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ON NORMAL SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR OPERATOR-DIFFERENTIAL EQUATIONS ON SEMI-AXIS IN WEIGHT SPACE

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Abstract. In the paper, the conditions of normal solvability of some boundary value problems are obtained for a class of operator-differential equations of elliptic type on a semi-axis in weight spaces. The principal part of this equation contains a multiple characteristics operator. All conditions are expressed only by the properties of the operators of the given equation.

1. INTRODUCTION

Many problems of mechanics, mathematical physics, theory of partial differential equations and others reduce to investigation of solvability of boundary value problems for operator- differential equations in different spaces. Note that some problems of theory of elasticity in a half-strip [1,2,3], the problems of theory of vibrations of mechanical systems, vibrations of an elastic cylinder [4] reduce to investigation of solvability of appropriate boundary value problems for operator-differential equations. For example, stress-strain state of a plate reduces to the solution of problems of theory of elasticity in a half-strip. This, in its turn is investigated with solvability of some boundary value problems for second or fourth order operator- differential equation. In the paper of Popkovich P.F. [2,3], Ustinov Yu.A. and Yudovich Yu.I. [1], Orazov [5], the boundary value problem of elasticity theory in a strip $t > 0, |x| \leq 1$ is reduced to the solvability of different boundary value problems for such equations, and solutions are obtained in the form of limits of decreasing elementary solutions of a homogeneous equation. Investigation of solvability of operator- differential equations are closely connected with some spectral problems of different type operator bundles [1-3,6-9,11]. In the paper [12], the relation of solvability of boundary value problems with exact values of

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the norm of intermediate derivatives operators is shown. This enables to choose a wider class of operator- differential equations for which the stated problem was well-posed. Finding of exact values of the norm of intermediate derivatives operators is of independent mathematical interest and has numerical applications in various fields of mathematical analysis [13,20,21,25], for example, in approximation theory [22,23]. In many problems, it is necessary to investigate not correctness of solvability of boundary value problems for operator-differential equations, but their Fredholm property, or Noether property (F- solvability) in some Sobolev spaces. Note that when the principal part has simple characteristics in an infinite domain, such problems were investigated in the papers [15,16,16,17,18] . It is difficult to investigate such problems in infinite domains by the reason that though the principal part is boundedly invertible in these spaces, and disturbed part is not relatively completely continuous in these spaces. Therefore, the studied problem is not Fredholm. By this reason, here another method for investigation of normal solvability of such problems is suggested. Therewith, the fact that the principal part of the inverse operator is a sum of integral operator whose kernel depends on difference and completely continuous operator, is very important. This representation enables to prove normal solvability of the suggested boundary value problem in some weight spaces. Investigation of such problems in weight spaces also have numerical applications [16,17,18]. In these problems, it is necessary to find dependence of weight exponent with lower boundary of the main operator of differential equation. When the principal part of the equation has simple characteristics, such problems are investigated by different methods in the papers [16,17].

2. Problem Statement

In a separable Hilbert space H, consider the boundary value problem

$$(1)\quad P\left(\frac{d}{dt}\right)u(t) \equiv \left(-\frac{d^2}{dt^2}+A^2\right)^n u(t) + \sum_{j=0}^{2n-1} A_{2n-j}u^{(j)}(t) = f(t),\ t \in R_+,$$

$$(2)\qquad u^{(j)}(0) = 0, \quad (j = \overline{0, n-1}).$$

Here$A = A^* > c\,E(c > 0),\ A_j(j = \overline{0, 2n-1})$ are linear operators in H, $f(t)$ and $u(t)$ are vector-valued functions determined in $R_+ = (0, +\infty)$ with values in H, the derivatives are understood in the sense of distributions [13].

Let $H_\alpha(\alpha \geq 0)$ be a Hilbert scale of the space H generated by the operator A, i.e. $H_\alpha = D(A^\alpha)$, $(x, y)_\alpha = (A^\alpha x, A^\alpha y)$, $x, y \in D(A^\alpha)$.

