_{n_i})$ also converges to $\mathcal{A}(\lambda, 1-\lambda; a, b) = \mathfrak{F}(0, a, b)$, we recall the equations (15) and (16). So

$$
\begin{aligned}
&\mathcal{R}_{\mu_{n_i}}(\lambda, 1-\lambda; a_{n_i}, b_{n_i}) \\
&= \left(\lambda\big(a_{n_i} + \frac{e}{\mu_{n_i}}\big)^{-1} + (1-\lambda)\big(b_{n_i} + \frac{e}{\mu_{n_i}}\big)^{-1}\right)^{-1} - \frac{e}{\mu_{n_i}} \\
&= \frac{d(b_{n_i} + \frac{e}{\mu_{n_i}})}{\lambda(1-\lambda)^2 d\big(\frac{1}{\lambda}(a_{n_i} + \frac{e}{\mu_{n_i}}) + \frac{1}{1-\lambda}(b_{n_i} + \frac{e}{\mu_{n_i}})\big)}\, a_{n_i} \\
&\quad + \frac{d(a_{n_i} + \frac{e}{\mu_{n_i}})}{\lambda^2(1-\lambda) d\big(\frac{1}{\lambda}(a_{n_i} + \frac{e}{\mu_{n_i}}) + \frac{1}{1-\lambda}(b_{n_i} + \frac{e}{\mu_{n_i}})\big)}\, b_{n_i} \\
&\quad + \left(\frac{\frac{1}{\lambda(1-\lambda)^2} d(b_{n_i} + \frac{e}{\mu_{n_i}}) + \frac{1}{\lambda^2(1-\lambda)} d(a_{n_i} + \frac{e}{\mu_{n_i}})}{d\big(\frac{1}{\lambda}(a_{n_i} + \frac{e}{\mu_{n_i}}) + \frac{1}{1-\lambda}(b_{n_i} + \frac{e}{\mu_{n_i}})\big)} - 1\right) \frac{e}{\mu_{n_i}}.
\end{aligned}
\tag{22}
$$

In addition,

$$
\begin{aligned}
d\big(a_{n_i} + \frac{e}{\mu_{n_i}}\big) &= d(a_{n_i}) + \frac{1}{\mu_{n_i}^2} + \frac{2(a_{n_i})_1}{\mu_{n_i}}, \\
d\big(b_{n_i} + \frac{e}{\mu_{n_i}}\big) &= d(b_{n_i}) + \frac{1}{\mu_{n_i}^2} + \frac{2(b_{n_i})_1}{\mu_{n_i}} \quad \text{and} \\
d\left(\frac{1}{\lambda}\big(a_{n_i} + \frac{e}{\mu_{n_i}}\big) + \frac{1}{1-\lambda}\big(b_{n_i} + \frac{e}{\mu_{n_i}}\big)\right) &= d\big(\frac{a_{n_i}}{\lambda} + \frac{b_{n_i}}{1-\lambda}\big) + \frac{1}{\lambda^2(1-\lambda)^2} \cdot \frac{1}{\mu_{n_i}^2} \\
&\quad + \frac{2[(1-\lambda)(a_{n_i})_1 + \lambda(b_{n_i})_1]}{\lambda^2(1-\lambda)^2} \cdot \frac{1}{\mu_{n_i}}.
\end{aligned}
\tag{23}
$$

From the above three equalities (23) and the continuities of determinant $x \mapsto d(x)$ and the projection map $x \mapsto x_1$, we have

$$
\begin{aligned}
&\lim_{i \to +\infty} \frac{d(b_{n_i} + \frac{e}{\mu_{n_i}})}{\lambda(1-\lambda)^2 d\big(\frac{1}{\lambda}(a_{n_i} + \frac{e}{\mu_{n_i}}) + \frac{1}{1-\lambda}(b_{n_i} + \frac{e}{\mu_{n_i}})\big)} = \lambda, \\
&\lim_{i \to +\infty} \frac{d(a_{n_i} + \frac{e}{\mu_{n_i}})}{\lambda^2(1-\lambda) d\big(\frac{1}{\lambda}(a_{n_i} + \frac{e}{\mu_{n_i}}) + \frac{1}{1-\lambda}(b_{n_i} + \frac{e}{\mu_{n_i}})\big)} = 1-\lambda \quad \text{and} \\
&\lim_{i \to +\infty} \left(\frac{\frac{1}{\lambda(1-\lambda)^2} d(b_{n_i} + \frac{e}{\mu_{n_i}}) + \frac{1}{\lambda^2(1-\lambda)} d(a_{n_i} + \frac{e}{\mu_{n_i}})}{d\big(\frac{1}{\lambda}(a_{n_i} + \frac{e}{\mu_{n_i}}) + \frac{1}{1-\lambda}(b_{n_i} + \frac{e}{\mu_{n_i}})\big)} - 1\right) \cdot \frac{1}{\mu_{n_i}} = 0,
\end{aligned}
\tag{24}
$$

