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CLASSIFICATION OF PSEUDO-UMBILICAL SLANT SURFACES IN LORENTZIAN COMPLEX SPACE FORMS

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Abstract. In this paper we prove that slant surfaces in a non-flat Lorentzian complex space form must be Lagrangian. By applying this result, we completely classify pseudo-umbilical slant surfaces in Lorentzian complex space forms. Our classification results state that there exist two families of pseudo-umbilical slant surfaces in Lorentzian complex plane $\mathbb{C}_1^2$, three families in complex projective plane CP_1^2, and three families in complex hyperbolic plane CH_1^2.

1. INTRODUCTION

Let $\tilde{M}_i^n(4c)$ be a simply-connected indefinite complex space form of complex dimension n and complex index i. Here, the complex index is defined as the complex dimension of the largest complex negative definite subspace of the tangent space. If $i = 1$, we say that $\tilde{M}_1^n(4c)$ Lorentzian. The curvature tensor $\tilde{R}$ of $\tilde{M}_i^n(4c)$ is given by

$$\begin{aligned}\tilde{R}(X,Y)Z = c\{&\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX \\ &-\langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ\}.\end{aligned} \tag{1.1}$$

Let $\mathbb{C}^n$ denote the complex number n-space with complex coordinates $z_1, ..., z_n$. The $\mathbb{C}^n$ endowed with $g_{s,n}$, i.e., the real part of the Hermitian form

$$b_{s,n}(z,\omega) = -\sum_{k=1}^{s} \bar{z}_k\omega_k + \sum_{j=s+1}^{n} \bar{z}_j\omega_j, \qquad z,\omega \in \mathbb{C}^n, \tag{1.2}$$

defines a flat indefinite complex space form with complex index s. We denote the pair $(\mathbb{C}^n, g_{s,n})$ by $\mathbb{C}_s^n$ briefly, which is the flat Lorentzian complex n-space. In particular, $\mathbb{C}_1^2$ is the flat complex Lorentzian plane.

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Consider the differentiable manifold:

$$S_2^{2n+1}(c) = \{z \in \mathbb{C}_1^{n+1};\ b_{1,n+1}(z,z) = c^{-1} > 0\},$$

which is an indefinite real space form of constant sectional curvature c. The Hopf-fibration

$$\pi : S_2^{2n+1}(c) \to CP_1^n(4c) : z \mapsto z \cdot \mathbb{C}^*$$

is a submersion and there exists a unique pseudo-Riemannian metrix of complex index one on $CP_1^n(4c)$ such that π is a Riemannian submersion.

The pseudo-Riemannian manifold $CP_1^n(4c)$ is a Lorentzian complex space form of positive holomorphic sectional curvature $4c$.

Analogously, if $c < 0$, consider

$$H_2^{2n+1}(c) = \{z \in \mathbb{C}_2^{n+1}; b_{1,n+1}(z,z) = c^{-1} < 0\},$$

which is an indefinite real space form of constant sectional curvature c. The Hopf-fibration

$$\pi : H_2^{2n+1}(c) \to CH_1^n(4c) : z \mapsto z \cdot \mathbb{C}^*$$

is a submersion and there exists a unique pseudo-Riemannian metrix of complex index one on $CH_1^n(4c)$ such that π is a Riemannian submersion.

The pseudo-Riemannian manifold $CH_1^n(4c)$ is a Lorentzian complex space form of negative holomorphic sectional curvature $4c$.

It's well-known that a complete simply-connected Lorentzian complex space form $\tilde{M}_i^n(4c)$ is holomorphic isometric to $\mathbb{C}_1^n$, $CP_1^n(4c)$, or $CH_1^n(4c)$, according to $c = 0$, $c > 0$ or $c < 0$, respectively.

A real surface in a Kähler surface with almost complex structure J is called slant if its Wirtinger angle is constant (see [2, 3, 13]). From J-action point of views, slant surfaces are the simplest and the most natural surfaces of a Lorentzian Kähler surface $(\tilde{M}, \tilde{g}, J)$. Slant surfaces arise naturally and play some important roles in the studies of surfaces of Kähler surfaces in the Lorentzian complex space forms, see [14].

In last years, the geometry of Lorentzian surfaces in Lorentzian complex space forms has been studied by a series of papers given by B. Y. Chen and other geometers, for instance [1, 5-13, 15]. Lorentzian geometry is a vivid field of mathematical research that represents the mathematical foundation of the general theory of relativity-which is probably one of the most successful and beautiful theories of physics. For Lorentzian surfaces in Lorentzian complex space forms, especially, Chen [7] proved a very interesting result that Ricci equation is a consequence of Gauss and Codazzi equations. This indicates that Lorentzian surfaces in Lorentzian complex space forms have much interesting properties, which is quite different from surfaces in Riemannian complex space forms.

A submanifold is called pseudo-umbilical if its shape operator with respect to the mean curvature vector is proportional to the identity map (see [2] for details). Pseudo-umbilical submanifolds are a natural generalization of minimal submanifolds. In [4], Chen completely classified pseudo-umbilical submanifolds in Riemannian complex space forms. The non-flat minimal slant surfaces in $\mathbb{C}_1^2$ were completely classified by Arslan-Carrazo-Chen-Murathan [1].

In this paper, we study pseudo-umbilical surfaces in Lorentzian complex space forms and give a complete classification of pseudo-umbilical slant surfaces in Lorentzian complex Euclidean plane $\mathbb{C}_1^2$, Lorentzian complex projective plane CP_1^2 and Lorentzian complex hyperbolic plane CH_1^2.

2. Preliminaries

Let M be a Lorentzian surface of a Lorentzian Kähler surface $\tilde{M}_1^2$ equipped with an almost structure J and metric $\tilde{g}$. Let $\langle\ ,\ \rangle$ denote the inner product associated with $\tilde{g}$.

We denote the Levi-Civita connections of M and $\tilde{M}_1^2$ by ∇ and $\tilde{\nabla}$, respectively. Gauss formula and Weingarten formula are given respectively by (see [2, 3])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to M and ξ normal to M, where h, A and D are the second fundamental form, the shape operator and the normal connection. It's well known that the second fundamental form h and the shape operator A are related by

$$\langle h(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle \tag{2.3}$$

for X, Y tangent to M and ξ normal to M.

A vector v is called spacelike (timelike) if $\langle v,v\rangle > 0 (\langle v,v\rangle < 0)$. A vector v is called lightlike if it is nonzero and it satisfies $\langle v,v\rangle = 0$.