By $L_{2,\gamma}(R_+; H)$ for $\gamma \in R = (-\infty, \infty)$ we denote a Hilbert space of the vector-function $f\,(t)$

$$\|f\|^2_{L_{2,\gamma}(R_+;H)} = \int_0^\infty \|f(t)\|_H \cdot e^{-2\gamma t} dt < \infty.$$

Further, we define the following spaces

$$W_{2,\gamma}^{2n}(R_+;H) = \left\{u(t)/u^{(2n)} \in L_{2,\gamma}(R_+;H), A^{2n}u \in L_{2,\gamma}(R_+;H)\right\}$$

and

$$\overset{\circ}{W}{}_{2,\gamma}^{2n}(R_+;H) = \left\{u(t)/u \in W_{2,\gamma}^{2n}(R_+;H),\ u^{(j)}(0) = 0,\ j = \overline{0, n-1}\right\}$$

with the norm

$$\|u\|_{W_{2,\gamma}^{2n}(R_+;H)} = \left(\left\|A^{2n}u\right\|_{L_{2,\gamma}(R_+;H)}^2 + \left\|u^{(2n)}\right\|_{L_{2,\gamma}(R_+;H)}^2\right)^{1/2}.$$

For $\gamma = 0$, we assume that $L_{2,0}(R_+;H) = L_2(R_+;H)$, $W_{2,0}^{2n}(R_+;H) = W_2^{2n}(R_+;H)$ and $\overset{\circ}{W}{}_2^{2n}(R_+;H) = \overset{\circ}{W}(R_+;H)$.

In sequel, by $L(X;Y)$ we'll denote a space of bounded operators acting from the space X to the space Y, and by $\sigma_\infty(H)$ we denote a set of completely continuous operators acting in H.

3. Auxiliary Facts

Definition 1. If for $f(t) \in L_{2,\gamma}(R_+;H)$ there exists a vector-function $u(t) \in \overset{\circ}{W}{}_{2,\gamma}^{2n}(R_+;H)$ that satisfies equation (1) almost everywhere in R_+, it will be said to be a regular solution of equation (1).

Definition 2. Let there exist the spaces $\tilde{L}_{2,\gamma}(R_+;H) \subset L_{2,\gamma}(R_+;H)$ and $\tilde{W}_{2,\gamma}^{2n}(R_+;H) \subset W_{2,\gamma}^{2n}(R_+;H)$ that have finitedimensional orthogonal completions in the spaces $L_{2,\gamma}(R_+;H)$ and $W_{2,\gamma}^{2n}(R_+;H)$, respectively and for any $f(t) \in \tilde{L}_{2,\gamma}(R_+;H)$ there exist a regular solution $u(t) \in \tilde{W}_{2,\gamma}^{2n}(R_+;H)$ of equation (1) that satisfies boundary condition (2) in the sense of convergence

$$\lim_{t\to 0}\left\|u^{(j)}(t)\right\|_{2n-j-1/2} = 0$$

and it hold the estimation

$$\|u\|_{W_{2,\gamma}^{2n}(R_+;H)} \le const\,\|f\|_{L_{2,\gamma}(R_+;H)}.$$

Then problem (1), (2) is said to be normally solvable.

Denote

$$P_0u(t) \equiv \left(-\frac{d^2}{dt^2} + A^2\right)^n u(t)$$

and

$$P_1 u(t) \equiv \sum_{j=0}^{2n-1} A_{2n-j} u^{(j)}(t),\ u(t) \in \overset{\circ}{W}{}^{2n}_{2,\gamma}(R_+;H).$$

After substitution $u(t)e^{-\gamma t} = v(t)$ we reduce problem (1), (2) to a boundary value problem in the space $W_2^{2n}(R_+;H)$

$$P_\gamma\left(\frac{d}{dt}\right)v(t) \equiv P_{0,\gamma}\left(\frac{d}{dt}+\gamma\right)v(t) + P_{1,\gamma}\left(\frac{d}{dt}+\gamma\right)v(t) = g(t),\ \ t \in R_+, \tag{3}$$

$$v^{(j)}(0) = 0, \tag{4}$$

where

$$\begin{aligned} P_{0,\gamma}\left(\frac{d}{dt}+\gamma\right)v(t) &= \left(-\left(\frac{d}{dt}+\gamma\right)^2 + A^2\right)^n v(t),\ v(t) \in \overset{\circ}{W}{}^{2n}_{2,\gamma}(R_+;H), \\ P_{1,\gamma}\left(\frac{d}{dt}+\gamma\right)v(t) &= \sum_{j=0}^{2n-1} A_{2n-j} v^{(j)}(t),\ v(t) \in \overset{\circ}{W}{}^{2n}_{2,\gamma}(R_+;H), \\ g(t) &= f(t)e^{-\gamma t} \in L_2(R_+;H). \end{aligned}$$