which implies that

$$\lim_{i\to+\infty} \mathcal{R}_{\mu_{n_i}}(\lambda, 1-\lambda; a_{n_i}, b_{n_i}) = \lambda a + (1-\lambda)b = \mathcal{A}(\lambda, 1-\lambda; a, b)$$

by (22). Therefore, we have

$$\lim_{k\to+\infty} \mathfrak{F}(\mu_k,\, a_k,\, b_k) = \mathcal{A}(\lambda, 1-\lambda; a, b) = \mathfrak{F}(0,\, a,\, b).$$

This completes our proof. ∎

Proposition 8. (Continuity at $+\infty$). *Let* $\lambda > 0$. *It holds that*

$$\lim_{\mu\to+\infty} \mathcal{R}_{\mu}(\lambda,\, 1-\lambda; a_\mu,\, b_\mu) = \mathcal{H}(\lambda,\, 1-\lambda; a,\, b)$$

where $\lim_{\mu\to+\infty}(a_\mu, b_\mu) = (a, b)$ *with* $a, b, a_\mu, b_\mu \succ 0$.

Proof. Replacing a and b by a_μ and b_μ, respectively, in (15) and (16) yields that

$$\begin{aligned} &\lim_{\mu\to+\infty} \frac{d(b_\mu + \frac{e}{\mu})}{\lambda(1-\lambda)^2 d\big(\frac{1}{\lambda}(a_\mu + \frac{e}{\mu}) + \dfrac{1}{1-\lambda}(b_\mu + \dfrac{e}{\mu})\big)}\, a_\mu \\ &= \frac{d(b)}{\lambda(1-\lambda)^2 d\big(\dfrac{a}{\lambda} + \dfrac{b}{1-\lambda}\big)}\, a, \\ &\lim_{\mu\to+\infty} \frac{d(a_\mu + \frac{e}{\mu})}{\lambda^2(1-\lambda) d\big(\frac{1}{\lambda}(a_\mu + \frac{e}{\mu}) + \frac{1}{1-\lambda}(b_\mu + \dfrac{e}{\mu})\big)}\, b_\mu \\ &= \frac{d(a)}{\lambda^2(1-\lambda) d\big(\frac{a}{\lambda} + \dfrac{b}{1-\lambda}\big)}\, b \quad \text{and} \\ &\lim_{\mu\to+\infty} \left(\frac{\frac{1}{\lambda(1-\lambda)^2} d(b_\mu + \frac{e}{\mu}) + \frac{1}{\lambda^2(1-\lambda)} d(a_\mu + \frac{e}{\mu})}{d\big(\frac{1}{\lambda}(a_\mu + \frac{e}{\mu}) + \frac{1}{1-\lambda}(b_\mu + \frac{e}{\mu})\big)} - 1 \right) \cdot \frac{e}{\mu} = 0 \end{aligned} \tag{25}$$

by the continuities of determinant $x \mapsto d(x)$ and the projection map $x \mapsto x_1$. Hence

$$\lim_{\mu\to+\infty} \mathcal{R}_{\mu}(\lambda,\, 1-\lambda; a_\mu,\, b_\mu) = \mathcal{H}(\lambda,\, 1-\lambda; a,\, b)$$

by means of (19) and (20). This completes the proof. ∎

Remark 1. Let us define $\mathcal{R}_{+\infty}(\lambda,\, 1-\lambda; a,\, b) = \mathcal{H}(\lambda,\, 1-\lambda; a,\, b)$ for $\lambda > 0$ and $a, b \in \mathrm{int}\mathcal{K}$. Then by the same argument above, it can be proved that the map $\mathfrak{F}(\mu,\, a,\, b) = \mathcal{R}_\mu(\lambda,\, 1-\lambda; a,\, b)$ is continuous at $+\infty$ in the sense that $\mathfrak{F}(\mu_\alpha,\, a_\alpha,\, b_\alpha)$

converges to $\mathfrak{F}(+\infty, a, b) = \mathcal{R}_{+\infty}(\lambda, 1-\lambda; a, b) = \mathcal{H}(\lambda, 1-\lambda; a, b)$ whenever the net $(\mu_\alpha, a_\alpha, b_\alpha) \in [0, +\infty) \times \text{int}\mathcal{K} \times \text{int}\mathcal{K}$ goes to $(+\infty, a, b)$.

