TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 3, pp. 1089-1099, June 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

NEW CRITERIA OF SOME BOUNDED APPROXIMATION PROPERTIES

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Abstract. The bounded (resp. bounded compact) approximation property is well known in the theory of Banach spaces. This paper is concerned with some bounded approximation properties in the more general setting. We establish various new criteria of bounded approximation properties.

1. Introduction and Resultss

Throughout this paper, X and Y are Banach spaces. We denote by τ the *topology of compact convergence* on $\mathcal{B}(X,Y)$, the space of bounded linear operators from X into Y, which is strictly weaker than the operator norm topology; for a net (T_α) in $\mathcal{B}(X,Y)$ and $T \in \mathcal{B}(X,Y)$,

$$T_\alpha \xrightarrow{\tau} T \text{ if and only if } \sup_{x\in K} \|T_\alpha x - Tx\| \longrightarrow 0$$

for each compact $K \subset X$; for $\mathcal{S} \subset \mathcal{B}(X,Y)$ and $T \in \mathcal{B}(X,Y)$,

$$T \in \overline{\mathcal{S}}^\tau \text{ if and only if for each compact } K \subset X \text{ and } \varepsilon > 0,$$

$$\text{there exists a } R \in \mathcal{S} \text{ so that } \sup_{x\in K} \|Rx - Tx\| < \varepsilon.$$

We denote by $\mathcal{A}(X,Y)$ a subspace of $\mathcal{B}(X,Y)$. For example, when $\mathcal{A} = \mathcal{K}$ or $\mathcal{A} = \mathcal{F}$, we denote by $\mathcal{K}(X,Y)$ and $\mathcal{F}(X,Y)$, respectively, the space of compact and bounded finite rank linear operators between X and Y. We define $\mathcal{A}(X,Y;\lambda)$ as the set $\{T \in \mathcal{A}(X,Y) : \|T\| \le \lambda\}$. A Banach space X is said to have the $\mathcal{A}$-*approximation property* ($\mathcal{A}$-AP) if

$$I_X \in \overline{\mathcal{A}(X,X)}^\tau,$$

Received June 24, 2009, accepted December 1, 2009.
Communicated by Bor-Luh Lin.
2000 *Mathematics Subject Classification*: Primary 46B28; Secondary 46B10.
Key words and phrases: Approximation property, Bounded approximation property.
This work was supported by the Korea Research Foundation Grant funded by the Korean Government (2009-0094068), (2010-0022118).

where I_X is the identity operator on X. For $\lambda \geq 1$, X is said to have the λ-$\mathcal{A}$-*bounded approximation property* (λ-$\mathcal{A}$-BAP) if

$$I_X \in \overline{\mathcal{A}(X, X; \lambda)}^{\tau}.$$

If $\mathcal{A} = \mathcal{F}$ (resp. $\mathcal{K}$), then the properties are well known as the *approximation property* and λ*-bounded approximation property* (resp. the *compact approximation property* and λ*-bounded compact approximation property*) (cf. Casazza [1, Sections 2, 3, 8]). The main purpose of this paper is to establish various criteria of the λ-$\mathcal{A}$-BAP. Also some similar results for the $\mathcal{A}$-AP are presented.

We are now ready to state main theorems of this paper, which are proved in the next section.

Theorem 1.1. *The following are equivalent.*

(a) *For every Banach space Y and $T \in \mathcal{B}(X, Y)$, $T \in \overline{\{TS : S \in \mathcal{A}(X, X; \lambda)\}}^{\tau}$.*

(b) *For every Banach space Y and $T \in \mathcal{K}(X, Y)$, $T \in \overline{\{TS : S \in \mathcal{A}(X, X; \lambda)\}}^{\tau}$.*

(c) *For every separable reflexive Banach space Y and $T \in \mathcal{K}(X, Y)$, $T \in \overline{\{TS : S \in \mathcal{A}(X, X; \lambda)\}}^{\tau}$.*

