GENERALIZED r-EXPONENTS OF PRIMITIVE DIGRAPHS

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Abstract. In this paper, as natural extensions of the generalized exponents of primitive digraphs introduced by R.A. Brualdi and B. Liu, we introduce the generalized r-exponents of primitive digraphs, where $1 \leq r \leq n$. Moreover, some properties and sharp upper bounds of the generalized r-exponents of primitive digraphs are obtained, respectively.

1. INTRODUCTION

Let D be a simple digraph, possibly with loops. A digraph D is called primitive if there exists a positive integer k such that for each ordered pair of vertices x and y (not necessarily distinct), there is a walk of length k from x to y. The smallest such k is called the primitive exponent of D, denoted by $\gamma(D)$. Notice that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of its cycles is 1 (see [6]).

In [1], R.A. Brualdi and B. Liu introduced the generalized exponents of primitive digraphs, which are generalizations of the exponents of primitive digraphs and the ergodic indices for the transition matrices of finite homogeneous Markov chains. It is known that the generalized exponents have been investigated systematically in recent years (see the survey [5]).

The generalized exponents have their interpretations in a model of memoryless communication system which is represented by a digraph D of order n (see [1]). From the interpretations, the generalized exponents characterize the shortest time that some information takes for all vertices (namely, n vertices) to know the initial bit information. It is natural to ask: how long is the shortest time that some information takes for r vertices to know the initial bit information, where $1 \leq r \leq n$. In order to answer this question, we introduce the generalized r-exponents of primitive digraphs

as follows. Let n, k, r be integers with $1 \leq k$, $r \leq n$, and D be a primitive digraph of order n throughout this paper. For $X \subseteq V(D)$ (the vertex set of D), let $R_t^D(X)$ denote the set of all vertices each of which can be reached by a walk of length t starting from some vertex in X.

Definition 1.1. The k-point r-exponent $p_r(D, k)$ of D is defined to be the smallest nonnegative integer p such that there exists $X \subseteq V(D)$ with $|X| = k$ and $|R_p^D(\{x\})| \geq r$ for every vertex $x \in X$.

Definition 1.2. The k-point r-same-exponent $s_r(D, k)$ of D is the smallest nonnegative integer p such that there exist $X \subseteq V(D)$ with $|X| = k$ and v_1, v_2, ... , $v_r \in V(D)$ which satisfy $R_p^D(\{x\}) \supseteq \{v_1, v_2, ... , v_r\}$ for every vertex $x \in X$.

Definition 1.3. The kth lower r-multiexponent $f_r(D, k)$ of D is the smallest nonnegative integer p such that there exists $X \subseteq V(D)$ with $|X| = k$ and $|R_p^D(X)| \geq r$.

Definition 1.4. The kth upper r-multiexponent $F_r(D, k)$ of D is the smallest nonnegative integer p such that for any $X \subseteq V(D)$ with $|X| = k$, $|R_p^D(X)| \geq r$.

Remark 1. For convenience, the parameters defined above are all called the generalized r-exponents. Obviously, $p_n(D, k) = s_n(D, k)$ is the k-point exponent of D, $p_n(D, n) = s_n(D, n)$ is the exponent of D, $f_n(D, k)$ is the kth lower multiexponent, and $F_n(D, k)$ is the kth upper multiexponent (see [1]).

In this paper, we shall study the sharp upper bounds of the generalized r-exponents of primitive digraphs respectively, that is,

$$p_r(n, k) := max\{p_r(D, k)\}, \ s_r(n, k) := max\{s_r(D, k)\},$$

$$f_r(n, k) := max\{f_r(D, k)\}, \ F_r(n, k) := max\{F_r(D, k)\},$$

where the maximums are taken over all primitive digraphs D of order n. Now some properties of the generalized r-exponents are listed as follows.

Proposition 1.5. *Let A be the adjacency matrix of a primitive digraph D of order n. Let k, r be integers with $1 \leq k$, $r \leq n$. Then*

(1) *$p_r(D, k)$ is the smallest power of A for which there exist k rows that each contains at least r positive entries;*

(2) *$s_r(D, k)$ is the smallest power of A for which there exists a $k \times r$ submatrix each of whose entries equals 1;*

(3) *$f_r(D, k)$ is the smallest power of A for which there are k rows having at most $n - r$ columns of all zeros;*

(4) $F_r(D, k)$ is the smallest power of A for which no set of k rows has $n-r+1$ columns of all zeros.

Proposition 1.6. *Let D be a primitive digraph of order n and $1 \leq r \leq n$. Then*

$$F_r(D, 1) = p_r(D, n), \ f_r(D, n) = F_r(D, n), \ f_r(D, 1) = p_r(D, 1) = s_r(D, 1).$$

It is easy to obtain the following relations of the generalized r-exponents.

$$s_r(D, 1) \leq s_r(D, 2) \leq \cdots \leq s_r(D, n-1) \leq s_r(D, n)$$
$$\| \qquad \text{VI} \qquad\qquad \text{VI} \qquad\qquad \text{VI}$$
$$p_r(D, 1) \leq p_r(D, 2) \leq \cdots \leq p_r(D, n-1) \leq p_r(D, n)$$
$$\text{VI} \qquad \text{VI} \qquad\qquad \text{VI} \qquad\qquad \|$$
$$F_r(D, n) \leq F_r(D, n-1) \leq \cdots \leq F_r(D, 2) \leq F_r(D, 1)$$
$$\| \qquad\quad \text{VI} \qquad\qquad \text{VI} \qquad\quad \text{VI}$$
$$f_r(D, n) \leq f_r(D, n-1) \leq \cdots \leq f_r(D, 2) \leq f_r(D, 1) = s_r(D, 1)$$

2. THE k-POINT r-EXPONENT

Lemma 2.1. *Let D be a primitive digraph of order n and $1 \leq k \leq n$. Then*

$$p_r(D, k) = 0 \ if \ and \ only \ if \ r = 1.$$

Proof. If $r = 1$, by Definition 1.1, we have $p_1(D, k) = 0$ immediately. Note that $R_0^D(\{v\}) = \{v\}$ for $v \in V(D)$. Hence $p_r(D, k) = 0$ implies $r = 1$. ∎

It follows from Lemma 2.1 that $p_1(n, k) = 0$, and it will suffice to study the k-point r-exponent with $r > 1$ in the following.