For each normal vector ξ of M at $x \in M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space T_xM. The mean curvature vector is defined by

$$H = \frac{1}{2}\text{trace } h. \tag{2.4}$$

A Lorentzian surface M in $\tilde{M}_1^2$ is called minimal if its mean curvature vector vanishes at each point on M. A Lorentzian surface M in $\tilde{M}_1^2$ is called quasi-minimal if its mean curvature vector is lightlike at each point on M. And, a

Lorentzian surface M in $\tilde{M}_1^2$ is called pseudo-umbilical if its shape operator A_H satisfies

$$A_H = \rho I,$$

where ρ is a nonzero function and I is the identity map.

For a Lorentzian surface M in a Lorentzian complex space form $\tilde{M}_1^2$, the Gauss and Codazzi and Ricci equations are given respectively by

$$\langle R(X,Y)Z,W\rangle = \langle \tilde{R}(X,Y)Z,W\rangle + \langle h(Y,Z),h(X,W)\rangle - \langle h(X,Z),h(Y,W)\rangle, \tag{2.5}$$

$$(\tilde{R}(X,Y)Z)^{\perp} = (\bar{\nabla}h)(X,Y,Z) - (\bar{\nabla}h)(Y,X,Z), \tag{2.6}$$

$$\langle R^D(X,Y)\xi,\eta\rangle = \langle \bar{R}(X,Y)\xi,\eta\rangle + \langle [A_\xi, A_\eta]X,Y\rangle, \tag{2.7}$$

where X,Y,Z,W are vectors tangent to M, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}h)(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z). \tag{2.8}$$

3. Basic Results for Lorentzian Slant Surfaces

Let M be a Lorentzian surface in a Lorentzian Kähler surface $(\tilde{M}_1^2, g, J)$. For each tangent vector X of M, we put

$$JX = PX + FX, \tag{3.1}$$

where PX and FX are the tangential and the normal components of JX.

On the Lorentzian surface M there exists a pseudo-orthonormal local frame $\{e_1, e_2\}$ such that

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0, \quad \langle e_1, e_2\rangle = -1. \tag{3.2}$$

It follows from (3.1), (3.2) and $\langle JX, JY\rangle = \langle X, Y\rangle$ that

$$Pe_1 = (\sinh\theta)e_1, \quad Pe_2 = -(\sinh\theta)e_2 \tag{3.3}$$

for some function θ. This function θ is called the Wirtinger angle of M.

When the Wirtinger angle θ is constant on M, the Lorentzian surface M is called a slant surface (cf. [3, 13]). In this case, θ is called the slant angle; the slant surface is then called θ-slant.

A θ-slant surface is called Lagrangian if $\theta = 0$ and proper slant if $\theta \neq 0$.

If we put

$$e_3 = (\text{sech } \theta)Fe_1, \quad e_4 = (\text{sech } \theta)Fe_2, \tag{3.4}$$

the we find from (3.1)-(3.4) that

$$Je_1 = \sinh\theta e_1 + \cosh\theta e_3, \qquad Je_2 = -\sinh\theta e_2 + \cosh\theta e_4, \tag{3.5}$$

$$Je_3 = -\cosh\theta e_1 - \sinh\theta e_3, \quad Je_4 = -\cosh\theta e_2 + \sinh\theta e_4, \tag{3.6}$$

$$\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = 0, \quad \langle e_3, e_4\rangle = -1. \tag{3.7}$$

We call such a frame $\{e_1, e_2, e_3, e_4\}$ an adapted pseudo-orthonormal frame for the Lorentzian surface M in $\tilde{M}^2_1$.

We need the following lemmas (see [13]).

Lemma 3.1. *If M is a slant surface in a Lorentzian Kähler surface $\tilde{M}^2_1$, then with respect to an adaped pseudo-orthonormal frame we have*

$$\nabla_X e_1 = \omega(X)e_1, \qquad \nabla_X e_2 = -\omega(X)e_2, \tag{3.8}$$

$$D_X e_3 = \Phi(X)e_3, \qquad D_X e_4 = -\Phi(X)e_4 \tag{3.9}$$

for some 1-forms ω, Φ on M.

For a Lorentzian surface M in $\tilde{M}^2_1$, we put

$$h(e_i, e_j) = h^3_{ij}e_3 + h^4_{ij}e_4, \tag{3.10}$$

where $\{e_1, e_2, e_3, e_4\}$ is an adapted pseudo-orthonormal frame and h is the second fundamental form of M.

Lemma 3.2. ([13]) *If M is a θ-slant surface in a Lorentzian Kähler surface $\tilde{M}^2_1$, then with respect to an adaped pseudo-orthonormal frame we have*

$$\omega_j - \Phi_j = 2h^3_{1j}\tanh\theta, \tag{3.11}$$

$$A_{FX}Y = A_{FY}X, \tag{3.12}$$

$$A_{e_3}e_j = h^3_{1j}e_1 + h^4_{1j}e_2, \qquad A_{e_4}e_j = h^3_{j2}e_1 + h^4_{j2}e_2, \tag{3.13}$$

for any $X, Y \in TM$ and $j = 1, 2$, where $\omega_j = \omega(e_j)$ and $\Phi_j = \Phi(e_j)$.

4. A Fundamental Theorem of Lorentzian Slant Surfaces

For Lorentzian slant surfaces in $\tilde{M}^2_1(4c)$, we have the following result

Theorem 4.1. *Every slant surface in a non-flat Lorentzian complex space form $\tilde{M}_1^2(4c)$ must be Lagrangian.*