(d) *X has the λ-$\mathcal{A}$-BAP.*

In view of Theorem 1.1(c), X has the λ-bounded approximation property (λ-BAP) if and only if for every separable reflexive Banach space Y and $T \in \mathcal{K}(X, Y)$, $T \in \overline{\{TS : S \in \mathcal{F}(X, X; \lambda)\}}^{\tau}$. Lima and Oja [7] introduced and investigated the *weak λ-bounded approximation property* (weak λ-BAP), which are formally weaker than λ-BAP, and showed that X has the weak λ-BAP if and only if for every separable reflexive Banach space Y and $T \in \mathcal{K}(X, Y)$, $T \in \overline{\{TS : S \in \mathcal{F}(X, X), \|TS\| \leq \lambda \|T\|\}}^{\tau}$ [7, Theorem 2.4]. It is not known whether the λ-BAP and weak λ-BAP are different. In view of the criteria of these properties, it seems like that they are slightly different. In [7], the authors conjectured that they are different.

The following is an unbounded version of Theorem 1.1.

Theorem 1.2. *The following are equivalent.*

(a) *For every Banach space Y and $T \in \mathcal{B}(X, Y)$, $T \in \overline{\{TS : S \in \mathcal{A}(X, X)\}}^{\tau}$.*

(b) *For every Banach space Y and $T \in \mathcal{K}(X, Y)$, $T \in \overline{\{TS : S \in \mathcal{A}(X, X)\}}^{\tau}$.*

(c) *For every separable reflexive Banach space Y and $T \in \mathcal{K}(X, Y)$, $T \in \overline{\{TS : S \in \mathcal{A}(X, X)\}}^{\tau}$.*

(d) *X has the $\mathcal{A}$-AP.*

In view of Theorem 1.2(c), X has the approximation property (AP) if and only if for every separable reflexive Banach space Y and $T \in \mathcal{K}(X, Y)$, $T \in \overline{\{TS : S \in \mathcal{F}(X, X)\}}^{\tau}$. In [7], for the dual space X^* of X, the authors showed that X^* has the AP if and only if X^* has the weak 1-BAP [7, Theorem 3.6]. Consequently, for the dual space X^*, for every separable reflexive Banach space Y and $T\in\mathcal{K}(X^*,Y)$, $T\in\overline{\{TS:S\in\mathcal{F}(X^*,X^*)\}}^{\tau}$ if and only if for every separable reflexive Banach space Y and $T\in\mathcal{K}(X^*,Y)$, $T\in\overline{\{TS:S\in\mathcal{F}(X^*,X^*),\|TS\|\le\|T\|\}}^{\tau}$. It is a long standing famous problem whether the AP and 1-BAP are equivalent for general dual spaces (cf. [1, Problem 3.8]).

Now interchanging domain and codomain spaces we have the same results.

Theorem 1.3. *The following are equivalent.*

(a) *For every Banach space Y and $T\in\mathcal{B}(Y,X)$, $T\in\overline{\{ST:S\in\mathcal{A}(X,X;\lambda)\}}^{\tau}$.*

(b) *For every Banach space Y and $T\in\mathcal{K}(Y,X)$, $T\in\overline{\{ST:S\in\mathcal{A}(X,X;\lambda)\}}^{\tau}$.*

(c) *For every separable reflexive Banach space Y and $T\in\mathcal{K}(Y,X)$, $T\in\overline{\{ST:S\in\mathcal{A}(X,X;\lambda)\}}^{\tau}$.*

(d) *X has the λ-$\mathcal{A}$-BAP.*

Theorem 1.4. *The following are equivalent.*

(a) *For every Banach space Y and $T\in\mathcal{B}(Y,X)$, $T\in\overline{\{ST:S\in\mathcal{A}(X,X)\}}^{\tau}$.*

(b) *For every Banach space Y and $T\in\mathcal{K}(Y,X)$, $T\in\overline{\{ST:S\in\mathcal{A}(X,X)\}}^{\tau}$.*

(c) *For every separable reflexive Banach space Y and $T\in\mathcal{K}(Y,X)$, $T\in\overline{\{ST:S\in\mathcal{A}(X,X)\}}^{\tau}$.*

(d) *X has the $\mathcal{A}$-AP.*

We now extend Theorems 1.3 and 1.4 as a version of the operator norm topology.