Let $L_{n, s}$ $(1 \leq s \leq n)$ be the digraph with the set $V = \{i \mid 1 \leq i \leq n\}$ of vertices and the set $E = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(n, 1)\} \cup \{(i, i) \mid 1 \leq i \leq s\}$ of arcs. Obviously, $L_{n, s}$ is a primitive digraph with s loop vertices, and

$$(1) \quad |R_t^{L_{n, s}}(\{i\})| = \begin{cases} min\{t+1, \ n\}, & if \ 1 \leq i \leq s, \\ 1, & if \ s+1 \leq i \leq n \ and \ t \leq n-i+1, \\ min\{t-n+i, \ n\}, & if \ s+1 \leq i \leq n \ and \ t > n-i+1. \end{cases}$$

Now a bound for the k-point r-exponent of a primitive digraph with loop vertices is given in the following theorem.

Theorem 2.2. *Let D be a primitive digraph of order n with s loop vertices, where $1 \leq s \leq n$. If $1 \leq k \leq n$ and $1 < r \leq n$, then*

$$p_r(D,\ k) \leq \begin{cases} r-1, & if\ k \leq s, \\ r-1+k-s, & if\ k > s. \end{cases}$$

Moreover, this bound can be attained and $L_{n,\ s}$ is one of its extremal digraphs.

Proof. If $k \leq s$, then let $X = \{x_1,\ x_2,\ \dots,\ x_k\} \subseteq V(D)$ $(|X| = k)$ be a subset of the s loop vertices. Since each $x_i \in X$ is a loop vertex, we have $|R_{r-1}^D(\{x_i\})| \geq r$, where $1 \leq i \leq k$. Therefore, $p_r(D,\ k) \leq r-1$.

If $k > s$, then let L be the set of all loop vertices. Since D is primitive, it is strongly connected, and then there is a set X^* of $k-s$ vertices whose distances to L are at most $k-s$. Let $X^* \cup L = X \subseteq V(D)$. Then $|X| = k$. Note that for each vertex $x \in X$, $R_{k-s}^D(\{x\}) \supseteq \{w_x\}$, where $w_x \in L$. Since w_x is a loop vertex, then $|R_{r-1}^D(\{w_x\})| \geq r$. It follows that $|R_{k-s+r-1}^D(\{x\})| \geq r$ for each vertex $x \in X$. Consequently, $p_r(D,\ k) \leq r-1+k-s$.

Moreover, it follows from (1) that $p_r(L_{n,\ s},\ k) = \begin{cases} r-1, & if\ k \leq s, \\ r-1+k-s, & if\ k > s. \end{cases}$

Hence the bound can be attained and $L_{n,\ s}$ is one of its extremal digraphs. ∎

Corollary 2.3. *Let D be a primitive digraph of order n and girth g. If $1 \leq k \leq n$ and $1 < r \leq n$, then $p_r(D,\ k) \leq \begin{cases} g(r-1), & if\ k \leq g, \\ g(r-1+k-g), & if\ k > g. \end{cases}$*

Proof. Let $D^{(g)}$ be the digraph whose vertices are the same as those of D such that there is an arc from vertex x to vertex y if and only if there is a walk in D from x to y of length g. Then $D^{(g)}$ is a primitive digraph having at least g loop vertices. By Theorem 2.2, $p_r(D,\ k) \leq g \cdot p_r(D^{(g)},\ k) \leq \begin{cases} g(r-1), & if\ k \leq g, \\ g(r-1+k-g), & if\ k > g. \end{cases}$ ∎

Now we want to determine the value $p_r(n,\ k)$. Let D_1 be the digraph with the set $V = \{i \mid 1 \leq i \leq n\}$ of vertices and the set $E = \{(i,\ i-1) \mid 2 \leq i \leq n\} \cup \{(1,\ n),\ (1,\ n-1)\}$ of arcs.

Lemma 2.4. *Let $n,\ k,\ r$ be integers with $1 \leq k \leq n$ and $1 < r \leq n$. Then*

$$p_r(D_1,\ k) = k + (n-1)(r-2).$$

Proof. Let $t = k + (n-1)(r-2)$. It is easy to verify that

$$|R_t^{D_1}(\{1\})| \geq r \ and \ |R_t^{D_1}(\{2\})| = |R_t^{D_1}(\{3\})| = \cdots = |R_t^{D_1}(\{k\})| = r.$$

By Definition 1.1, we have $p_r(D_1, k) \le k + (n-1)(r-2)$.