It holds the following

Theorem 1. *Let $A \geq \mu_0 E\,(\mu_0 > 0)$ and $|\gamma| < \mu_0$. Then the operator $P_{0,\gamma}$ isomorphically maps the space $\overset{\circ}{W}{}^{2n}_2(R_+;H)$ onto $L_2(R_+;H)$, and the solution of the equation $P_{0,\gamma} v_0(t) = g(t)$ has the following form*

$$v_0(t) = \int_0^\infty K(t-s)g(s)ds + \sum_{j=0}^{n-1}(t(A+\gamma E))^j e^{-(A+\gamma E)t}\int_0^\infty K_j(s)g(s)ds, \tag{5}$$

where

$$K(t-s) = \begin{cases} \sum\limits_{k=0}^{n} q_n (t-s)^k A^{k+1} e^{-(A+\gamma E)(t-s)} \cdot A^{-2n+1}, & t-s > 0 \\ \sum\limits_{k=0}^{n} p_n (t-s)^k A^{k+1} e^{-(A-\gamma E)(t-s)} \cdot A^{-2n+1}, & t-s < 0, \end{cases} \tag{6}$$

and p_n, q_n are some constant numbers, the operators $\mathrm{K}_j g(t) = \int_0^\infty \mathrm{K}_j(s)g(s)ds$, $j = \overline{0, n-1}$ are continuous operators from $L_2(R_+;H)$ from $H_{2n-1/2}$, i.e. $\mathrm{K}_j : L\left(L_2(R_+;H) \to H_{2n-1/2}\right)$.

Proof. Let $\hat{g}(\xi)$ be a Fourier transformation of the vector-function $g(t)$ continued on a negative semi-axis as a zero vector-function

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} g(\xi) e^{-i\xi t} d\xi.$$

Then

$$v_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(-(i\xi+\gamma)^2 + A^2\right)^n \hat{g}(\xi) e^{i\xi t} d\xi,\ t \in R = (-\infty, \infty)$$

satisfies the equation $P_{0,\gamma}(d/dt+\gamma)v(t) = g(t)$ almost everywhere on $L_2(R_+;H)$. Show that for $|\gamma| < \mu_0$

$$v_1(t) \in W_2^{2n}(R;H)\ (R = (-\infty, +\infty)).$$

By the Plancherel theorem, it suffices to prove that $A^{2n}\hat{v}_1(\xi) \in L_2\left(R_+;H\right)$ and $\xi^{2n}\hat{v}_1(\xi) \in L_2\left(R_+;H\right)$, since

$$\begin{aligned} \|v_1\|^2_{W_2^{2n}(R_+;H)} &= \left\|A^{2n}v_1\right\|^2_{L_2(R_+;H)} + \left\|v_1^{(2n)}\right\|^2_{L_2(R_+;H)} \\ &= \left\|A^{2n}\hat{v}_1(\xi)\right\|^2_{L_2(R_+;H)} + \left\|\xi^{2n}v_2^{(2n)}(\xi)\right\|^2_{L_2(R_+;H)}. \end{aligned}$$

It is obvious that

$$\begin{aligned} \left\|A^{2n}\hat{v}_1\right\|_{L_2(R;H)} &= \left\|A^{2n}\left(-(i\xi+\gamma)^2 + A^2\right)^{-n}\hat{g}(\xi)\right\|_{L_2(R;H)} \\ &\le \sup_{\xi\in R}\left\|A^{2n}\left(-(i\xi+\gamma)^2 + A^2\right)^{-n}\right\|_{H\to H} \cdot \|\hat{g}(\xi)\|_{L_2(R;H)}. \end{aligned}$$

On the other hand, for any $\xi \in R$, it follows from the spectral expansion of A that

$$\begin{aligned} \left\|A^{2n}\left(-(i\xi+\gamma)^2 + A^2\right)^{-n}\right\| &= \sup_{\sigma\in\sigma(A)}\left\|\sigma^{2n}\left((\xi^2+\sigma^2-\gamma^2)^2 - 2i\xi\sigma\right)^{-n}\right\| \\ &\le \sup_{\sigma\ge\mu_0}\left|\sigma^{2n}(\xi^2+\sigma^2-\gamma^2)\right| \\ &\le \sup_{\sigma\ge\mu_0}\frac{\sigma^{2n}}{(\sigma^2-\gamma^2)^n} \le \frac{\mu_0^{2n}}{(\mu_0^2-\gamma^2)^n} < \infty. \end{aligned}$$