Corollary 1.5. *The following are equivalent.*

(a) *For every Banach space Y and $T\in\mathcal{K}(Y,X)$, $T\in\overline{\{ST:S\in\mathcal{A}(X,X;\lambda)\}}$.*

(b) *For every separable reflexive Banach space Y and $T\in\mathcal{K}(Y,X)$, $T\in\overline{\{ST:S\in\mathcal{A}(X,X;\lambda)\}}$.*

(c) *X has the λ-$\mathcal{A}$-BAP.*

Proof. (a) $\Longrightarrow$(b) is clear and (b)$\Longrightarrow$(c) follows by Theorem 1.3(c)$\Longrightarrow$(d).

(c) $\Longrightarrow$(a) Let Y be a Banach space and let $T\in\mathcal{K}(Y,X)$. Let $\varepsilon>0$. Then by the assumption there exists $S\in\mathcal{A}(X,X;\lambda)$ so that $\|ST-T\|=\sup_{y\in B_Y}\|STy-Ty\|<\varepsilon$. Hence $T\in\overline{\{ST:S\in\mathcal{A}(X,X;\lambda)\}}$. ■

Corollary 1.6. *The following are equivalent.*

(a) *For every Banach space Y and $T\in\mathcal{K}(Y,X)$, $T\in\overline{\{ST:S\in\mathcal{A}(X,X)\}}$.*

(b) *For every separable reflexive Banach space Y and $T\in\mathcal{K}(Y,X)$, $T\in\overline{\{ST:S\in\mathcal{A}(X,X)\}}$.*

(c) *X has the $\mathcal{A}$-AP.*

Proof. (a)$\Longrightarrow$(b) is clear and (b)$\Longrightarrow$(c) follows by Theorem 1.4(c)$\Longrightarrow$(d). And (c)$\Longrightarrow$(a) follows by the proof of Corollary 1.5(c)$\Longrightarrow$(a). ■

In general, Theorems 1.1 and 1.2 cannot be extended as the version of the operator norm topology. Indeed, there exists a Banach space Z so that Z has the 1-BAP but the dual space Z^* fails to have the AP (cf. [1, Proposition 1.4]). The well known result of Grothendieck [4] is that the dual space X^* has the AP if and only if for every Banach space Y and $T \in \mathcal{K}(X, Y)$, $T \in \overline{\mathcal{F}(X, Y)}$ (cf. Lindenstrauss and Tzafriri [8, Theorem 1.e.5]). Therefore it does not hold that for every Banach space Y and $T \in \mathcal{K}(Z, Y)$, $T \in \overline{\{TS : S \in \mathcal{F}(Z, Z)\}}$.

We next characterize the λ-$\mathcal{A}$-BAP in terms of the trace of finite rank operators. For $T \in \mathcal{F}(X, X)$ we denote by tr(T) the trace of T. Recall that for $T = \sum_{i=1}^{n} x_i^*(\cdot)x_i \in \mathcal{F}(X, X)$

$$\text{tr}(T) = \sum_{i=1}^{n} x_i^*(x_i)$$

and that if $T \in \mathcal{B}(X, Y), S \in \mathcal{B}(Y, X)$, and one of these operators has finite rank, then

$$\text{tr}(TS) = \text{tr}(ST)$$

(cf. Diestel, Jarchow, and Tonge [3, Lemma 6.1]). We now have

Theorem 1.7. *X has the λ-$\mathcal{A}$-BAP if and only if for every $T \in \mathcal{F}(X, X)$ $|tr(T)| \le \sup\{|tr(TR)| : R \in \mathcal{A}(X, X; \lambda)\}$.*

In the final criterion of λ-$\mathcal{A}$-BAP, we use finite-dimensional subspaces of X.

Theorem 1.8. *Suppose $\mathcal{F} \subset \mathcal{A}$. Then X has the λ-$\mathcal{A}$-BAP if and only if for every finite-dimensional subspace F of X and $\varepsilon > 0$, there exists a $T \in \mathcal{A}(X, X; \lambda + \varepsilon)$ so that $Tx = x$ for all $x \in F$.*

Let $\mathcal{N}$ be the family of equivalent norms $|||\cdot|||$ on X which of the form $|||x||| = \|x\| + Md(x, Z)$. Here M ranges over nonnegative constants, Z ranges over finite dimensional subspaces of X, and $d(x, Z) = \inf\{\|x - z\| : z \in Z\}$. Then we have