On the other hand, let μ be a nonnegative integer with $\mu \le t - 1$. Note that

$$|R_\mu^{D_1}(\{i\})| < r \ for \ each \ integer \ i \in \{k, \ k+1, \ \dots, \ n\}.$$

Therefore, $p_r(D_1, k) \ge k + (n-1)(r-2)$ and the proof is finished. ∎

Let D be a primitive digraph of order n and C be a directed cycle in D. Denote by $d^+(v)$ the out-degree of $v \in V(D)$, $P_C(x \to y)$ the unique directed path in C from $x \in V(C)$ to $y \in V(C)$, respectively. Especially, if $x = y$, then $P_C(x \to y)$ denotes the directed cycle C. Let $l(P)$ be the length of a directed path P.

Lemma 2.5. *Let D be a primitive digraph of order n and girth g. Let C_g be a directed cycle of length g in D. Then for $1 \le k \le n$ and $1 < r \le n$,*

$$p_r(D, k) \le g(r-2) + k.$$

Proof. We consider the following two cases.

Case 1. $1 \le k \le g$.

To begin with, we claim that there exists a vertex $v_g \in V(C_g)$ with $d^+(v_g) \ge 2$ such that there is a directed path P of length $l(P)$ from v_g to a vertex $u \in V(C_g)$ with $(V(P) - \{v_g, \ u\}) \cap V(C_g) = \emptyset$ satisfying $l(P) \ne l(P_{C_g}(v_g \to u))$.

By contradiction. Suppose for any vertex $v \in V(C_g)$ with $d^+(v) \ge 2$, and any path P from v to a vertex $u \in V(C_g)$ with $(V(P) - \{v, \ u\}) \cap V(C_g) = \emptyset$, we have $l(P) = l(P_{C_g}(v \to u))$. Note that D is primitive, in particular, strongly connected. Hence we conclude that the lengths of all elementary cycles in D are multiples of $l(C_g)$, which is a contradiction to the fact that the greatest common divisor of the lengths of all elementary cycles of D is 1 (see [6]).

Let $C_g = (v_1, \ v_2, \ \dots, \ v_g, \ v_1)$, and $X = \{v_{g-k+1}, \ \dots, \ v_{g-1}, \ v_g\}$ ($|X| = k$). Since $k \le g$ and $l(P) \ne l(P_{C_g}(v_g \to u))$, then each vertex of $v_i \in X$ can reach two different vertices (one is v_{i+k-g}, and the other is denoted by u_i) by some walks of length k, where $g - k + 1 \le i \le g$.

Note that the vertices of C_g in $D^{(g)}$ are loop vertices. Hence in $D^{(g)}$, $v_{i+k-g} \in V(C_g)$ is a loop vertex and there is an arc from v_{i+k-g} to u_i, where $g-k+1 \le i \le g$. Let $X_i = \{v_{i+k-g}, \ u_i\}$ ($g - k + 1 \le i \le g$). Then for each integer i with $g - k + 1 \le i \le g$, $|R_{r-2}^{D^{(g)}}(X_i)| \ge r$. Hence $|R_{g(r-2)}^D(X_i)| \ge r$, and then $|R_{k+g(r-2)}^D(\{v_i\})| \ge r$ for each vertex $v_i \in X$, where $g-k+1 \le i \le g$. Therefore, $p_r(D, \ k) \le g(r-2) + k$.

Case 2. $g < k \le n$.

Since D is strongly connected, there exists a set X of k vertices such that $V(C_g) \subseteq X$ and each vertex of $X - V(C_g)$ can reach some vertex of C_g by a walk of length $k - g$. Note that the vertices of C_g in $D^{(g)}$ are loop vertices.

Hence $|R_{r-1}^{D^{(g)}}(\{v\})| \geq r$, and then $|R_{g(r-1)}^{D}(\{v\})| \geq r$ for each vertex $v \in V(C_g)$. Therefore, each vertex of X can reach r vertices by some walks of length $(k-g) + g(r-1) = k + g(r-2)$. Thus we obtain that $p_r(D,\ k) \leq k + g(r-2)$.   ∎

Theorem 2.6. *Let $n,\ k,\ r$ be integers with $1 \leq k \leq n$ and $1 < r \leq n$. Then*

$$p_r(n,\ k) = k + (n-1)(r-2).$$

In addition, D_1 is one of the extremal digraphs.

Proof. Let D be a primitive digraph of order $n \geq 2$ and girth g. Then $g \leq n-1$. By Lemma 2.5, we have

$$p_r(D,\ k) \leq k + g(r-2) \leq k + (n-1)(r-2).$$

Besides, it follows from Lemma 2.4 that $p_r(D_1,\ k) = k + (n-1)(r-2)$. The proof of Theorem 2.6 is finished.   ∎

Remark 2. From Theorem 2.6, the k-point exponent $p_n(n,\ k) = k + (n–1)(n–2)$, which is one of the main results in [1].

3. The k-Point r-Same-exponen

Lemma 3.1. *Let D be a primitive digraph of order n and $1 \leq k \leq n$. Then*

$$s_r(D,\ k) = 0 \text{ if and only if } k = r = 1.$$

Proof. If $k = r = 1$, it follows from Definition 1.2 that $sp_1(D,\ 1) = 0$. Conversely, if $s_r(D,\ k) = 0$, by contradiction, suppose that $k > 1$ or $r > 1$.

Case 1. If $k > 1$, then for any k different vertices $x_1,\ \ldots,\ x_k$ of D, $R_0^D(\{x_i\}) = \{x_i\}$. However, $x_i \neq x_j$ if $i \neq j$. It implies $s_r(D,\ k) > 0$, a contradiction.

Case 2. If $r > 1$, by Lemma 2.1, we have $p_r(D,\ k) > 0$. It is obvious that $s_r(D,\ k) \geq p_r(D,\ k) > 0$, also a contradiction.   ∎

By Lemma 3.1, we have $s_r(n,\ k) = 0$ if $k = r = 1$. Now we consider the k-point r-same-exponent with $k + r > 2$.