Thus,

$$\left\|A^{2n}\hat{v}_1(\xi)\right\|^2_{L_2(R;H)} \le \frac{\mu_0^{2n}}{(\mu_0^2-\gamma^2)^n} \cdot \|\hat{g}(\xi)\|_{L_2(R;H)} = const\,\|g(t)\|_{L_2(R_+;H)},$$

i.e. $A^{2n}v_1(t) \in L_2(R_+;H)$. It is similarly proved that $v_1^{(2n)}(t) \in L_2(R;H)$. Moreover,

$$\left\|v_1^{2n}(t)\right\|_{W_2^{2n}(R;H)} \le const\,\|g(t)\|_{L_2(R_+;H)}.$$

Thus,

$$v_1(t) \in W_2^n(R;H)$$

and

$$\|v(t)\|_{W_2^{2n}(R;H)} \leq const\,\|g(t)\|_{L_2(R_+;H)}\,.$$

Now, let's find representation for $v_1(t)$. Since

$$\begin{aligned} v_1(t) &= \int_0^\infty \left(\frac{1}{2\pi} \int_0^\infty \left(-\left(i\xi+\gamma\right)^2 E + A^2 \right)^{-n} e^{i\xi(t-s)} d\xi \right) g(s) ds \\ &\equiv \int_0^\infty \mathrm{K}\left(t-s\right) g\left(s\right) ds, \end{aligned}$$

we'll find the form of the operator

$$\mathrm{K}(t-s) = \frac{1}{2\pi} \int_0^\infty \left(-\left(i\xi+\gamma\right)^2 E + A^2 \right)^{-n} e^{i\xi(t-s)} d\xi.$$

It is obvious that

$$\mathrm{K}(t-s) = -\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(-\left(\eta+\gamma\right)^2 E + A^2 \right)^{-n} e^{\eta(t-s)} d\eta.$$

For $\sigma \in \sigma(A)$, we have

$$\begin{aligned} \mathrm{K}(\sigma; t-s) &= -\frac{1}{2\pi} \int_{-i\infty}^{i\infty} \left(-\left(\eta+\gamma\right)^2 + \sigma^2 \right)^{-n} e^{\eta(t-s)} d\eta \\ &= -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{e^{\eta(t-s)}}{(\eta-(\sigma-\gamma)^n(\eta-((\sigma+\gamma))^n} d\eta. \end{aligned}$$

Let $t-s>0$, then

$$\begin{aligned} &-\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{e^{\eta(t-s)}}{(\eta-(\sigma-\gamma)^n)(\eta+(\sigma+\gamma))^n} d\eta \\ &= -\underset{\eta=-(\sigma+\gamma)}{Res} \frac{e^{\eta(t-s)}}{(\eta-(\sigma-\gamma))^n(\eta+(\sigma+\gamma))^n} \\ &= -\lim_{t\to(\sigma+\gamma)} \frac{1}{(\eta-1)!} \cdot \frac{d^{n-1}}{d\eta^{n-1}} \left(\frac{e^{\eta(t-s)}}{(\eta-(\sigma-\gamma))^n} \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\frac{d^{n-1}}{d\eta^{n-1}} \cdot e^{\eta(t-s)} \cdot (\eta-(\sigma-\gamma))^{-n} \\ &= \sum_{k=0}^{n-1} C_{n-1}^k (t-s)^k e^{\eta(t-s)} \left(\eta-(\sigma-\gamma)^{-n}\right)^{n-1+k} \end{aligned}$$

$$= \sum_{k=0}^{n-1} C_{n-1}^{k}(t-s)^{k}\left(\eta-(\sigma-\gamma)\right)^{k}\cdot(-1)^{n-k}n(n+1)$$
$$\cdot\ldots\cdot(n+k-1)\left(\eta-(\sigma-\gamma)\right)^{-2n+1}.$$

Then

$$-\lim_{t\to(\sigma+\gamma)}\frac{1}{(\eta-1)!}\cdot\frac{d^{n-1}}{d\eta^{n-1}}\cdot\frac{e^{\eta(t-s)}}{(\eta-(\sigma-\gamma))^{n}}=\sum_{k=0}^{n-1}q_n(t-s)^{k}\sigma^{k}e^{-(\sigma+\gamma)t}\cdot\sigma^{-2n+1},$$

where q_n are constant numbers. We similarly have that for $t-s<0$

$$-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{e^{\eta(t-s)}}{(\eta-(\sigma-\gamma)^{n}(\eta-(\sigma+\gamma))^{n}}d\eta$$
$$=\underset{\eta=\sigma-\gamma}{Res}\frac{e^{\eta(t-s)}}{(\eta-(\sigma-\gamma))^{n}(\eta-(\sigma+\gamma))^{n}}$$
$$=\frac{1}{(\eta-1)!}\cdot\frac{d^{n-1}}{d\eta^{n-1}}\cdot\left(-\lim_{t\to\sigma-\gamma}\frac{e^{\eta(t-s)}}{(\eta-(\sigma+\gamma))^{n}}\right)$$
$$=\sum_{k=0}^{n-1}P_n(t-s)^{k}\sigma^{k}e^{(\sigma-\gamma)(t-s)}\sigma^{-2n+1}.$$