Proof. Assume that M is a θ-slant surface in a Lorentzian complex space form $\tilde{M}_1^2(4c)$. Let $\{e_1, e_2, e_3, e_4\}$ be an adapted pseudo-orthonormal frame on M. By applying (3.2) and the total symmetry of $\langle h(X,Y), FZ\rangle$, we obtain

$$\begin{aligned} h(e_1,e_1) &= \beta Fe_1 + \lambda Fe_2,\ h(e_1,e_2) \\ &= \alpha Fe_1 + \beta Fe_2,\ h(e_2,e_2) = \gamma Fe_1 + \alpha Fe_2 \end{aligned} \tag{4.1}$$

for some real-valued functions $\alpha, \beta, \gamma, \lambda$. Then it follows from Lemma 3.1, (4.1) and Codazzi equation (2.6) that

$$\begin{aligned} (\tilde{R}(e_1,e_2)e_1)^\perp &= (\bar{\nabla}_{e_1}h)(e_1,e_2) - (\bar{\nabla}_{e_2}h)(e_1,e_1) \\ &= e_1(\alpha)Fe_1 + e_1(\beta)Fe_2 + \alpha\Phi_1 Fe_1 - \beta\Phi_1 Fe_2 \\ &\quad -e_2(\beta)Fe_1 - e_2(\lambda)Fe_2 - \beta\Phi_2 Fe_1 + \lambda\Phi_2 Fe_2 \\ &\quad +2\omega_2(\beta Fe_1 + \lambda Fe_2), \end{aligned} \tag{4.2}$$

$$\begin{aligned} (\tilde{R}(e_2,e_1)e_2)^\perp &= (\bar{\nabla}_{e_2}h)(e_1,e_2) - (\bar{\nabla}_{e_1}h)(e_2,e_2) \\ &= e_2(\alpha)Fe_1 + e_2(\beta)Fe_2 + \alpha\Phi_2 Fe_1 - \beta\Phi_2 Fe_2 \\ &\quad -e_1(\gamma)Fe_1 - e_1(\alpha)Fe_2 - \gamma\Phi_1 Fe_1 + \alpha\Phi_1 Fe_2 \\ &\quad -2\omega_1(\gamma Fe_1 + \alpha Fe_2). \end{aligned} \tag{4.3}$$

On the other hand, by applying (1.1) and (3.5)-(3.6) we have

$$(\tilde{R}(e_1,e_2)e_1)^\perp = 3c(\sinh\theta)Fe_1, \quad (\tilde{R}(e_2,e_1)e_2)^\perp = 3c(\sinh\theta)Fe_2. \tag{4.4}$$

Thus, combining (4.2)-(4.4) and comparing coefficients give

$$-3c\sinh\theta + e_1(\alpha) - e_2(\beta) + \alpha\Phi_1 - \beta\Phi_2 + 2\beta\omega_2 = 0, \tag{4.5}$$

$$e_1(\beta) - e_2(\lambda) - \beta\Phi_1 + \lambda\Phi_2 + 2\lambda\omega_2 = 0, \tag{4.6}$$

$$-3c\sinh\theta + e_2(\beta) - e_1(\alpha) - \beta\Phi_2 + \alpha\Phi_1 - 2\alpha\omega_1 = 0, \tag{4.7}$$

$$e_2(\alpha) - e_1(\gamma) + \alpha\Phi_2 - \gamma\Phi_1 - 2\gamma\omega_1 = 0. \tag{4.8}$$

Combining (4.5) with (4.7) we obtain

$$-6c\sinh\theta + 2\alpha(\Phi_1 - \omega_1) - 2\beta(\Phi_2 - \omega_2) = 0. \tag{4.9}$$

From Lemma 3.2 and (4.1) we have

$$\omega_1 - \Phi_1 = 2\beta\sinh\theta, \quad \omega_2 - \Phi_2 = 2\alpha\sinh\theta. \tag{4.10}$$

Consequently, (4.9) becomes

$$-6c\sinh\theta = 0,$$

which implies that $\theta = 0$, from the assumption $c \neq 0$. ■

Theorem 4.2. *Every pseudo-umbilical slant surface in a Lorentzian complex space form $\tilde{M}_1^2(4c)$ has constant Gaussian curvature c.*

Proof. Let M be a pseudo-umbilical θ-slant surface in a Lorentzian complex space form $\tilde{M}_1^2(4c)$. There is a pseudo-orthonormal local frame field $\{\hat{e}_1, \hat{e}_2\}$ such that

$$\langle \hat{e}_1, \hat{e}_1\rangle = \langle \hat{e}_2, \hat{e}_2\rangle = 0, \qquad \langle \hat{e}_1, \hat{e}_2\rangle = -1, \tag{4.11}$$

$$H = -h(\hat{e}_1, \hat{e}_2). \tag{4.12}$$

Assume that $h(\hat{e}_1, \hat{e}_2) = \hat{\alpha}F\hat{e}_1 + \hat{\beta}F\hat{e}_2$ for some real-valued functions $\hat{\alpha}$, $\hat{\beta}$. Since M is not minimal, without loss of generality, we assume $\hat{\alpha}$ is not vanishing. By putting $e_1 = \hat{\alpha}\hat{e}_1$, $e_2 = \hat{\alpha}^{-1}\hat{e}_2$, we have

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0, \qquad \langle e_1, e_2\rangle = -1, \tag{4.13}$$

$$h(e_1, e_2) = Fe_1 + \beta F e_2, \tag{4.14}$$

where $\beta = \hat{\beta}\hat{\alpha}^{-1}$. Similar as in the proof of Theorem 4.1, we have

$$\begin{aligned} h(e_1, e_1) &= \beta F e_1 + \lambda F e_2, \quad h(e_1, e_2) \\ &= Fe_1 + \beta F e_2, \quad h(e_2, e_2) = \gamma F e_1 + F e_2. \end{aligned} \tag{4.15}$$

Then the mean curvature vector is given by

$$H = -h(e_1, e_2) = -Fe_1 - \beta F e_2.$$

It follows from (2.3), (3.4), (3.7) and (4.15) that

$$A_H = \begin{pmatrix} -2\beta\cosh^2\theta & -(1+\beta\gamma)\cosh^2\theta \\ -(\lambda+\beta^2)\cosh^2\theta & -2\beta\cosh^2\theta \end{pmatrix}. \tag{4.16}$$

Hence, from the assumption that M is pseudo-umbilical, we have

$$1 + \beta\gamma = 0, \qquad \lambda + \beta^2 = 0. \tag{4.17}$$

On the other hand, it follows from (4.15) and Gauss equation (2.5) that

$$K = c + (-\beta + \lambda\gamma)\cosh^2\theta. \tag{4.18}$$

Substituting (4.17) into (4.18), we find $K = c$. This completes the proof of Theorem 4.2. ■

5. Classification of Pseudo-umbilical Slant Surfaces in $\mathbb{C}_1^2$

In this section we completely classify pseudo-umbilical slant surfaces in $\mathbb{C}_1^2$.

Theorem 5.1. *Up to rigid motions of $\mathbb{C}_1^2$, every pseudo-umbilical slant surface in $\mathbb{C}_1^2$ is given by one of the following two families.*

(1) A flat Lagrangian surface defined by

$$L(x,y) = \Big(\frac{1}{2b}e^{(ib+b)x+(-1+i)y} + \frac{1}{2}e^{(ib-b)x+(1+i)y}, \frac{1}{2b}e^{(ib+b)x+(-1+i)y} - \frac{1}{2}e^{(ib-b)x+(1+i)y}\Big)$$

with $b \in \mathbb{R} \setminus 0$.