Theorem 3.2. *Let D be a primitive digraph of order n with nonzero trace. If $1 \leq k,\ r \leq n$ and $k + r > 2$, then*

$$s_r(D,\ k) \leq r + k - 2.$$

Moreover, this bound is best possible and $L_{n,\ 1}$ is one of its extremal digraphs.

Proof. Because the trace of D is nonzero, there exists a loop vertex w in D. Since D is strongly connected, there exists a set X^* of $k-1$ vertices whose distances to w are at most $k-1$. Let $X^* \cup \{w\} = X \subseteq V$. Then $|X| = k$.

For each vertex $x \in X$, since w is a loop vertex, $R_{k-1}^D(\{x\}) \supseteq \{w\}$. In addition, $R_{r-1}^D(\{w\}) \supseteq \{v_1, \ldots, v_r\}$, where $v_1, \ldots, v_r \in V(D)$. Hence $R_{k+r-2}^D(\{x\}) \supseteq \{v_1, \ldots, v_r\}$ for each vertex $x \in X$. Therefore, $s_r(D, k) \leq r+k-2$. Moreover, it is easy to see that $s_r(L_{n,1}, k) = r+k-2$. This completes the proof. ∎

In the following, we focus on the value $s_r(n, k)$.

Lemma 3.3. *Let n, k, r be integers with $1 \leq k$, $r \leq n$ and $k+r > 2$. Then*

$$s_r(D_1, k) = \begin{cases} (k+r-3)(n-1)+1, & if\ k+r \leq n+1, \\ (n-2)(n-1)+(k+r-n), & if\ k+r \geq n+1. \end{cases}$$

Proof. Now we consider the following two cases.

Case 1. $k+r \leq n+1$. Let $t = (k+r-3)(n-1)+1$. Then

(2) $R_t^{D_1}(\{m\}) = \{m-1,\ m,\ \ldots,\ k+r-5+m,\ k+r-4+m\ (mod\ n)\}\ (2 \leq m \leq n)$,

(3) $\quad R_t^{D_1}(\{1\}) = \{n-1,\ n,\ \ldots,\ k+r-4+n,\ k+r-3+n\ (mod\ n)\}$.

Note that $R_t^{D_1}(\{1\}) \cap R_t^{D_1}(\{n\}) \cap \cdots \cap R_t^{D_1}(\{n-k+2\}) \supseteq \{n-1,\ n,\ \ldots,\ n+r-2\ (mod\ n)\}$ and $|\{n-1,\ n,\ \ldots,\ n+r-2\ (mod\ n)\}| = r$. Take $X = \{1,\ n,\ n-1,\ \ldots,\ n-k+2\}$ with $|X| = k$. It follows from Definition 1.2 that

$$s_r(D_1, k) \leq (k+r-3)(n-1)+1.$$

On the other hand, for each nonnegative integer $\mu \leq t-1$, it follows from (2) and (3) that there do not exist $X = \{x_1,\ x_2,\ \ldots,\ x_k\} \subseteq V(D_1)$ and $v_1,\ v_2,\ \ldots,\ v_r \in V(D_1)$ such that $R_\mu^{D_1}(\{x_i\}) \supseteq \{v_1,\ v_2,\ \ldots,\ v_r\}$ for each vertex $x_i \in X$ $(1 \leq i \leq k)$. Hence $s_r(D_1, k) \geq (k+r-3)(n-1)+1$.

Case 2. $k+r \geq n+1$. Let $k+r-n = i$. Then $1 \leq i \leq n$. Let $t = (n-2)(n-1)+(k+r-n) = (n-2)(n-1)+i$. It is easy to verify that

(4) $R_t^{D_1}(\{m\}) = \{1,\ 2,\ \ldots,\ n\}\ (1 \leq m \leq i),\ R_t^{D_1}(\{i+1\}) = \{1,\ 2,\ \ldots,\ n-1\}$,

(5) $\quad R_t^{D_1}(\{m\}) = \{1,\ 2,\ \ldots,\ n\} - \{m-i-1\}\ (i+2 \leq m \leq n)$.

Note that $R_t^{D_1}(\{1\}) \cap R_t^{D_1}(\{2\}) \cap \cdots \cap R_t^{D_1}(\{k\}) \supseteq \{k-i,\ k-i+1,\ \ldots,\ n-1\}$ and $|\{k-i,\ k-i+1,\ \ldots,\ n-1\}| = n-k+i = r$. Take $X = \{1,\ 2,\ \ldots,\ k\}$ with $|X| = k$. By Definition 1.2, $s_r(D_1, k) \leq (n-2)(n-1)+(k+r-n)$.

On the other hand, for each nonnegative integer $\mu \leq t-1$, it follows from (4) and (5) that there do not exist $X = \{x_1, x_2, \ldots, x_k\} \subseteq V(D_1)$ and $v_1, v_2, \ldots, v_r \in V(D_1)$ such that $R_\mu^{D_1}(\{x_i\}) \supseteq \{v_1, v_2, \ldots, v_r\}$ for each vertex $x_i \in X$ ($1 \leq i \leq k$). Hence $s_r(D_1, k) \geq (n-2)(n-1)+(k+r-n)$.