Using spectral expansion of the operator A, we get (6). Then, the general solution of the equation $P_{0,\gamma}(d/dt+\gamma)v(t)=g(t)$ from the space $W_2^n(R_+;H)$will be of the form

$$v(t)=\int_0^\infty K(t-s)g(s)ds+\sum_{j=0}^{n-1}t^j(A+\gamma E)^j e^{-(A+\gamma E)t}\varphi_j,$$

where $\varphi_j\in H_{2n-1/2}$, $j=\overline{0,n-1}$ [15,20]. For determination of the unknown vectors φ_j we'll use the condition (4). It is obvious that for a vector-function $v_1(t)\in W_2^{2n}(R_+;H)$, then $v_1^{(j)}(0)\in H_{2n-j-1/2}$. Therefore, $0=v(0)=\varphi_0\in H_{2n-1/2}$. On the other hand,

$$\varphi_0=v(0)=-\int_0^\infty \mathrm{K}(-s)g(s)ds=\mathrm{K}_0 g.$$

Here, the operator $\mathrm{K}_0\in L(L_2(R_+;H);H_{2n-1/2})$. We can similarly define the remaining

$$\varphi_j=\int_0^\infty \mathrm{K}_j(s)g(s)ds\equiv \mathrm{K}_j g,$$

where $\mathrm{K}_j\in L(L_2(R_+;H);H_{2n-1/2})$. Thus,

$$v(t)=\int_0^\infty \mathrm{K}(t-s)g(s)ds+\sum_{j=0}^{n-1}\left(t(A+\gamma E)\right)^j e^{-(A+\gamma)t}\int_0^\infty \mathrm{K}_j(s)g(s)ds.$$

The theorem is proved.

4. Basic Result

Now, engage in solvability of problem (1), (2) and prove the main theorem.

Theorem 2. *Let $A \geq \mu_0 E\,(\mu_0 > 0)$, $A^{-1} \in \sigma_\infty(H)$, $|\gamma| < \mu_0$ and on the axis $Re\lambda = \gamma$ the resolvent $P^{-1}(\lambda)$ exists, moreover, on the axis $Re\lambda = \gamma$ are finite*

$$\left\| P_0(\lambda)P^{-1}(\lambda) \right\| \leq const.$$

If the operators $B_j = A_j \cdot A^{-j} (j = \overline{1, 2n})$ are completely continuous in H, the problem (1), (2) *is normally solvable.*

Proof. It is obvious that normal solvability of problem (1),(2) is equivalent with normal solvability with problem (3),(4). Therefore, we prove normal solvability of problem (3),(4). By theorem 1, the operator $P_{0,\gamma}$ realizes isomorphism between the spaces $\overset{\circ}{W}{}_2^{2n}(R_+; H)$ and $L_2(R_+; H)$. Then assuming $\omega = P_{0,\gamma}v_1$, $v_1 \in \overset{\circ}{W}{}_2^{2n}(R_+; H)$, $\omega \in L_2(R_+; H)$, from the equation (3) in the space $L_2(R_+; H)$ for determining $\omega(t)$, we get the following equivalent integro-differential equation

$$\begin{aligned} &\omega(t) + \sum_{j=0}^{2n-1} B_{2n-j} A^{2n-j} \left(\frac{d}{dt} + \gamma \right)^j \left(\int_0^\infty \mathrm{K}(t-s)\omega(s)ds \right) \\ &+ \sum_{j=0}^{2n-1} B_{2n-j} A^{2n-j} \left(\frac{d}{dt} + \gamma \right)^j \sum_{p=0}^{n-1} t^p (A + \gamma E)^p e^{-(A+\gamma)t} \int_0^\infty \mathrm{K}_p(s)\omega(s)ds. \end{aligned} \tag{7}$$

All first we prove complete continuity of the second term in $L_2(R_+; H)$. Let

$$Q_{j,\gamma} = A^{2n-j} \left(\frac{d}{dt} + \gamma \right)^j \sum_{p=0}^{n-1} t^p (A + \gamma E)^p e^{-(A+\gamma)t} \int_0^\infty \mathrm{K}_p(s)\omega(s)ds.$$