(2) A flat θ-slant surface defined by

$$\begin{aligned} L = \Big(&(1+i)e^{\big(\sinh\theta+\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\big)x+\big(-\sinh\theta-\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\big)y} \\ &+(-m+ni)e^{\big(\sinh\theta-\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\big)x+\big(-\sinh\theta+\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\big)y}, \\ &(1+i)e^{\big(\sinh\theta+\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\big)x+\big(-\sinh\theta-\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\big)y} \\ &+(n+mi)e^{\big(\sinh\theta-\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\big)x+\big(-\sinh\theta+\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\big)y}\Big) \end{aligned}$$

with $m = -\frac{c}{a^2+4}$ and $n = -\frac{ac\sinh\theta}{(a^2+4)\sqrt{4\cosh^2\theta+a^2}}$, where $a \in \mathbb{R}$ and $c \in \mathbb{R} \setminus 0$.

Conversely, locally every pseudo-umbilical Lorentzian θ-slant surface in $\mathbb{C}_1^2$ is congruent to the two families of surfaces defined above.

Proof. Let M be a pseudo-umbilical θ-slant surface in $\mathbb{C}_1^2$. Since (4.17) implies that $\beta \neq 0$, M is not quasi-minimal. In this case, it follows from (4.10) and (4.17) that (4.5)-(4.8) become

$$e_2(\beta) - \omega_1 - \beta\omega_2 = 0, \tag{5.1}$$

$$e_1(\beta) + 2\beta e_2(\beta) - \beta\omega_1 - 3\beta^2\omega_2 + 4\beta^2\sinh\theta = 0, \tag{5.2}$$

$$e_1(\beta) - \beta^2\omega_2 - 3\beta\omega_1 + 4\beta^2\sinh\theta = 0. \tag{5.3}$$

Substituting (5.1) and (5.3) into (5.2) yields

$$4\beta\omega_1 = 0. \tag{5.4}$$

Since $\beta \neq 0$, we have $\omega_1 = 0$. Then (5.1)-(5.3) become

$$e_2(\beta) = \beta\omega_2, \qquad e_1(\beta) = \beta^2\omega_2 - 4\beta^2 \sinh\theta. \tag{5.5}$$

we divide it into two cases:
Case (A): β is a constant, say b. In this case, it follows from (5.5) that $\omega_2 = \sinh\theta = 0$, which implies that M is Lagrangian. Therefore we have $\nabla_{e_i} e_j = 0$ for $i, j = 1, 2$. There exist local coordinates $\{x, y\}$ such that

$$g = -dx \otimes dy - dy \otimes dx, \quad \frac{\partial}{\partial x} = e_1, \quad \frac{\partial}{\partial y} = e_2. \tag{5.6}$$

By applying (4.15), (4.17) and Gauss formula (2.1), we obtain that the immersion satisfies

$$L_{xx} = ibL_x - ib^2 L_y, \tag{5.7}$$

$$L_{xy} = iL_x + ibL_y, \tag{5.8}$$

$$L_{yy} = -\frac{1}{b} iL_x + iL_y. \tag{5.9}$$

Equations (5.7) and (5.8) reduce to

$$L_{xxx} = 2ibL_{xx} + 2b^2 L_x, \tag{5.10}$$

whose characteristic polynomial equation is given by

$$r^3 - 2ibr^2 - 2b^2 r = 0. \tag{5.11}$$

After solving this equation, we obtain the immersion in the form

$$L(x, y) = A(y)e^{(ib+b)x} + B(y)e^{(ib-b)x} + C(y) \tag{5.12}$$

for $\mathbb{C}^2_1$-valued functions A, B and C. Substituting (5.12) into (5.7)-(5.9), we find

$$A(y) = c_1 e^{(-1+i)y}, \quad B(y) = c_2 e^{(1+i)y}, \quad C(y) = c_3 \tag{5.13}$$

for constant vectors c_i in $\mathbb{C}^2_1$, where $i = 1, 2, 3$. Combining these with (5.12) shows that the immersion is congruent to

$$L(x, y) = c_1 e^{(ib+b)x+(-1+i)y} + c_2 e^{(ib-b)x+(1+i)y}. \tag{5.14}$$

By applying (5.6),(5.14) and the Lagrangian condition, we obtain

$$\langle c_1, c_1\rangle = \langle c_2, c_2\rangle = \langle c_1, ic_2\rangle = 0, \quad \langle c_1, c_2\rangle = -\frac{1}{2b}. \tag{5.15}$$

Hence we may choose $c_1 = (\frac{1}{2b}, \frac{1}{2b})$ and $c_2 = (\frac{1}{2}, -\frac{1}{2})$. Combining these with (5.15) yields Case (1).

Case (B): β is not constant. In this case, it follows from the first equation of (5.5) and Lemma 3.1 that $[\beta^{-1}e_1, e_2] = 0$. Therefore there exist local coordinates $\{x, y\}$ such that

(5.16) $$\frac{\partial}{\partial x} = \beta^{-1}e_1, \qquad \frac{\partial}{\partial y} = e_2.$$

Using these local coordinates, (5.5) implies that

(5.17) $$\beta_x = \beta_y - 4\beta \sinh\theta.$$

Solving equation (5.17), we have

(5.18) $$\beta = f(x+y)e^{2\sinh\theta(y-x)}$$

for a nonzero function f depending on variable $x+y$. Hence, the metric tensor is given by

(5.19) $$g = -f^{-1}(x+y)e^{2\sinh\theta(x-y)}(dx \otimes dy + dy \otimes dx),$$

and the Levi-Civita connection of g satisfies

(5.20) $$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = (-\frac{f_x}{f} + 2\sinh\theta)\frac{\partial}{\partial x}, \ \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0, \ \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = (-\frac{f_y}{f} - 2\sinh\theta)\frac{\partial}{\partial y}.$$

Moreover, it follows from (4.15), (4.17) and (5.16) that

(5.21) $$\begin{aligned} h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = F\frac{\partial}{\partial x} - F\frac{\partial}{\partial y}, \quad h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \\ = F\frac{\partial}{\partial x} + F\frac{\partial}{\partial y}, \quad h(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = -F\frac{\partial}{\partial x} + F\frac{\partial}{\partial y}. \end{aligned}$$