Corollary 1.9. *Suppose $\mathcal{F} \subset \mathcal{A}$. Then X has the λ-$\mathcal{A}$-BAP if and only if for every $|||\cdot||| \in \mathcal{N}$ $(X, |||\cdot|||)$ has the λ-$\mathcal{A}$-BAP.*

Proof. We only need to prove the "only if" part. Let $|||\cdot||| \in \mathcal{N}$ with $|||x||| = \|x\| + Md(x, Z)$ for $x \in X$. Let F be a finite-dimensional subspace of X and $\varepsilon > 0$. Since X has the λ-$\mathcal{A}$-BAP, by Theorem 1.8 there exists a $T \in \mathcal{A}(X, X; \lambda+\varepsilon)$ so that $Tx = x$ for all $x \in \text{span}(F \cup Z)$. If $x \in X$ and $z \in Z$, then

$$d(Tx, Z) = d(Tx + z, Z) = d(T(x + z), Z) \leq \|T\|\|x + z\|.$$

It follows that $d(Tx, Z) \leq \|T\|d(x, Z)$ for every $x \in X$. Now let $x \in X$ with $|||x||| \leq 1$. Then

$$|||Tx||| = \|Tx\| + Md(Tx, Z) \leq \|T\|\|x\| + M\|T\|d(x, Z) = \|T\||||x||| \leq \|T\| \leq \lambda+\varepsilon.$$

Thus $T \in \mathcal{A}((X, |||\cdot|||), (X, |||\cdot|||); \lambda+\varepsilon)$. Hence $(X, |||\cdot|||)$ has the λ-$\mathcal{A}$-BAP. ■

2. Proofs of Theorems

The approximation property is one of important properties in Banach space theory. Grothendieck [4] systematically investigated the approximation property and showed the following which is one of main tools of the proofs of Theorems 1.1, 1.2, 1.3, and 1.4. For a concrete proof one may see [1, Proposition 2.4] or Choi and Kim [2, Lemma 3.1].

Fact. $(\mathcal{B}(X, Y), \tau)^*$consists of all functionals f of the form $f(T) = \sum_n y_n^*(Tx_n)$, where $\{x_n\} \subset X$, $\{y_n^*\} \subset Y^*$, and $\sum_n \|x_n\|\|y_n^*\| < \infty$.

Other main tools of the proofs of Theorems 1.1, 1.2, 1.3, and 1.4 are the following.

Lemma 2.1. (6, Lemmas 1.1 and 2.1). *If K is a balanced convex and compact set in the unit ball B_X of X, then there exist a separable reflexive Banach space Z, a map $J : Z \longrightarrow X$ such that J is the inclusion, compact, $\|J\| \leq 1$, and $K \subset B_Z$.*

Lemma 2.2. *Suppose that $(V, \mathcal{T})$ is a locally convex space and $x \in V$. If C is a balanced convex subset of V, then $x \in \overline{C}^{\mathcal{T}}$ if and only if for every $f \in (V, \mathcal{T})^*$ $|f(x)| \leq \sup_{y \in C} |f(y)|$.*

Proof. By continuity the"only if"part is clear. To show the"if"part, suppose $x \notin \overline{C}^{\mathcal{T}}$. Then by an application of the separation theorem (cf. [9, Theorem 2.2.28]) there exists a $f \in (V, \mathcal{T})^*$ so that $\text{Re}f(x) < t < \text{Re}f(y)$ for all $y \in C$. Consider $-f$. Then we may assume that $\text{Re}f(x) > t > \text{Re}f(y)$ for all $y \in C$. Since C is balanced,

$$|f(x)| > t \geq \sup_{y \in C} \text{Re}f(y) = \sup_{y \in C} |f(y)|.$$

This completes the proof. ■

Lemma 2.3. ([9, Corollary 2.2.20]). *Suppose that $(V,\mathcal{T})$ is a locally convex space and $x \in V$. If W is a subspace of V, then $x \in \overline{W}^{\mathcal{T}}$ if and only if for every $f \in (V,\mathcal{T})^*$ satisfying $f(y) = 0$ for all $y \in W$, we have $f(x) = 0$.*

We are now ready to prove Theorems 1.1, 1.2, 1.3, and 1.4.

Proof of Theorem 1.1.