Therefore, $s_r(D_1, k) = \begin{cases} (k+r-3)(n-1)+1, & if\ k+r \leq n+1, \\ (n-2)(n-1)+(k+r-n), & if\ k+r \geq n+1. \end{cases}$ ∎

Let D_2 be the digraph with the set $V = \{i \mid 1 \leq i \leq n\}$ of vertices and the set $E = \{(i, i-1) \mid 2 \leq i \leq n\} \cup \{(n, n-2), (1, n), (1, n-1)\}$ of arcs.

Corollary 3.4. *Let n, k, r be integers with $1 \leq k$, $r \leq n$ and $k+r > 2$. Then*

$$s_r(D_2, k) \leq \begin{cases} (k+r-3)(n-1)+1, & if\ k+r \leq n+1, \\ (n-2)(n-1)+(k+r-n), & if\ k+r \geq n+1. \end{cases}$$

Proof. Let $A(D_1)$ and $A(D_2)$ be the adjacent matrices of D_1 and D_2, respectively. Clearly, $A(D_2) = A(D_1) + B$, where B is a nonnegative matrix. By Proposition 1.5, $s_r(D_1, k)$ is the smallest power of $A(D_1)$ for which there exists a $k \times r$ submatrix each of whose entries equals 1. It follows that there exists a $k \times r$ submatrix each of whose entries equals 1 in $A(D_2)^{s_r(D_1, k)} = (A(D_1)+B)^{s_r(D_1, k)}$. Hence we obtain that $s_r(D_2, k) \leq s_r(D_1, k)$. ∎

Lemma 3.5. *Let D be a primitive digraph of order n and girth g. Let C_g be a directed cycle of length g in D. If $1 \leq k$, $r \leq n$ and $k+r > 2$, then*

$$s_r(D, k) \leq \begin{cases} g(n-1), & if\ k \leq g, \\ g(n-2)+k, & if\ k \geq g. \end{cases}$$

Proof. We discuss two cases as follows.

Case 1. $k \leq g$. Note that the vertices of C_g in $D^{(g)}$ are loop vertices. Take $X \subset V(C_g)$ with $|X| = k$. Since each vertex of X is a loop vertex in $D^{(g)}$, it follows that $R_{n-1}^{D^{(g)}}(\{x\}) = V(D)$, and then $R_{g(n-1)}^D(\{x\}) = V(D)$ for each vertex $x \in X$. Then by Definition 1.2, $s_r(D, k) \leq g(n-1)$.

Case 2. $k \geq g$. Since D is strongly connected, there exists a set X of k vertices such that $V(C_g) \subset X$ and each vertex of $X - V(C_g)$ can reach some vertex of C_g by a walk of length $k - g$. Note that the vertices of C_g in $D^{(g)}$ are loop vertices. Hence $R_{n-1}^{D^{(g)}}(\{v\}) = V(D)$, and then $R_{g(n-1)}^D(\{v\}) = V(D)$ for each vertex $v \in V(C_g)$. It follows that $R_{(k-g)+g(n-1)}^D(\{x\}) = V(D)$ for each vertex $x \in X$. Consequently, we have $s_r(D, k) \leq g(n-2)+k$. ∎

Theorem 3.6. *Let n, k, r be integers with $1 \leq k$, $r \leq n$ and $k+r \geq n+2$. Then*

$$s_r(n,\ k) = (n-2)(n-1) + (k+r-n).$$

In addition, D_1 is one of its extremal digraphs.

Proof. Let D be a primitive digraph of order n and girth g. If D is isomorphic to D_1 or D_2, by Lemma 3.3 and Corollary 3.4, we have

$$s_r(D,\ k) \leq (n-2)(n-1) + (k+r-n)\ for\ k+r \geq n+2.$$

If D is not isomorphic to D_1 or D_2, then $g \leq n-2$. By Lemma 3.5,

$$s_r(D,\ k) \leq \begin{cases} g(n-1) \leq (n-2)(n-1) & if\ k \leq g, \\ g(n-2) + k \leq (n-2)(n-2) + k & if\ k \geq g. \end{cases}$$

It follows that $s_r(D,\ k) \leq (n-2)(n-1) + max\{0,\ k+2-n\}$. Note that $k+r \geq n+2$ and $1 \leq k \leq n$. Hence $r \geq 2$ and then $max\{0,\ k+2-n\} \leq k+r-n$. Thus $s_r(D,\ k) \leq (n-2)(n-1) + (k+r-n)$ for $k+r \geq n+2$.

In addition, by Lemma 3.3, D_1 is one of its extremal digraphs.   ■

Lemma 3.7. *Let D be a primitive digraph of order n and girth g. If $1 \leq k,\ r \leq n$ and $k+r > 2$, then $s_r(D,\ k) \leq g(r+k-3) + 1$.*

Proof. We have the following two case.

Case 1. $k = 1$. Note that $s_r(D,\ 1) = p_r(D,\ 1)$. By Lemma 2.5,

$$s_r(D,\ 1) \leq g(r-2) + 1.$$

Case 2. $k > 1$. Let C_g be a directed cycle of length g in D, and z be a vertex in C_g such that the in-degree $d^-(z) \geq 2$. Let A be the adjacency matrix of D, and D^T be the associated digraph of A^T, where A^T is the transposed matrix of A.

In D^T, note that z is a vertex of a directed cycle C_g^T of length g and $d^+(z) \geq 2$. Suppose there are arcs from z to the vertices x and y in D^T, where $x \in V(C_g^T)$. Let $S = R_1^{D^T}(\{z\})$. Then $S \supseteq \{x,\ y\}$. Hence in $[D^T]^{(g)}$, x is a loop vertex and there is an arc from x to y. It follows that $|R_{k-2}^{[D^T]^{(g)}}(S)| \geq k$, and then $|R_{g(k-2)}^{D^T}(S)| \geq k$.