By applying (5.20), (5.21) and Gauss formula (2.1), we have the following PDE system:

(5.22) $$L_{xx} = (-\frac{f_x}{f} + i + \sinh\theta)L_x - (i + \sinh\theta)L_y,$$

(5.23) $$L_{xy} = (i - \sinh\theta)L_x + (i + \sinh\theta)L_y,$$

(5.24) $$L_{yy} = -(i - \sinh\theta)L_x + (-\frac{f_y}{f} + i - \sinh\theta)L_y.$$

The compatibility condition of this system is given by

(5.25) $$(\frac{f_x}{f})_y = 0.$$

Solving this equation gives

$$f = c^{-1} e^{a(x+y)} \tag{5.26}$$

for some constants $a \in \mathbb{R}$ and $c \in \mathbb{R} \setminus 0$. Hence (5.18) and (5.19) become

$$\beta = c^{-1} e^{(a-2\sinh\theta)x+(a+2\sin\theta)y}, \tag{5.27}$$

$$g = -ce^{(2\sinh\theta-a)x-(2\sinh\theta+a)y}(dx \otimes dy + dy \otimes dx), \tag{5.28}$$

Using (5.26), combining (5.23) with (5.22) and (5.24), we have

$$L_{xx} + L_{xy} = (-a+2i)L_x, \quad L_{xy} + L_{yy} = (-a+2i)L_y. \tag{5.29}$$

After solving these two equations in (5.29), we obtain

$$L_x = P(x-y)e^{(-\frac{a}{2}+i)(x+y)}, \quad L_y = Q(x-y)e^{(-\frac{a}{2}+i)(x+y)} \tag{5.30}$$

for vector-valued functions P, Q in $\mathbb{C}^2_1$. Substituting the two equations in (5.30) into (5.22) and (5.24) respectively, we have

$$P_x + (\frac{a}{2} - \sinh\theta)P = -(i + \sinh\theta)Q, \tag{5.31}$$

$$Q_x - (\frac{a}{2} + \sinh\theta)Q = (i - \sinh\theta)P. \tag{5.32}$$

By differentiating equation (5.32) with respect to x and using (5.31) and (5.32) again, we obtain

$$Q_{xx} - 2\sinh\theta Q_y - (1 + \frac{a^2}{4})Q = 0. \tag{5.33}$$

Solving this linear equation (5.33) gives

$$Q = c_1 e^{\left(\sinh\theta + \sqrt{\cosh^2\theta + \frac{a^2}{4}}\right)(x-y)} + c_2 e^{\left(\sinh\theta - \sqrt{\cosh^2\theta + \frac{a^2}{4}}\right)(x-y)} \tag{5.34}$$

for constant vectors c_1 and c_2. It follows from (5.32) and (5.34) that

$$\begin{aligned} P &= -c_1 \frac{(i+\sinh\theta)}{\cosh^2\theta}\left(\sqrt{\cosh^2\theta + \frac{a^2}{4}} - \frac{a}{2}\right)e^{\left(\sinh\theta + \sqrt{\cosh^2\theta + \frac{a^2}{4}}\right)(x-y)} \\ &\quad + c_2 \frac{(i+\sinh\theta)}{\cosh^2\theta}\left(\sqrt{\cosh^2\theta + \frac{a^2}{4}} + \frac{a}{2}\right)e^{\left(\sinh\theta - \sqrt{\cosh^2\theta + \frac{a^2}{4}}\right)(x-y)}. \end{aligned} \tag{5.35}$$

Hence the second equation of (5.30) becomes

$$L_y = c_1 e^{\left(\sinh\theta+\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)x+\left(-\sinh\theta-\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)y} + c_2 e^{\left(\sinh\theta-\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)x+\left(-\sinh\theta+\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)y}. \tag{5.36}$$

By integrating equation (5.36), we obtain that the immersion is congruent to

$$L = c_3 e^{\left(\sinh\theta+\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)x+\left(-\sinh\theta-\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)y} + c_4 e^{\left(\sinh\theta-\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)x+\left(-\sinh\theta+\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)y} + A(x), \tag{5.37}$$

where $A(x)$ is a vector-valued function and

$$c_3 = \frac{c_1}{-\sinh\theta-\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i}, \quad c_4 = \frac{c_2}{-\sinh\theta+\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i}.$$

By applying (5.35) and substituting (5.37) into the first equation of (5.30), we find

$$A'(x) = 0.$$

Hence A is a constant vector and the immersion is congruent to

$$L(x,y) = c_3 e^{\left(\sinh\theta+\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)x+\left(-\sinh\theta-\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)y} + c_4 e^{\left(\sinh\theta-\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)x+\left(-\sinh\theta+\sqrt{\cosh^2\theta+\frac{a^2}{4}}-\frac{a}{2}+i\right)y}. \tag{5.38}$$

It follows from (5.28) and (5.38) that

$$\begin{aligned} &\langle c_3, c_3\rangle = \langle c_4, c_4\rangle = 0, \quad \langle c_3, c_4\rangle = -\frac{c}{a^2+4}, \\ &\langle c_3, ic_4\rangle = -\frac{ac\sinh\theta}{(a^2+4)\sqrt{4\cosh^2\theta+a^2}}. \end{aligned} \tag{5.39}$$

If we put

$$m = -\frac{c}{a^2+4}, \qquad n = -\frac{ac\sinh\theta}{(a^2+4)\sqrt{4\cosh^2\theta + a^2}},$$

then we may choose $c_3 = (1+i, 1+i)$ and $c_4 = (-m+ni, n+mi)$, combining these with (5.38) yields Case (2).

On the other hand, the converse can be verified by a long straightforward computation. This completes the proof of Theorem 5.1. ■

6. Classification of Lorentzian Pseudo-umbilical Slant Surfaces in $CP^2_1(4)$ and $CH^2_1(-4)$

The following results classify all the pseudo-umbilical slant surfaces in Lorentzian complex projective plane $CP^2_1(4)$ and Lorentzian complex hyperbolic plane $CH^2_1(-4)$.

Since the proofs of Theorems 6.1 and 6.2 require similar arguments, we will prove them together. Let ε be the constant sectional curvature of the ambient space, that is $\varepsilon = 1$ for $CP^2_1(4)$ and $\varepsilon = -1$ for $CH^2_1(-4)$.