(a)$\Longrightarrow$(b) and (b)$\Longrightarrow$(c) are clear.

(c)$\Longrightarrow$(d) Let (x_n) and (x_n^*) be sequences in X and X^*, respectively, so that $\sum_n \|x_n\|\|x_n^*\| < \infty$. We may assume that for every n $\|x_n^*\| \le 1$, $x_n^* \longrightarrow 0$, and $\sum_n \|x_n\| < \infty$. Consider the balanced convex and compact $\overline{co}\{\alpha x_n^* : |\alpha| \le 1, n = 1,2,3,\cdots\} \subset B_{X^*}$. Then by Lemma 2.1 there exist a separable reflexive Banach space Z with $\overline{co}\{\alpha x_n^* : |\alpha| \le 1, n = 1,2,3,\cdots\} \subset B_Z$ and the inclusion, compact operator $J : Z \longrightarrow X^*$ with $\|J\| \le 1$. By the assumption

$$J^*Q_X \in \overline{\{J^*Q_XS : S \in \mathcal{A}(X,X;\lambda)\}}^{\tau} \subset \mathcal{B}(X,Z^*),$$

where $Q_X : X \longrightarrow X^{**}$ is the natural isometric imbedding. Since $\sum_n \|Q_Z(x_n^*)\| \|x_n\| = \sum_n \|x_n^*\|_Z\|x_n\| \le \sum_n \|x_n\| < \infty$, by **Fact** $\sum_n Q_Z(x_n^*)(\,\cdot\, x_n) \in (\mathcal{B}(X,Z^*),\tau)^*$.

We now have

$$\begin{aligned}
\Big|\sum_n x_n^*(x_n)\Big| &= \Big|\sum_n (Q_Xx_n)(x_n^*)\Big| \\
&= \Big|\sum_n (Q_Xx_n)J(x_n^*)\Big| \\
&= \Big|\sum_n (J^*Q_Xx_n)(x_n^*)\Big| \\
&= \Big|\sum_n Q_Z(x_n^*)(J^*Q_Xx_n)\Big| \\
&\le \sup\Big\{\Big|\sum_n Q_Z(x_n^*)(J^*Q_XSx_n)\Big| : S \in \mathcal{A}(X,X;\lambda)\Big\} \\
&= \sup\Big\{\Big|\sum_n (J^*Q_XSx_n)(x_n^*)\Big| : S \in \mathcal{A}(X,X;\lambda)\Big\} \\
&= \sup\Big\{\Big|\sum_n (Q_XSx_n)Jx_n^*\Big| : S \in \mathcal{A}(X,X;\lambda)\Big\} \\
&= \sup\Big\{\Big|\sum_n (Q_XSx_n)x_n^*\Big| : S \in \mathcal{A}(X,X;\lambda)\Big\} \\
&= \sup\Big\{\Big|\sum_n x_n^*(Sx_n)\Big| : S \in \mathcal{A}(X,X;\lambda)\Big\}.
\end{aligned}$$

From **Fact** and Lemma 2.2 $I_X \in \overline{\mathcal{A}(X,X;\lambda)}^{\tau}$. Hence X has the λ-$\mathcal{A}$-BAP.

(d)$\Longrightarrow$(a) Let Y be a Banach space and let $T \in \mathcal{B}(X,Y)$. Let $K \subset X$ be compact and $\varepsilon > 0$. Then by the assumption there exists $S \in \mathcal{A}(X,X;\lambda)$ so that $\|T\| \sup_{x\in K} \|Sx - x\| < \varepsilon$. Thus

$$\sup_{x\in K} \|TSx - Tx\| \leq \|T\| \sup_{x\in K} \|Sx - x\| < \varepsilon.$$

Hence $T \in \overline{\{TS : S \in \mathcal{A}(X,X;\lambda)\}}^{\tau}$. ■

Proof of Theorem 1.2. (a)$\Longrightarrow$(b) and (b)$\Longrightarrow$(c) are clear.

