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EXISTENCE OF SOLUTIONS FOR IMPULSIVE ANTI-PERIODIC BOUNDARY VALUE PROBLEMS OF FRACTIONAL ORDER

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Abstract. In this paper, we prove the existence of solutions for impulsive differential equations of fractional order $q \in (1, 2]$ with anti-periodic boundary conditions in a Banach space. Our study is based on the contraction mapping principle and Krasnoselskii's fixed point theorem.

1. Introduction

Fractional differential equations have recently gained much importance and attention. The study of fractional differential equations ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. Fractional differential equations appear naturally in a number of fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. For examples and details, see [1-3, 5, 7, 10, 15-16, 18-19, 23-24, 26] and the references therein.

The theory of impulsive differential equations of integer order has found its extensive applications in realistic mathematical modelling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory and applications of impulsive differential equations, we refer the reader to the references [17, 25, 27, 33].

Anti-periodic problems have recently received considerable attention as anti-periodic boundary conditions appear in numerous situations. Examples include anti-periodic trigonometric polynomials in the study of interpolation problems [11], anti-periodic wavelets [8], difference equations [6], ordinary, partial and abstract

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differential equations [13-14, 20, 22, 28-32], and impulsive differential equations [4, 12, 21], etc. For some more application of anti-periodic boundary conditions in physics, see [9] and the references therein.
In this paper, we study an anti-periodic boundary value problem for impulsive differential equations of fractional order

$$(1.1)\quad \begin{cases} {}^cD^q x(t) = f(t, x(t)),\ 1 < q \le 2,\ t \in J_1 = [0,T] \setminus \{t_1, t_2, ..., t_p\}, \\ \Delta x(t_k) = \mathcal{I}_k(x(t_k^-)),\ \Delta x'(t_k) = \mathcal{J}_k(x(t_k^-)),\ t_k \in (0,T),\ k = 1,2,...,p, \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \end{cases}$$

where ${}^cD$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $J = [0,T], \mathcal{I}_k,\ \mathcal{J}_k : \mathbb{R} \to \mathbb{R}$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{h \to 0^+} x(t_k + h),\ x(t_k^-) = \lim_{h \to 0^-} x(t_k + h),\ k = 1, 2, ..., p$ for $0 = t_0 < t_1 < t_2 < ... < t_p < t_{p+1} = T$.

2. Preliminaries

We define $PC(J, \mathbb{R}) = \{x : J \to \mathbb{R};\ x \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, 1, 2, ..., p+1$ and $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^-) = x(t_k),\ k = 1, 2, ..., p\}$, and $PC^1(J, \mathbb{R}) = \{x' \in PC(J, \mathbb{R});\ x'(t_k^+),\ x'(t_k^-)$ exist and x' is left continuous at $t_k,$ for $k = 1, 2, ..., p\}$. Note that $PC^1(J, \mathbb{R})$ is a Banach space with the norm $\|x\| = \sup_{t \in J}\{\|x(t)\|_{PC}, \|x'(t)\|_{PC}\}$

Definition 2.1. A function $x \in PC^1(J, \mathbb{R})\}$ with its Caputo derivative of order q existing on J is a solution of (1.1) if it satisfies (1.1).

We need the following result to prove the existence of at least one solution of (1.1).

Theorem 2.1. (Krasnoselskii's Theorem) Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

To study the nonlinear problem (1.1), we first consider the associated linear problem and obtain its solution.

Lemma 2.1. For a given $\sigma \in PC[0,T]$, a function x is a solution of the following linear impulsive boundary value problem

$$(2.1)\quad \begin{cases} {}^cD^q x(t) = \sigma(t),\ 1 < q \le 2,\ t \in J_1 = [0,T] \setminus \{t_1, t_2, ..., t_p\}, \\ \Delta x(t_k) = \mathcal{I}_k(x(t_k^-)),\ \Delta x'(t_k) = \mathcal{J}_k(x(t_k^-)),\ t_k \in (0,T),\ k = 1,2,...,p, \\ x(0) = -x(T), \qquad x'(0) = -x'(T), \end{cases}$$

if and only if x is a solution of the impulsive fractional integral equation

$$x(t) =$$

$$(2.2)\quad \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \frac{1}{2}\int_{t_p}^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)ds \\ \displaystyle + \frac{(T-2t)}{4}\int_{t_p}^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds \\ \displaystyle - \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \mathcal{I}_k(x(t_k^-))\Big) \\ \displaystyle - \frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds + \mathcal{J}_k(x(t_k^-))\Big), \\ \hfill t \in [0, t_1], \\ \displaystyle\int_{t_k}^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \frac{1}{2}\int_{t_k}^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)ds \\ \displaystyle + \frac{(T-2t)}{4}\int_{t_k}^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds \\ \displaystyle - \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \mathcal{I}_k(x(t_k^-))\Big) \\ \displaystyle - \frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds + \mathcal{J}_k(x(t_k^-))\Big) \\ \displaystyle + \sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \mathcal{I}_k(x(t_k^-))\Big) \\ \displaystyle + \sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds + \mathcal{J}_k(x(t_k^-))\Big),\ t \in (t_k, t_{k+1}]. \end{cases}$$