We must prove complete continuity of the operator $B_{2n-j}Q_{j,\gamma}$. For simplicity, we consider the case $j = 0$, the remaining cases are similarly considered. Then

$$Q_{0,\gamma} = A^{2n} \sum_{p=0}^{n-1} t^p (A + \gamma E)^p e^{-(A+\gamma E)t} \int_0^\infty \mathrm{K}_p(s)\omega(s)ds.$$

Show that $Q_{0,\gamma}$ is a bounded operator in $L_2(R_+; H)$. Since $\varphi_p = \int_0^\infty \mathrm{K}_p(s)g(s)ds \in H_{2n-1/2}$ and

$$\|\varphi_p\|_{2n-\frac{1}{2}} \leq const \, \|\omega(t)\|_{L_2(R_+;H)},$$

$$S(t)\,\varphi_p \equiv t^p (A + \gamma E)^p e^{-(A+\gamma)t} \varphi_p \in W_2^{2n}(R_+; H),$$

i.e. $S(t) \in L\left(H_{2n-\frac{1}{2}}, W_2^{2n}(R_+;H)\right)$, then using the above-mentioned inequalities, we get

$$\|Q_{0,\gamma}\|_{L_2(R_+;H)} = \left\|A^{2n}S(t)\varphi_p\right\|_{L_2(R_+;H)} \le \|S(t)\varphi_p\|_{W_2^n(R_+;H)}$$
$$\le const\,\|\varphi_p\|_{2n-\frac{1}{2}} \le const\,\|\omega(t)\|_{L_2(R_+;H)},$$

i.e. $Q_{0,\gamma}$ is a bounded operator in $L_2(R_+;H)$. Denote

$$Q_{0,\gamma,n} = B_{2n}P_mQ_{0,\gamma},$$

where P_m is an ortprojector onto the first m eigen vector of the operators $A\,(A\varphi_l = \lambda_l\varphi_l, l = \overline{1,m})$. Then, it is obvious that

$$\begin{aligned}
&Q_{j,\gamma,n}g \\
&= \sum_{l=1}^{m}\lambda_s^{\frac{1}{2}}\sum_{p=0}^{n-1}t^p(\lambda_l+\gamma)^pe^{-(\lambda_l+\gamma)t}\left(A^{2n-\frac{1}{2}}\int_0^\infty \mathrm{K}_p(s)\omega(s)ds,\varphi_l\right)B_{2n}\varphi_l \\
&= \sum_{l=1}^{m}\lambda_l^{\frac{1}{2}}\sum_{p=0}^{n-1}\left(A^{2n-\frac{1}{2}}\int_0^\infty \mathrm{K}_p(s)\omega(s)ds,\varphi_l\right)\left(t^p(\lambda_l+\gamma)^pe^{-(\lambda_l+\gamma)t}B_{2n}\varphi_l\right) \\
&= \sum_{l=1}^{m}\lambda_l^{\frac{1}{2}}\sum_{p=0}^{n-1}\left(\omega(s),T_p^*\varphi_l\right)t^p(\lambda_l+\gamma)^pe^{-(\lambda_l+\gamma)t}B_{2n}\varphi_l,
\end{aligned}$$

where

$$T_p^* = \left(A^{2n-\frac{1}{2}}\mathrm{K}_p\right)^* \in L(H;L_2(R_+;H)).$$

i.e. $Q_{0,\gamma,n}$ is a finite-dimensional operator. On the other hand, it follows from complete continuity of the operator B_{2n} that $\|B_{2n} - B_{2n}P_m\| \to 0$ as $m \to \infty$ therefore

$$\|Q_{0,\gamma} - P_mQ_{0,\gamma,n}\|_{L_2(R_+;H)\to L_2(R_+;H)}$$
$$\le \|B_{2n} - B_{2n}P_m\|\cdot\|Q_{0,\gamma}\|_{L_2(R_+;H)\to L_2(R_+;H)} \to 0,$$

as $m \to \infty$, therefore, $Q_{0,\gamma}$ is a completely continuous operator. Thus, it follows from equality (7) that for proving normal solvability of the given problem, it suffices to prove normal solvability of the problem