Theorem 6.1. *Let M be a Lorentzian pseudo-umbilical θ-slant surface in $CP^2_1(4)$. Then M is Lagrangian, with Gaussian curvature 1, and the immersion is congruent to $\pi \circ L$, where $\pi : S^5_2(1) \to CP^2_1(4)$ is the Hopf-fibration and $L : M \to S^5_2(1) \in \mathbb{C}^3_1$ is locally one of the following three families of surfaces:*
(1) *A Lagrangian surface defined by*

$$L(s,t) = \frac{1}{\sqrt{a^2+4}}\Big(2\cosh(\frac{1}{2}\sqrt{a^2+4}t)\frac{\sqrt{a^2\mid b\mid}}{\mid 1-be^{as}\mid}e^{(\frac{a}{2}+i)s}, \\ 2\sinh(\frac{1}{2}\sqrt{a^2+4}t)\frac{\sqrt{a^2\mid b\mid}}{\mid 1-be^{as}\mid}e^{(\frac{a}{2}+i)s}, -2i - \frac{abe^{as}+a}{be^{as}-1}\Big)$$

with $a, b \in \mathbb{R}\setminus 0$.
(2) *A Lagrangian surface defined by*

$$L(s,t) = \frac{1}{\sqrt{1-c^2}}\Big(\cosh(\sqrt{1-c^2}t)\frac{\mid c\mid e^{is}}{\mid\cos(cs)\mid}, \\ \sinh(\sqrt{1-c^2}t)\frac{\mid c\mid e^{is}}{\mid\cos(cs)\mid}, -i + c\tan(cs)\Big)$$

with $0 <\mid c\mid< 1$ and $c \in \mathbb{R}$.
(3) *A Lagrangian surface defined by*

$$L(s,t) = \frac{1}{\sqrt{c^2-1}}\Big(-i+\tan(cs), \sin(\sqrt{c^2-1}t) \\ \frac{\mid c\mid e^{is}}{\mid\cos(cs)\mid}, \cos(\sqrt{c^2-1}t)\frac{\mid c\mid e^{is}}{\mid\cos(cs)\mid}\Big)$$

with $| c |> 1$ *and* $c \in \mathbb{R}$.

Theorem 6.2. *Let* M *be a Lorentzian pseudo-umbilical* θ*-slant surface in* $CH_1^2(-4)$. *Then* M *is Lagrangian, with Gaussian curvature -1, and the immersion is congruent to* $\pi \circ L$, *where* $\pi : H_3^5(-1) \to CH_1^2(-4)$ *is the Hopf-fibration and* $L : M \to H_3^5(-1) \in \mathbb{C}_2^3$ *is locally one of the following three families of surfaces:*
(1) *A Lagrangian surface defined by*

$$L(s,t) = \frac{1}{\sqrt{a^2+4}}\Big(2i + \frac{abe^{as}+a}{be^{as}-1}, 2\cosh(\frac{1}{2}\sqrt{a^2+4}t)\frac{\sqrt{a^2 \mid b \mid}}{\mid 1-be^{as} \mid}e^{(\frac{a}{2}+i)s}, 2\sinh(\frac{1}{2}\sqrt{a^2+4}t)\frac{\sqrt{a^2 \mid b \mid}}{\mid 1-be^{as} \mid}e^{(\frac{a}{2}+i)s}\Big)$$

with $a, b \in \mathbb{R} \setminus 0$.
(2) *A Lagrangian surface defined by*

$$L(s,t)=\frac{1}{\sqrt{1-c^2}}\Big(i-c\tan(cs), \cosh(\sqrt{1-c^2}t)\frac{\mid c \mid e^{is}}{\mid \cos(cs) \mid}, \sinh(\sqrt{1-c^2}t)\frac{\mid c \mid e^{is}}{\mid \cos(cs) \mid}\Big)$$

with $0 <| c |< 1$ *and* $c \in \mathbb{R}$.
(3) *A Lagrangian surface defined by*

$$L(s,t)=\frac{1}{\sqrt{c^2-1}}\Big(\sin(\sqrt{c^2-1}t)\frac{\mid c \mid e^{is}}{\mid \cos(cs) \mid}, \cos(\sqrt{c^2-1}t)\frac{\mid c \mid e^{is}}{\mid \cos(cs) \mid}, i-\tan(cs)\Big)$$

with $| c |> 1$ *and* $c \in \mathbb{R}$.

Proof. Assume that M is a Lorentzian pseudo-umbilical θ-slant surface in $CP_1^2(4)$ or $CH_1^2(-4)$. From Theorem 4.1, we conclude that M must be Lagrangian. Then similar as in Theorem 5.1, we have

$$\beta \neq 0, \quad \omega_1 = 0, \quad e_2(\beta) = \beta\omega_2, \quad e_1(\beta) = \beta^2\omega_2. \tag{6.1}$$

Suppose that β is constant. Since $\beta \neq 0$, from the third equation of (6.1) we have $\omega_2 = 0$. This implies that M is flat, which contradicts to Theorem 4.2. Hence β is nonconstant. It follows from the third equation of (6.1) and Lemma 2.1 that $[\beta^{-1}e_1, e_2] = 0$. Therefore there exist local coordinates $\{x, y\}$ such that

$$\frac{\partial}{\partial x} = \beta^{-1}e_1, \ \frac{\partial}{\partial y} = e_2. \tag{6.2}$$

By (6.2), the third and the fourth equations of (6.1) imply that

$$\beta_y = \beta_x. \tag{6.3}$$

From (6.3) we may assume $\beta = f(x+y)$, where f is a nonconstant function depending on $x+y$. Therefore the metric tensor is given by

$$g = -f^{-1}(x+y)(dx \otimes dy + dy \otimes dx), \tag{6.4}$$

and the Levi-Civita connection is given by

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = -\frac{f_x}{f}\frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0, \quad \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\frac{f_y}{f}\frac{\partial}{\partial y}. \tag{6.5}$$

Moreover, it follows from (5.21) that

$$\begin{aligned} &h(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = J\frac{\partial}{\partial x} - J\frac{\partial}{\partial y}, \quad h(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \\ &= J\frac{\partial}{\partial x} + J\frac{\partial}{\partial y}, \quad h(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = -J\frac{\partial}{\partial x} + J\frac{\partial}{\partial y}. \end{aligned} \tag{6.6}$$