(c)$\Longrightarrow$(d) Let (x_n) and (x_n^*) be sequences in X and X^*, respectively, so that $\sum_n \|x_n\|\|x_n^*\| < \infty$ and $\sum_n x_n^*(Sx_n) = 0$ for every $S \in \mathcal{A}(X,X)$. By the assumption and the proof of Theorem 1.1 $J^*Q_X \in \overline{\{J^*Q_XS : S \in \mathcal{A}(X,X)\}}^{\tau} \subset \mathcal{B}(X,Z^*)$. By the proof of Theorem 1.1

$$\sum_n Q_Z(x_n^*)(J^*Q_XSx_n) = \sum_n x_n^*(Sx_n) = 0$$

for every $S \in \mathcal{A}(X,X)$. Since $J^*Q_X \in \overline{\{J^*Q_XS : S \in \mathcal{A}(X,X)\}}^{\tau}$, by **Fact** $\sum_n Q_Z(x_n^*)(J^*Q_Xx_n) = 0$. By the proof of Theorem 1.1, $\sum_n x_n^*(x_n) = 0$. From **Fact** and Lemma 2.3 $I_X \in \overline{\mathcal{A}(X,X)}^{\tau}$. Hence X has the $\mathcal{A}$-AP.

(d)$\Longrightarrow$(a) See the proof of Theorem 1.1(d)$\Longrightarrow$(a). ■

Proof of Theorem 1.3. (a)$\Longrightarrow$(b) and (b)$\Longrightarrow$(c) are clear.

(c)$\Longrightarrow$(d) Let (x_n) and (x_n^*) be sequences in X and X^*, respectively, so that $\sum_n \|x_n\|\|x_n^*\| < \infty$. We may assume that for every n $\|x_n\| \leq 1$, $x_n \longrightarrow 0$, and $\sum_n \|x_n^*\| < \infty$. Consider the balanced convex and compact $\overline{co}\{\alpha x_n : |\alpha| \leq 1, n = 1,2,3,\cdots\} \subset B_X$. Then by Lemma 2.1 there exist a separable reflexive Banach space Z with $\overline{co}\{\alpha x_n : |\alpha| \leq 1, n = 1,2,3,\cdots\} \subset B_Z$ and the inclusion, compact operator $J : Z \longrightarrow X$ with $\|J\| \leq 1$. By the assumption

$$J \in \overline{\{SJ : S \in \mathcal{A}(X,X;\lambda)\}}^{\tau} \subset \mathcal{B}(Z,X).$$

By **Fact** $\sum_n x_n^*(\,\cdot\, x_n) \in (\mathcal{B}(Z,X),\tau)^*$ and we have

$$\begin{aligned}\Big|\sum_n x_n^*(x_n)\Big| &= \Big|\sum_n x_n^* J x_n\Big| \\ &\leq \sup\Big\{\Big|\sum_n x_n^*(SJx_n)\Big| : S \in \mathcal{A}(X,X;\lambda)\Big\} \\ &= \sup\Big\{\Big|\sum_n x_n^*(Sx_n)\Big| : S \in \mathcal{A}(X,X;\lambda)\Big\}.\end{aligned}$$

From **Fact** and Lemma 2.2 $I_X \in \overline{\mathcal{A}(X,X;\lambda)}^{\tau}$. Hence X has the λ-$\mathcal{A}$-BAP.

(d)$\Longrightarrow$(a) Let Y be a Banach space and let $T \in \mathcal{B}(Y,X)$. Let $K \subset Y$ be compact and $\varepsilon > 0$. Then by the assumption there exists $S \in \mathcal{A}(X,X;\lambda)$ so that

$$\sup_{y \in K} \|STy - Ty\| < \varepsilon.$$

Hence $T \in \overline{\{ST : S \in \mathcal{A}(X,X;\lambda)\}}^{\tau}$. ∎

Proof of Theorem 1.4. (a)$\Longrightarrow$(b) and (b)$\Longrightarrow$(c) are clear.

(c)$\Longrightarrow$(d) Let (x_n) and (x_n^*) be sequences in X and X^*, respectively, so that $\sum_n \|x_n\|\|x_n^*\| < \infty$ and $\sum_n x_n^*(Sx_n) = 0$ for every $S \in \mathcal{A}(X,X)$. By the assumption and the proof of Theorem 1.3, $J \in \overline{\{SJ : S \in \mathcal{A}(X,X)\}}^{\tau} \subset \mathcal{B}(Z,X)$. Also

$$\sum_n x_n^*(SJx_n) = \sum_n x_n^*(Sx_n) = 0$$

for every $S \in \mathcal{A}(X,X)$. Since $J \in \overline{\{SJ : S \in \mathcal{A}(X,X)\}}^{\tau}$, $\sum_n x_n^*(x_n) = \sum_n x_n^*(Jx_n) = 0$. From **Fact** and Lemma 2.3 $I_X \in \overline{\mathcal{A}(X,X)}^{\tau}$. Hence X has the $\mathcal{A}$-AP.