Now we are ready to state the following result.

Theorem 6. (Limits). *For any* $a_1, \ldots, a_m \succ 0$, $\lambda_i > 0$ *with* $\sum_{i=1}^m \lambda_i = 1$ *and* $\mu > 0$, *we have*

(i) $\lim_{\mu\to 0^+} \mathcal{R}_\mu(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m) = \mathcal{A}(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m)$

(ii) $\lim_{\mu\to +\infty} \mathcal{R}_\mu(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m) = \mathcal{H}(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m)$

Proof. (i) Theorem 3 implies that (i) holds for $m = 2$. Suppose that the assertion is true for $m-1$. Employing Propositions 6 and 7 with Corollary 2, we see that

$$\begin{aligned}
&\lim_{\mu\to 0^+} \mathcal{R}_\mu(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m) \\
&= \lim_{\mu\to 0^+} \mathcal{R}_\mu\big(1-\lambda_m, \lambda_m; \mathcal{R}_\mu(\frac{\lambda_1}{1-\lambda_m}, \ldots, \frac{\lambda_{m-1}}{1-\lambda_m}; a_1, \ldots, a_{m-1}), a_m\big) \\
&= \mathcal{A}\big(1-\lambda_m, \lambda_m; [\lim_{\mu\to 0^+} \mathcal{R}_\mu(\frac{\lambda_1}{1-\lambda_m}, \ldots, \frac{\lambda_{m-1}}{1-\lambda_m}; a_1, \ldots, a_{m-1})], a_m\big) \\
&= \mathcal{A}\big(1-\lambda_m, \lambda_m; \mathcal{A}(\frac{\lambda_1}{1-\lambda_m}, \ldots, \frac{\lambda_{m-1}}{1-\lambda_m}; a_1, \ldots, a_{m-1}), a_m\big) \\
&= \mathcal{A}(\lambda_1, \ldots, \lambda_m; a_1, \ldots, a_m).
\end{aligned}$$

(ii) Using the same argument, we get the result with the aid of Proposition 8. ■

References

1. M. Baes, Convexity and differentiability properties of spectral functions and spectral mappings on Euclidean Jordan algebras, *Linear Algebra Appl.*, **422** (2007), 664-700.
2. H. H. Bauschke, E. Matouskova and S. Reich, Projection and proximal point methods: convergence results and counterexamples, *Nonlinear Anal. TMA*, **56** (2004), 715-738.
3. H. H. Bauschke, Y. Lucet and X. Wang, Primal-dual symmetric intrinsic methods for finding antiderivatives of cyclically monotone operators, *SIAM J. Control Optim.*, **46** (2007), 2031-2051.
4. H. H. Bauschke, Y. Lucet and M. Trienis, How to transform one convex function continuously into another, *SIAM Rev.*, **50** (2008), 115-132.
5. H. H. Bauschke, R. Goebel, Y. Lucet and X. Wang, The proximal average: basic theory, *SIAM J. Optim.*, **19** (2008), 766-785.
6. H. H. Bauschke and X. Wang, The kernel average for two convex functions and its applications to the extension and representation of monotone operators, *Trans. Amer. Math. Soc.*, **361** (2009), 5947-5965.

7. H. H. Bauschke, Y. Lucet and M. Trienis, The piecewise linear-quadratic model for computational convex analysis, *Comput. Optim. Appl.*, **43** (2009), 95-118.

8. H. H. Bauschke, S. M. Moffat and X. Wang, The Resolvent average for positive semidefinite matrices, *Linear Algebra Appl.*, **432** (2010), 1757-1771.

9. J.-S. Chen, The convex and monotone functions associated with second-order cone, *Optimization*, **55** (2006), 363-385.

10. J. Faraut and A. Koranyi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.

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