It follows from (6.5), (6.6) and Gauss formula (2.1) that lift $L : M^2_1 \to \mathbb{C}^3_i$ of the immersion of M into $CP^2_1(4)$ for $i = 1$ and $CH^2_1(-4)$ for $i = 2$, satisfies

$$L_{xx} = (-\frac{f_x}{f} + i)L_x - iL_y, \tag{6.7}$$

$$L_{xy} = iL_x + iL_y + \varepsilon f^{-1}L, \tag{6.8}$$

$$L_{yy} = -iL_x + (i - \frac{f_y}{f})L_y. \tag{6.9}$$

If we put

$$s = x+y, \qquad t = x-y, \tag{6.10}$$

then f is a function depending only on s, and (6.7)-(6.9) become

$$L_{ss} = (i - \frac{f'}{2f})L_s + \frac{\varepsilon}{2f}L, \tag{6.11}$$

$$L_{st} = (i - \frac{f'}{2f})L_t, \tag{6.12}$$

$$L_{tt} = (-i - \frac{f'}{2f})L_s - \frac{\varepsilon}{2f}L. \tag{6.13}$$

The compatibility condition of this system is given by

$$(-\frac{f'}{f})' = \varepsilon f^{-1}. \tag{6.14}$$

Solving this nonlinear autonomous ordinary equation, we obtain two kinds of solutions, which are given by

$$f(s) = \frac{\varepsilon(1 - be^{as})^2}{2a^2 be^{as}}, \tag{6.15}$$

and

$$f(s) = \frac{\varepsilon}{2c^2} \cos^2(cs + d), \tag{6.16}$$

where a, b, $c \in \mathbb{R} \setminus 0$ and $d \in \mathbb{R}$. It should be noted that we can let $d = 0$ by replacing s by $s - d/c$.

Solving (6.12) gives

$$L(s,t) = A(t) f(s)^{-\frac{1}{2}} e^{is} + B(s) \tag{6.17}$$

for vector-valued functions A, B in $\mathbb{C}^3_i$, where $i = 1$ or 2. Substituting (6.17) into (6.11)-(6.13) gives

$$B'' = (i - \frac{f'}{2f})B' + \frac{\varepsilon}{2f} B, \tag{6.18}$$

$$\frac{\varepsilon}{2f} B = (-i - \frac{f'}{2f})B', \tag{6.19}$$

$$A'' = (\frac{f'^2}{4f^2} - \frac{\varepsilon}{2f} + 1)A. \tag{6.20}$$

It follows from (6.18) and (6.19) that

$$B(s) = c_1 \varepsilon(-2i - f' f^{-1}) \tag{6.21}$$

for constant vector c_1.

We note that, from (6.4), (6.10) and the Lagrangian condition, the immersion L should satisfy the following conditions:

$$\begin{aligned} &\langle L_s, L_t \rangle = \langle L_s, iL_t \rangle = 0, \quad \langle L_s, L_s \rangle = -\frac{1}{2f(s)}, \\ &\langle L_t, L_t \rangle = \frac{1}{2f(s)}, \quad \langle L, L \rangle = \varepsilon. \end{aligned} \tag{6.22}$$

Depending on two different kinds of solutions of f, we divide it into two cases. Case (A): f takes the solution in the form (6.15). Without loss of generality, we may assume $f(x + y) > 0$ by choosing the sign of the constant b. Substituting (6.15) into (6.20) and (6.21) gives

$$A''(t) = (\frac{a^2}{4} + 1)A(t), \tag{6.23}$$

$$B(s) = c_1 \varepsilon(-2i - \frac{abe^{as} + a}{be^{as} - 1}). \tag{6.24}$$

Solving (6.23) gives

$$A(t) = c_2 e^{\sqrt{\frac{a^2}{4}+1}t} + c_3 e^{-\sqrt{\frac{a^2}{4}+1}t}. \tag{6.25}$$

Substituting (6.15), (6.24) and (6.25) into (6.17), we find that the immersion is congruent to

$$\begin{aligned} L(s,t) =& \big(c_2 e^{\sqrt{\frac{a^2}{4}+1}t} + c_3 e^{-\sqrt{\frac{a^2}{4}+1}t}\big)\frac{\sqrt{2a^2 \mid b \mid}}{\mid 1 - be^{as} \mid} e^{(\frac{a}{2}+i)s} \\ &+ c_1\varepsilon(-2i - \frac{abe^{as}+a}{be^{as}-1}), \end{aligned} \tag{6.26}$$

where c_1, c_2 and c_3 are constant vectors in $\mathbb{C}^3_1$ or $\mathbb{C}^3_2$ depending on $\varepsilon = 1$ or -1, respectively. It follows from (6.26) that the conditions in (6.22) reduce to

$$\begin{array}{lll} \langle c_1, c_1\rangle = \dfrac{\varepsilon}{a^2+4}, & \langle c_1, c_2\rangle = 0, & \langle c_1, ic_2\rangle = 0, \\ \langle c_2, c_2\rangle = 0, & \langle c_1, c_3\rangle = 0, & \langle c_1, ic_3\rangle = 0, \\ \langle c_3, c_3\rangle = 0, & \langle c_2, c_3\rangle = -\dfrac{1}{a^2+4}, & \langle c_2, ic_3\rangle = 0. \end{array} \tag{6.27}$$

Case (A.1): $\varepsilon = 1$. In this case, c_1, c_2, $c_3 \in \mathbb{C}^3_1$, we may choose

$$\begin{aligned} c_1 &= \frac{1}{\sqrt{a^2+4}}(0,0,1), \quad c_2 = \frac{1}{\sqrt{2(a^2+4)}}(1,1,0), \\ c_3 &= \frac{1}{\sqrt{2(a^2+4)}}(1,-1,0). \end{aligned} \tag{6.28}$$

Combining these with (6.26) gives Case (1) of Theorem 6.1.
Case (A.2): $\varepsilon = -1$. In this case, c_1, c_2, $c_3 \in \mathbb{C}^3_2$, we may choose

$$\begin{aligned} c_1 &= \frac{1}{\sqrt{a^2+4}}(1,0,0), \quad c_2 = \frac{1}{\sqrt{2(a^2+4)}}(0,1,1), \\ c_3 &= \frac{1}{\sqrt{2(a^2+4)}}(0,1,-1). \end{aligned} \tag{6.29}$$

Combining these with (6.26) gives Case (1) of Theorem 6.2.
Case (B): f takes the solution in the form (6.16). By applying a suitable translation in s, we may assume $d = 0$. Substituting (6.16) into (6.20) and (6.21) respectively gives

$$A''(t) = (1-c^2)A(t), \tag{6.30}$$

$$B(s) = c_1\varepsilon(-2i + 2c\tan(cs)). \tag{6.31}$$

Solving (6.30) gives

$$A(t)=\begin{cases} c_2e^{\sqrt{1-c^2}t}+c_3e^{-\sqrt{1-c^2}t}, & \text{if } 0<\mid c\mid<1;\\ c_2t+c_3, & \text{if } \mid c\mid=1;\\ c_2\sin(\sqrt{c^2-1}t)+c_3\cos(\sqrt{c^2-1}t), & \text{if } \mid c\mid>1\end{cases} \tag{6.32}$$

for constant vectors c_1, c_2 and c_3.