(d)$\Longrightarrow$(a) See the proof of Theorem 1.3(d)$\Longrightarrow$(a). ∎

Recall the *weak operator topology* (*wo*) on $\mathcal{B}(X,Y)$; for a net (T_α) in $\mathcal{B}(X,Y)$ and $T \in \mathcal{B}(X,Y)$,

$$T_\alpha \xrightarrow{wo} T \text{ if and only if } y^*(T_\alpha x) \longrightarrow y^*(Tx)$$

for each $x \in X$ and $y^* \in Y^*$; $(\mathcal{B}(X,Y), wo)^*$ consists of all functionals f of the form $f(T) = \sum_{n=1}^m y_n^*(Tx_n)$, where $\{x_n\}_{n=1}^m \subset X$ and $\{y_n^*\}_{n=1}^m \subset Y^*$.

To show Theorem 1.7, we need the following lemma.

Lemma 2.4. (cf. Kim [5, Corollary 2.9]). *If $\mathcal{C}$ is a bounded convex set in $\mathcal{B}(X,Y)$, then $\overline{\mathcal{C}}^{wo} = \overline{\mathcal{C}}^{\tau}$.*

Proof of Theorem 1.7. By Lemma 2.4 X has the λ-$\mathcal{A}$-BAP if and only if $I_X \in \overline{\mathcal{A}(X,X;\lambda)}^{wo}$. Also by Lemma 2.2 $I_X \in \overline{\mathcal{A}(X,X;\lambda)}^{wo}$ if and only if for every $f \in (\mathcal{B}(X,X), wo)^*$ $|f(I_X)| \le \sup\{|f(R)| : R \in \mathcal{A}(X,X;\lambda)\}$. But for every $f \in (\mathcal{B}(X,X), wo)^*$ $|f(I_X)| \le \sup\{|f(R)| : R \in \mathcal{A}(X,X;\lambda)\}$ if and only if for every $\{x_n\}_{n=1}^m \subset X$ and $\{x_n^*\}_{n=1}^m \subset X^*$ $|\sum_{n=1}^m x_n^*(x_n)| \le \sup\{|\sum_{n=1}^m x_n^*(Rx_n)| : R \in \mathcal{A}(X,X;\lambda)\}$ if and only if for every $T \in \mathcal{F}(X,X)$ $|\mathrm{tr}(T)| \le \sup\{|\mathrm{tr}(TR)| : R \in \mathcal{A}(X,X;\lambda)\}$. ∎

To show the final theorem, we need the following lemma.

Lemma 2.5. (cf. [3, pp. 146-147]). *If E is an n-dimensional subspace of a normed space X, then there exists a projection P from X onto E with $\|P\| \le n$.*

Proof of Theorem 1.8. Suppose that X has the λ-$\mathcal{A}$-BAP. Let F be a finite-dimensional subspace of X and $\varepsilon > 0$. Choose $0 < \delta < 1$ so that