Consequently, there exist k vertices (denoted by $u_1,\ u_2,\ \ldots,\ u_k$) which are reachable from z by some walks of length exactly $g(k-2) + 1$ in D^T. In other words, each vertex of $X = \{u_1,\ u_2,\ \cdots,\ u_k\}$ ($|X| = k$) can reach the vertex z by some walks of length exactly $g(k-2) + 1$ in D. Notice that z is a loop vertex in $D^{(g)}$. Hence there exist r vertices $v_1,\ v_2,\ \ldots,\ v_r \in V(D)$ such that $R_{r-1}^{D^{(g)}}(\{z\}) \supseteq \{v_1,\ v_2,\ \ldots,\ v_r\}$, and then $R_{g(r-1)}^D(\{z\}) \supseteq \{v_1,\ v_2,\ \ldots,\ v_r\}$.

Therefore, each vertex of $X = \{u_1,\ u_2,\ \ldots,\ u_k\}$ ($|X| = k$) can reach these r vertices $v_1,\ v_2,\ \ldots,\ v_r$ by some walks of length exactly $g(r+k-3) + 1$ in D. Then we have $s_r(D,\ k) \leq g(r+k-3) + 1$.   ■

Theorem 3.8. *Let n, k, r be integers, $1 \leq k$, $r \leq n$ and $2 < k + r \leq n + 1$.*
Then

$$s_r(n, \ k) = (k + r - 3)(n - 1) + 1.$$

Moreover, D_1 is one of its extremal digraphs.

Proof. By Lemma 3.3, $s_r(D_1, \ k) = (k + r - 3)(n - 1) + 1$ for $2 < k + r \leq n + 1$.
Let D be a primitive digraph of order n and girth g. Then $g \leq n - 1$. By Lemma
3.7,

$$s_r(D, \ k) \leq g(r + k - 3) + 1 \leq (n - 1)(r + k - 3) + 1.$$

This completes the proof of Theorem 3.8. ∎

4. The kTH Lower r-Multiexponent

Lemma 4.1. *Let D be a primitive digraph of order n and $1 \leq k$, $r \leq n$. Then*

$$f_r(D, \ k) = 0 \ if \ and \ only \ if \ k \geq r.$$

Proof. If $k \geq r$, it follows from Definition 1.3 that $f_r(D, \ k) = 0$. Conversely,
for any $X \subseteq V(D)$ with $|X| = k$, $R_0^D(X) = X$. Hence $f_r(D, \ k) = 0$ implies
$k \geq r$. ∎

By Lemma 4.1, if $k \geq r$, then $f_r(n, \ k) = 0$. Hence it will suffice to investigate
the kth lower r-multiexponent with $n \geq r > k \geq 1$.

Theorem 4.2. *Let D be a primitive digraph of order n and girth g, and $1 \leq g \leq k < r \leq n$. Then $f_r(D, \ k) \leq r - k$, and this bound can be attained.*

Proof. Let Y be the set of all vertices of a directed cycle of length g. Since D
is strongly connected, there exists a set X of k vertices such that $Y \subseteq X$ and each
vertex of $X - Y$ can be reached from Y by some walks whose vertices belong to X
(If $k = g$, then $X = Y$.). Therefore, $|R_{r-k}^D(X)| \geq r$, and then $f_r(D, \ k) \leq r - k$.

Let $W_{n, \ g, \ t}$ ($1 \leq t \leq g < n$) be the digraph with the set $V = \{i \mid 1 \leq i \leq n\}$
of vertices and the set $E = \{(i, \ i + 1) \mid 1 \leq i \leq n - 1\} \cup \{(g, \ 1), (n, \ t)\}$ of
arcs, where t is the smallest positive integer such that $g.c.d.(g, \ n - t + 1) = 1$
(the greatest common divisor). (Since there exists a nonnegative integer h such that
$gh < n \leq g(h + 1)$, suppose $n - t + 1 = hg + 1$. Note that $g.c.d.(g, \ n - t + 1) = g.c.d.(g, \ hg + 1) = 1$. Hence $W_{n, \ g, \ t}$ do exist.)

Obviously, $W_{n, \ g, \ t}$ is a primitive digraph with girth g. It is easy to verify that
$f_r(W_{n, \ g, \ t}, \ k) = r - k$ for $g \leq k < r$. The proof is finished. ∎

Corollary 4.3. *Let D be a primitive digraph of order n with nonzero trace,
and $1 \leq k < r \leq n$. Then $f_r(D, \ k) \leq r - k$ and this bound is best possible.*

Proof. If the trace of D is nonzero, then the girth of D equals 1. By Theorem 4.2, we have $f_r(D,\ k) \le r - k$ immediately. On the other hand, it is easy to see that $f_r(L_{n,\ 1},\ k) = r - k$. Then we obtain the desired result. ∎

Theorem 4.4. *Let D be a primitive digraph of order n and girth g, $1 \le k < min\{r,\ g\} \le n$. Then $f_r(D,\ k) \le 1 + g(r - k - 1)$.*

Proof. If $1 \le k < min\{r,\ g\} \le n$, then let $\gamma_g = (x_1,\ ...\ ,\ x_g,\ x_1)$ be a directed cycle of length g. Because D is a primitive digraph, there exists a vertex, say x_1 of γ_g, such that there is an arc from x_1 to a vertex z not on γ_g. Let $X = \{x_1,\ x_2,\ ...\ ,\ x_k\}$, and let $Y = R_1^D(X)$. Thus $Y \supseteq \{z,\ x_2,\ ...\ ,\ x_{k+1}\}$.