Case (B.1): $0<\mid c\mid<1$. Substituting (6.16), (6.31) and the first equation of (6.32) into (6.17), we find that the immersion is congruent to

$$L(s,t)=\big(c_2e^{\sqrt{1-c^2}t}+c_3e^{-\sqrt{1-c^2}t}\big)\frac{\sqrt{2c^2}e^{is}}{\mid\cos(cs)\mid}+c_1\varepsilon(-2i+2c\tan(cs)), \tag{6.33}$$

where c_1, c_2 and c_3 are constant vectors in $\mathbb{C}^3_1$ or $\mathbb{C}^3_2$ depending on $\varepsilon=1$ or -1, respectively. It follows from (6.33) that the conditions in (6.22) reduce to

$$\begin{aligned} &\langle c_1,c_1\rangle=-\frac{\varepsilon}{4(c^2-1)}, && \langle c_1,c_2\rangle=0, && \langle c_1,ic_2\rangle=0,\\ &\langle c_2,c_2\rangle=0, && \langle c_1,c_3\rangle=0, && \langle c_1,ic_3\rangle=0,\\ &\langle c_3,c_3\rangle=0, && \langle c_2,c_3\rangle=\frac{\varepsilon}{4(c^2-1)}, && \langle c_2,ic_3\rangle=0. \end{aligned} \tag{6.34}$$

Case (B.1.1): $\varepsilon=1$. In this case, c_1, c_2, $c_3\in\mathbb{C}^3_1$, we may choose

$$c_1=\tfrac{1}{2\sqrt{1-c^2}}(0,0,1),\ c_2=\tfrac{1}{2\sqrt{2(1-c^2)}}(1,1,0),\ c_3=\tfrac{1}{2\sqrt{2(1-c^2)}}(1,-1,0). \tag{6.35}$$

Combining these with (6.33) gives Case (2) of Theorem 6.1.

Case (B.1.2): $\varepsilon=-1$. In this case, c_1, c_2, $c_3\in\mathbb{C}^3_2$, we may choose

$$c_1=\tfrac{1}{2\sqrt{1-c^2}}(1,0,0),\ c_2=\tfrac{1}{2\sqrt{2(1-c^2)}}(0,1,1),\ c_3=\tfrac{1}{2\sqrt{2(1-c^2)}}(0,1,-1). \tag{6.36}$$

Combining these with (6.33) gives Case (2) of Theorem 6.2.

Case (B.2): $\mid c\mid>1$. Substituting (6.16), (6.31) and the third equation of (6.32) into (6.17), we find that the immersion is congruent to

$$L(s,t)=(c_2\sin(\sqrt{c^2-1}t)+c_3\cos(\sqrt{c^2-1}t))\frac{\sqrt{2c^2}e^{is}}{\mid\cos(cs)\mid}+c_1\varepsilon(-2i+2\tan(cs)), \tag{6.37}$$

where c_1, c_2 and c_3 are constant vectors in $\mathbb{C}^3_1$ or $\mathbb{C}^3_2$ depending on $\varepsilon=1$ or -1, respectively. It follows from (6.37) that the conditions in (6.22) reduce to

$$\begin{aligned} &\langle c_1,c_1\rangle=\frac{-\varepsilon}{4(c^2-1)}, && \langle c_1,c_2\rangle=0, && \langle c_1,ic_2\rangle=0,\\ &\langle c_2,c_2\rangle=\frac{\varepsilon}{2(c^2-1)}, && \langle c_1,c_3\rangle=0, && \langle c_1,ic_3\rangle=0,\\ &\langle c_3,c_3\rangle=\frac{\varepsilon}{2(c^2-1)}, && \langle c_2,c_3\rangle=0, && \langle c_2,ic_3\rangle=0. \end{aligned} \tag{6.38}$$

Case (B.2.1): $\varepsilon = 1$. In this case, c_1, c_2, $c_3 \in \mathbb{C}^3_1$, we may choose

$$\begin{aligned} &c_1 = \frac{1}{2\sqrt{c^2-1}}(1,0,0), \quad c_2 = \frac{1}{\sqrt{2(c^2-1)}}(0,1,0), \\ &c_3 = \frac{1}{\sqrt{2(c^2-1)}}(0,0,1). \end{aligned} \tag{6.39}$$

Combining these with (6.37), we obtain Case (3) of Theorem 6.1.
Case (B.2.2): $\varepsilon = -1$. In this case, c_1, c_2, $c_3 \in \mathbb{C}^3_2$, we may choose

$$\begin{aligned} &c_1 = \frac{1}{2\sqrt{c^2-1}}(0,0,1), \quad c_2 = \frac{1}{\sqrt{2(c^2-1)}}(1,0,0), \\ &c_3 = \frac{1}{\sqrt{2(c^2-1)}}(0,1,0). \end{aligned} \tag{6.40}$$

Combining these with (6.37), we obtain Case (3) of Theorem 6.2.
Case (B.3): $\mid c \mid = 1$. Substituting (6.16), (6.31) and the second equation of (6.32) into (6.17), we find that the immersion is congruent to

$$L(x,y) = (c_2 t + c_3)\frac{\sqrt{2}e^{is}}{\mid \cos s \mid} + c_1\varepsilon(-2i + 2\tan s), \tag{6.41}$$

where c_1, c_2 and c_3 are constant vectors in $\mathbb{C}^3_1$ or $\mathbb{C}^3_2$ depending on $\varepsilon = 1$ or -1, respectively. From (6.41) and the third condition of (6.22), we find that

$$\langle c_2, c_2 \rangle \quad = \varepsilon/2. \tag{6.42}$$

Combining this with (6.41), we find the condition $\langle L, L \rangle = \varepsilon$ does not hold. Therefore this case is impossible. This completes the proof of Theorem 6.1 and Theorem 6.2. ∎

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