$$\frac{\delta}{1-\delta}\lambda\text{dim}(F) < \varepsilon.$$

Since B_F is compact in X, there exists a $S \in \mathcal{A}(X, X; \lambda)$ so that $\sup_{x\in B_F} \|Sx - x\| < \delta$. So $\|S|_F - i_F\| < \delta < 1$, where $i_F : F \longrightarrow X$ is the inclusion. If $S|_F y = 0$, then $\|y\| = \|(S|_F - i_F)y\| \le \|S|_F - i_F\|\|y\|$. Thus $(1 - \|S|_F - i_F\|)\|y\| \le 0$ and so $y = 0$. Therefore $S|_F$ is injective and $\|S|_F\| \le \|S|_F - i_F\| + \|i_F\| < \delta + 1$. Consider $S|_F^{-1} : S(F) \longrightarrow F$. If $x = S|_F y \in S|_F(F) = S(F)$, then $\|y\| \le \|i_F y - S|_F y\| + \|S|_F y\| \le \delta\|y\| + \|x\|$ and so $\|y\| < \|x\|/(1-\delta)$. Consequently $\|S|_F^{-1} x\| = \|y\| < \|x\|/(1 - \delta)$. By Lemma 2.5 there exists a projection $P : S(X) \longrightarrow S(F)$ with $\|P\| \le \text{dim}(S(F)) = \text{dim}(F)$. Put $T = (S|_F^{-1} P + i_{S(X)} - P)S$. Then $T \in \mathcal{A}(X, X)$ since $\mathcal{F} \subset \mathcal{A}$, and for every $x \in F$

$$Tx = S|_F^{-1} PSx + Sx - PSx = x.$$

Let $x \in X$ and choose $z \in F$ with $PSx = Sz$. Then we have

$$\begin{aligned}
\|Tx - Sx\| &= \|(S|_F^{-1} P + i_{S(X)} - P)Sx - Sx\| \\
&= \|S|_F^{-1} PSx - PSx\| \\
&= \|z - Sz\| \\
&= \|(i_F - S|_F)z\| \\
&\le \delta\|z\| \\
&= \delta\|S|_F^{-1} PSx\| \\
&\le \delta\|S|_F^{-1}\|\|P\|\|Sx\| \\
&< \frac{\delta}{1-\delta}\text{dim}(F)\|S\|\|x\|
\end{aligned}$$

Thus

$$\|T - S\| \le \frac{\delta}{1-\delta}\text{dim}(F)\|S\| \le \frac{\delta}{1-\delta}\text{dim}(F)\lambda.$$

Hence

$$\|T\| \le \frac{\delta}{1-\delta}\text{dim}(F)\lambda + \lambda \le \lambda + \varepsilon.$$

To show the converse, let K be a compact set in X, $\varepsilon > 0$, and $\delta > 0$. Choose $\gamma > 0$ so that $(\lambda + \delta + 1)\gamma < \varepsilon$. Let $x_1, \cdots, x_m$ be a γ-net of K. Then by the assumption there exists a $T \in \mathcal{A}(X, X; \lambda + \delta)$ so that $Tx_i = x_i$ for $1 \le i \le m$. Now let $x \in K$. Then for some $1 \le i_0 \le m$ $\|x - x_{i_0}\| < \gamma$ and so

$$\begin{aligned}
\|Tx - x\| &\le \|Tx - Tx_{i_0}\| + \|Tx_{i_0} - x_{i_0}\| + \|x_{i_0} - x\| \\
&\le \|T\|\|x - x_{i_0}\| + \|x_{i_0} - x\| \\
&< (\lambda + \delta + 1)\gamma < \varepsilon.
\end{aligned}$$

We have shown that for every compact $K \subset X, \varepsilon > 0$, and $\delta > 0$, there exists $T \in \mathcal{A}(X, X; \lambda + \delta)$ so that $\sup_{x \in K} \|Tx - x\| < \varepsilon$. Now let K be a compact set in X and $\varepsilon > 0$. Choose $\delta > 0$ so that

$$\frac{\delta}{\lambda + \delta} \sup_{x \in K} \|x\| < \frac{\varepsilon}{2}.$$

Then there exists a $T \in \mathcal{A}(X, X; \lambda + \delta)$ so that $\sup_{x \in K} \|Tx - x\| < \varepsilon/2$. Consider $\lambda/(\lambda + \delta)T \in \mathcal{A}(X, X; \lambda)$. Then we have

$$\begin{aligned} \sup_{x \in K} \|\frac{\lambda}{\lambda + \delta} Tx - x\| &= \sup_{x \in K} \|\frac{\lambda}{\lambda + \delta} Tx - \frac{\lambda + \delta}{\lambda + \delta} x\| \\ &\leq \frac{\lambda}{\lambda + \delta} \sup_{x \in K} \|Tx - x\| + \frac{\delta}{\lambda + \delta} \sup_{x \in K} \|x\| \\ &< \sup_{x \in K} \|Tx - x\| + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Hence X has the λ-$\mathcal{A}$-BAP. ■

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