(8) $$\omega(t) + P_{1,\gamma}(d/dt)\int_{-\infty}^{\infty}\mathrm{K}_\gamma(t-s)\omega(s)ds = g(t)$$

in the space $L_2(R;H)$. To this end, we introduce the denotation

$$W(t) = \begin{cases} \omega(t), & t>0 \\ \omega_1(t) = \omega(-t), & t<0 \end{cases},\quad G(t) = \begin{cases} g(t), & t>0 \\ g_1(t) = g(-t), & t<0. \end{cases}$$

Since $L_2\left(R;H\right)=L_2\left(R_+;H\right)\oplus L_2\left(R_+;H\right)$, we consider the following equation in the space $L_2\left(R_+;H\right)$

$$W(t)+P_{1,\gamma}\left(d/dt\right)\int_0^\infty \mathrm{K}_\gamma(t-s)W(s)ds=G(t),\ t\in R$$

that is equivalent to the following system

$$\begin{cases} \omega(t)+P_{1,\gamma}(d/dt)\displaystyle\int_0^\infty \mathrm{K}_\gamma(t-s)\omega(s)ds+P_{1,\gamma}(d/dt)\int_0^\infty \mathrm{K}_\gamma(t+s)\omega_1(s)ds=g(t) \\ \omega_1(t)+P_{1,\gamma}(d/dt)\displaystyle\int_0^\infty \mathrm{K}_\gamma(t+s)\omega(s)ds+P_{1,\gamma}(d/dt)\int_0^\infty \mathrm{K}_\gamma(t-s)\omega_1(s)ds=g_1(t). \end{cases}$$

We write this system in the form

$$\begin{pmatrix} E-\mathrm{K}_{11} & \mathrm{K}_{12} \\ \mathrm{K}_{21} & E-\mathrm{K}_{22} \end{pmatrix} W=\left[\begin{pmatrix} E-\mathrm{K}_{11} & 0 \\ 0 & E-\mathrm{K}_{22} \end{pmatrix}+\begin{pmatrix} 0 & \mathrm{K}_{12} \\ \mathrm{K}_{21} & 0 \end{pmatrix}\right]W=G,$$

where

$$\begin{aligned} K_{11}\omega &= \int_0^\infty P_{1,\gamma}(d/dt)K_\gamma(t-s)\omega(s)ds;\quad K_{12}\omega_1 \\ &= \int_0^\infty P_{1,\gamma}(d/dt)K_\gamma(t+s)\omega_1(s)ds, \\ K_{22}\omega_1 &= \int_0^\infty P_{1,\gamma}(d/dt)K_\gamma(t-s)\omega_1(s)ds;\ \ K_{21}\omega \\ &= \int_0^\infty P_{1,\gamma}(d/dt)K_\gamma(t+s)\omega(s)ds. \end{aligned}$$

It follows from the condition of the theorem that

$$\begin{pmatrix} E-\mathrm{K}_{11} & \mathrm{K}_{12} \\ \mathrm{K}_{21} & E-\mathrm{K}_{22} \end{pmatrix} W=G$$

is correctly and uniquely solvable in $L_2(R;H)$. Really, after the Fourier transformation we get

$$\left(E+P_1(i\xi+\gamma)P_0^{-1}(i\xi+\gamma)\right)\hat{W}(\xi)=\hat{G}(\xi)$$

or

$$\left(P_0(i\xi+\gamma)+P_1(i\xi+\gamma)\right)P_0^{-1}(i\xi+\gamma)\hat{W}(\xi)=\hat{G}(\xi).$$

Consequently,

$$\left(P(i\xi+\gamma)P_0^{-1}(i\xi+\gamma)\right)\hat{W}(\xi)=\hat{G}(\xi)$$

or

$$\hat{W}(\xi)=P_0(i\xi+\gamma)P^{-1}(i\xi+\gamma)\hat{G}(\xi).$$

Since $\left\|P(\lambda)P^{-1}(\lambda)\right\|^{-1} \leq const\ \ for\ \ \lambda = i\xi + \gamma$, then

$$\left\|\hat{W}(\xi)\right\|_{L_2(R;H)} \leq const \left\|\hat{G}(\xi)\right\|_{L_2(R;H)} = const \left\|G(t)\right\|_{L_2(R;H)}.$$

Consequently, $W(t) \in L_2(R;H)$. Now, prove that the operators K_{12} and K_{21} are completely continuous in $L_2(R_+;H)$. If it is so, then the operator

$$(9) \qquad \begin{pmatrix} E-\mathrm{K}_{11} & 0 \\ 0 & E-\mathrm{K}_{22} \end{pmatrix} = \begin{pmatrix} E-\mathrm{K}_{11} & \mathrm{K}_{12} \\ \mathrm{K}_{21} & E-\mathrm{K}_{22} \end{pmatrix} - \begin{pmatrix} 0 & \mathrm{K}_{12} \\ \mathrm{K}_{21} & 0 \end{pmatrix},$$

will be Fredholm in the space $L_2(R_+;H) \oplus L_2(R_+;H)$. This means that equation (8) will be normally solvable in $L_2(R_+;H)$.