In $D^{(g)}$, $x_2,\ ...\ ,\ x_{k+1}$ are loop vertices and there is an arc from x_2 to the vertex z. It follows that $|R_{r-k-1}^{D^{(g)}}(Y)| \ge r$, and then $|R_{g(r-k-1)}^D(Y)| \ge r$. Therefore, $|R_{1+g(r-k-1)}^D(X)| \ge r$, and this completes the proof. ∎

Corollary 4.5. *Let n, k, r be integers with $1 \le k < r \le n$. Then*

$$f_r(n,\ k) \le 1 + (n-1)(r-k-1),\ f_r(n,\ 1) = 1 + (n-1)(r-2),\ f_r(n,\ r-1) = 1.$$

Proof. Note that the girth of a primitive digraph of order n satisfies $g \le n - 1$. Then by Theorems 4.2 and 4.4, we have

$$f_r(n,\ k) \le max\{r - k,\ 1 + (n-1)(r-k-1)\} = 1 + (n-1)(r-k-1).$$

Moreover, by Proposition 1.6 and Theorem 2.6, $f_r(n,\ 1) = p_r(n,\ 1) = 1 + (n-1)(r-2)$. If $k = r - 1 < r$, then $f_r(n,\ r-1) \le 1 + (n-1)(r-1-k) = 1$. Since $1 \le k < r \le n$, it follows from Lemma 4.1 that $f_r(n,\ r-1) = 1$. ∎

Let g be the girth of D. If $k \ge g$, it follows from Theorem 4.2 that we obtain a sharp bound for the kth lower r-multiexponent. However, if $k < g$ and $k \mid g$, the bound in the following theorem is better than that of Theorem 4.4.

Theorem 4.6. *Let D be a primitive digraph of order n and girth g, $1 \le k < r \le n$. If $k \mid g$, then $f_r(D,\ k) \le 1 + \frac{g}{k}(r - k - 1)$.*

Proof. Let γ_g be a directed cycle of length g. Since D is primitive, then there exists $x_1 \in V(\gamma_g)$ such that there is an arc from x_1 to a vertex z, where $z \notin V(\gamma_g)$. Let $x_1,\ x_2,\ ...\ ,\ x_k \in V(\gamma_g)$ such that the distance between x_i and x_{i+1} $(1 \le i \le k\ (mod\ k))$ is $\frac{g}{k}$ since $k \mid g$. Let $X = \{x_1,\ x_2,\ ...\ ,\ x_k\}$ and $Y = R_1^D(X)$. Then $Y \supseteq \{z,\ x_1^*,\ x_2^*,\ ...\ ,\ x_k^*\}$, where there is an arc from x_i to $x_i^* \in V(\gamma_g)$ $(1 \le i \le k)$, and the distance between x_i^* and x_{i+1}^* $(1 \le i \le k\ (mod\ k))$ is also $\frac{g}{k}$.

Now we consider the strongly connected digraph $D^{(\frac{g}{k})}$ whose vertices are the same as those of D and arcs correspond to the walks of length $\frac{g}{k}$ in D. Thus $(x_1^*,\ ...\ ,\ x_k^*,\ x_1^*)$ is a cycle of length k and $(x_k^*,\ z)$ is an arc in $D^{(\frac{g}{k})}$.

Hence $|R_{r-k-1}^{D^{(\frac{g}{k})}}(Y)| \geq r$, and then $|R_{1+\frac{g}{k}(r-k-1)}^{D}(X)| \geq r$. It follows that $f_r(D, k) \leq 1 + \frac{g}{k}(r - k - 1)$ if $k \mid g$. ∎

5. THE kTH UPPER r-MULTIEXPONENT

Lemma 5.1. *Let D be a primitive digraph of order n and $1 \leq k$, $r \leq n$. Then*

$$F_r(D, k) = 0 \ if \ and \ only \ if \ k \geq r.$$

Proof. If $k \geq r$, by Definition 1.4, then $F_r(D, k) = 0$. On the other hand, for any $X \subseteq V(D)$ with $|X| = k$, $R_0^D(X) = X$. Hence $F_r(D, k) = 0$ implies $k \geq r$. ∎

By Lemma 5.1, if $k \geq r$, then $F_r(n, k) = 0$. Therefore, $F_r(D, k)$ need to be studied when $1 \leq k < r \leq n$.

Theorem 5.2. *Let D be a primitive digraph of order n with s loop vertices, and $1 \leq k < r \leq n$. Then*

$$F_r(D, k) \leq n - s - k + r.$$

Moreover, this bound can be attained and $L_{n, s}$ is one of its extremal digraphs.

Proof. Let X be any set of k vertices of D.

Case 1. $k - n + s > 0$. Then X contains at least $k - n + s$ loop vertices. Since $r > k \geq k - n + s$, we have $|R_{r-k-s+n}^D(X)| \geq r$. Thus $F_r(D, k) \leq n - s - k + r$.

Case 2. $k - n + s \leq 0$. If X contains a loop vertex, then it is obvious that $F_r(D, k) \leq r - 1 \leq n - s - k + r$.

Now suppose that X contains no loop vertices. Then there exists $x \in X$ such that $R_{n-s-k+1}^D(\{x\}) \supseteq \{w\}$, where w is a loop vertex. It follows that $|R_{r-1}^D(\{w\})| \geq r$ and then $|R_{(n-s-k+1)+(r-1)}^D(\{x\})| \geq r$. Hence $F_r(D, k) \leq n - s - k + r$.

Now considering the digraph $L_{n, s}$, by (1), it is easy to prove that $F_r(L_{n, s}, k) = n - s - k + r$ if $k < r$. This completes the proof. ∎

Corollary 5.3. *Let D be a primitive digraph of order n and girth g, $1 \leq k < r \leq n$. Then $F_r(D, k) \leq g(n - g - k + r)$.*

Proof. Note that $D^{(g)}$ is a primitive digraph with at least g loop vertices. By Theorem 5.2, we have $F_r(D, k) \leq g \cdot F_r(D^{(g)}, k) \leq g(n - g - k + r)$. ∎

Theorem 5.4. *Let D be a primitive digraphs of order n and girth g. If $1 \leq k < r \leq n$, then $F_r(D, k) \leq \begin{cases} g(r - 1), & if \ n - k - g + 1 \leq 0, \\ g(r - 2) + (n - k + 1), & if \ n - k - g + 1 > 0. \end{cases}$*

Proof. Let X be any set of k vertices and C_g be a directed cycle of length g.

Case 1. If $n - k - g + 1 \leq 0$, then $X \cap V(C_g) \neq \emptyset$. Taking a consideration on $D^{(g)}$, each vertex of C_g is a loop vertex in $D^{(g)}$. Therefore, $|R_{r-1}^{D^{(g)}}(X \cap V(C_g))| \geq r$, and then $|R_{g(r-1)}^{D}(X)| \geq |R_{g(r-1)}^{D}(X \cap V(C_g))| \geq r$. Hence $F_r(D, k) \leq g(r-1)$.

Case 2. If $n - k - g + 1 > 0$, since C_g is a directed cycle, then there exists a vertex $v \in V(C_g)$ such that $R_{n-k-g+1}^{D}(X) \supseteq \{v\}$. Since $v \in V(C_g)$ is a loop vertex in $D^{(g)}$, we have $|R_{r-1}^{D^{(g)}}(\{v\})| \geq r$, and then $|R_{g(r-1)}^{D}(\{v\})| \geq r$. Therefore, $|R_{(n-k-g+1)+g(r-1)}^{D}(X)| \geq r$, and then $F_r(D, k) \leq g(r-2) + (n-k+1)$. ∎

Corollary 5.5. *Let n, k, r be integers with $2 \leq k < r \leq n$. Then*

$$F_r(n, 1) = 1 + (n-1)(r-1), \text{ and } 1 + (n-1)(r-k) \leq F_r(n, k) \leq (n-1)(r-1).$$

Proof. Since the girth of a primitive digraph of order n satisfies $g \leq n - 1$, it follows from Theorem 5.4 that $F_r(n, k) \leq \begin{cases} 1 + (n-1)(r-1), & \text{if } k = 1, \\ (n-1)(r-1), & \text{if } k \geq 2. \end{cases}$

Now we consider the primitive digraph D_1. Directly computing, it is easy to show that $F_r(D_1, k) = 1 + (n-1)(r-k)$. Hence $F_r(n, 1) = 1 + (n-1)(r-1)$ and $F_r(n, k) \geq 1 + (n-1)(r-k)$ for $k \geq 2$. ∎

Remark 3. Note that the kth upper multiexponent $F_n(n, k) = 1 + (n-1)(n-k)$ (see [1]). We conjecture that $F_r(n, k) = 1 + (n-1)(r-k)$. What's more, the case of $k = 1$ has been proved in Corollary 5.5.

Theorem 5.6. *Let D be a primitive digraph of order n, $A(D) = (a_{ij})$ be the adjacency matrix of D, and $1 \leq k < r \leq n$. If there exist distinct vertices i, j, h such that $a_{ij} = a_{ji} = a_{jh} = a_{hi} = 1$, then $F_r(D, k) \leq n + r - k$.*

Proof. Let X be any set of k vertices of D, and let $Z = \{i, j\}$. Note that $a_{ij} = a_{ji} = a_{jh} = a_{hi} = 1$. Then $|R_{r+1}^{D}(\{i\})| \geq r$ and $|R_{r+1}^{D}(\{j\})| \geq r$.

If $k = n - 1$, then $Z \cap X \neq \emptyset$. Hence $F_r(D, k) \leq r + 1 = n - k + r$.

If $k \leq n - 2$ and $Z \cap X \neq \emptyset$, then $F_r(D, k) \leq r + 1 \leq n - k + r$.

Now suppose that $k \leq n - 2$ and $Z \cap X = \emptyset$. Then there exists a vertex $x \in X$ such that $R_{n-k-1}^{D}(\{x\}) \cap Z \neq \emptyset$. Note that $|R_{r+1}^{D}(\{i\})| \geq r$ and $|R_{r+1}^{D}(\{j\})| \geq r$. Hence $|R_{n-k+r}^{D}(X)| \geq |R_{(n-k-1)+(r+1)}^{D}(\{x\})| \geq r$. Therefore, $F_r(D, k) \leq n - k + r$. ∎

Remark 4. If the digraph D satisfies the conditions of Theorem 5.6 and $k < r - 1$, then $F_r(D, k) \leq n + r - k \leq 1 + (n-1)(r-k)$. Moreover, it follows from Theorem 5.2 that if D contains loop vertices, then $F_r(D, k) \leq n - 1 + r - k \leq 1 + (n-1)(r-k)$. Hence the conjecture for $F_r(n, k)$ is true for these two cases.

Remark 5. Note that the generalized n-exponents (if $r = n$) of primitive digraphs are the generalized exponents of primitive digraphs defined in [1]. Therefore, the results in [1] can be obtained immediately from this paper.

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