It is obvious that the kernel K_{21} is of the form

$$\begin{aligned} P_{1,\gamma}(d/dt)\mathrm{K}_\gamma(t+s) &= \sum_{j=0}^{2n-1} A_{2n-j}(d/dt+\gamma)\mathrm{K}_\gamma(t+s) \\ &= \sum_{j=0}^{2n-1} B_{2n-j}A^{2n-j}(d/dt+\gamma)^j\mathrm{K}_\gamma(t+s). \end{aligned}$$

Considering the form of the kernel $\mathrm{K}_\gamma(t+s)$, we easily see that

$$\left\|A^{2n-j}(d/dt+\gamma)^j\mathrm{K}_\gamma(t+s)\right\|_{H\to H} \leq \frac{c_j(\gamma)}{t+s}.$$

Thus, an integral operator T_j with kernel $A^{2n-j}(d/dt+\gamma)^j\mathrm{K}_\gamma(t+s)$ is a bounded operator from $L_2(R_+;H)$ to $L_2(R_+;H)$ [24]. Prove that an integral operator T_jP_m with kernel $B_{2n-j}A^{2n-j}(d/dt+\gamma)^j\mathrm{K}_\gamma(t+s)$ will be a completely continuous operator in $L_2(R_+;H)$. It is obvious that the kernel

$$\begin{aligned} &B_{2n-j}P_mA^{2n-j}\left(d/dt+\gamma\right)\mathrm{K}_\gamma\left(t+s\right) \\ &= B_{2n-j}\sum_{l=1}^{m}\sum_{j=0}^{2n-1}(d/dt+\gamma)^j\sum_{k=0}^{n}q_n(t+s)^k\lambda_l^k e^{-(\lambda_l+\gamma)(t+s)}(\cdot,\varphi_l)\lambda_l^{-2n+1} \\ &= \sum_{l=1}^{m}\sum_{j=0}^{2n-1}(d/dt+\gamma)^j\sum_{k=0}^{n}q_n(t+s)^k\lambda_l^{-2n+k-1}e^{-(\lambda_l+\gamma)(t+s)}(\cdot,\varphi_l)B_{2n-j}\varphi_l \end{aligned}$$

generates a finite-dimensional operator in $L_2(R_+;H)$. Since, B_{2n-j}, as $m\to\infty$

$$\left\|T_j - T_jP_m\right\|_{L_2(R_+;H)\to L_2(R_+;H)} = \left\|\left(B_{2n-j}-B_{2n-j}P_m\right)T_j\right\|_{L_2(R_+;H)\to L_2(R_+;H)}$$

$$\leq \left\|T_j\right\|_{L_2(R_+;H)\to L_2(R_+;H)},\ \ \left\|B_{2n-j}-B_{2n-j}P_m\right\| \to 0.$$

Consequently,K_{21} is a completely continuous operator in $L_2(R_+; H)$.Thus, normal solvability of problem (3),(4) and consequently, normal solvability of problem (1),(2) follows from (9). The theorem is proved.

Apply the obtained result to the most interesting case $n = 2$.

Example. Let $n = 2$. Then, we obtain boundary value problem (1),(2) in the form

$$\left(-\frac{d^2}{dt^2} + A^2\right)^2 u(t) + \sum_{j=0}^{3} A_{n-j} u^{(j)}(t) = f(t),\ t \in R_+ = (0, +\infty)\,, \tag{10}$$

$$u\,(0) = u^{'}\,(0) = 0. \tag{11}$$

Applying theorem 2, we get the following theorem.

Theorem 3. *Let* $A \geq \mu_0 E\,(\mu_0 > 0)$, $A^{-1} \in \sigma_\infty\,(H)$, $|\gamma| < \mu_0$ *and on the axis* $Re\lambda = \gamma$ *the resolvent* $P^{-1}(\lambda)$ *exists, moreover,*

$$\left\|P_0(\lambda)P^{-1}(\lambda)\right\| \leq const.$$

If the operators $B_j = A_j \cdot A^{-j} (j = \overline{1,4})$ are completely continuous in H, the problem (10), (11) is normally solvable.

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