Proof. Suppose that x is a solution of (2.1). Then, for some constants $b_0, b_1 \in \mathbb{R}$, we have

$$(2.3)\quad x(t) = I_{t_0^+}^q \sigma(t) - b_0 - b_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - b_0 - b_1 t,\ t \in [0, t_1].$$

For some constants $c_0, c_1 \in \mathbb{R}$, we can write

$$x(t) = I_{t_1^+}^q \sigma(t) - c_0 - c_1(t-t_1) = \int_{t_1}^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - c_0 - c_1(t-t_1),\ t \in (t_1, t_2].$$

Using the impulse conditions $\Delta x(t_1) = x(t_1^+) - x(t_1^-) = \mathcal{I}_1(x(t_1^-))$ and $\Delta x'(t_1) = x'(t_1^+) - x'(t_1^-) = \mathcal{J}_1(x(t_1^-))$, we find that

$$-c_0 = \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - b_0 - b_1t_1 + \mathcal{I}_1(x(t_1^-)),$$
$$-c_1 = \int_0^{t_1} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds - b_1 + \mathcal{J}_1(x(t_1^-)).$$

Thus,

$$\begin{aligned} x(t) = & \int_{t_1}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - b_0 - b_1t + \mathcal{I}_1(x(t_1^-)) \\ & +(t-t_1)\Big[\int_0^{t_1} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds + \mathcal{J}_1(x(t_1^-))\Big],\ t\in(t_1,t_2]. \end{aligned}$$

Repeating the process in this way, the solution $x(t)$ for $t\in(t_k,t_{k+1}]$ cane be written as

$$\begin{aligned} & x(t) \\ & = \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - b_0 - b_1t \\ & + \sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \mathcal{I}_k(x(t_k^-))\Big) \\ & + \sum_{0<t_k<t}\Big[(t-t_k)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds + \mathcal{J}_k(x(t_k^-))\Big)\Big], t\in(t_k,t_{k+1}]. \end{aligned} \tag{2.4}$$

Applying the anti-periodic boundary conditions $x(0)=-x(T),\ x'(0)=-x'(T)$, the values of b_0, b_1 are given by

$$\begin{aligned} b_0 = & \frac{1}{2}\int_{t_p}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \frac{T}{4}\int_{t_p}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds \\ & +\frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \mathcal{I}_k(x(t_k^-))\Big) \\ & +\frac{1}{2}\sum_{0<t_k<T}(\frac{T}{2}-t_k)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds + \mathcal{J}_k(x(t_k^-))\Big), \\ b_1 = & \frac{1}{2}\int_{t_p}^{T} \frac{(T-s)^{q-2}}{\Gamma(q)}\sigma(s)ds + \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q)}\sigma(s)ds + \mathcal{J}_k(x(t_k^-))\Big). \end{aligned}$$

Substituting the values of b_0, b_1 in (2.3) and (2.4), we obtain (2.2). Conversely, we assume that x is a solution of the impulsive fractional integral equation (2.2). It follows by a direct computation that x given by (2.2) satisfies the fractional linear anti-periodic boundary value problem (2.1). This completes the proof.

Remark. 2.1. The first three terms of the solution (2.2) correspond to the solution for the problem without impulses [3]. The solution for the associated homogeneous problem with impulses and anti-periodic boundary conditions can be obtained by taking $\sigma = 0$ in (2.2).

3. Main Results

Theorem 3.1. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function and $I_k,\ J_k : \mathbb{R} \to \mathbb{R}$ are continuous functions. Assume that there exist positive constants L_1, L_2, L_3, M_2, M_3 such that*

$(\mathbf{A_1})$ $\|f(t,x) - f(t,y)\| \le L_1\|x-y\|,\ \forall t \in [0,T],\ x,y \in \mathbb{R}$;

$(\mathbf{A_2})$ $\|\mathcal{I}_k(x) - \mathcal{I}_k(y)\| \le L_2\|x-y\|,\ \|\mathcal{J}_k(x) - \mathcal{J}_k(y)\| \le L_3\|x-y\|$ *with* $\|\mathcal{I}_k(x)\| \le M_2,\ \|\mathcal{J}_k(x)\| \le M_3,\ \forall x,y \in \mathbb{R},\ k = 1,2,...,p.$

Further $L_1T^q\Big(\frac{3(1+p)}{2\Gamma(q+1)} + \frac{1+7p}{4\Gamma(q)}\Big) + \frac{p}{4}(6L_2 + 7TL_3) < 1$, *with* $L_1 \le \frac{1}{2}[T^q\{\frac{3(1+p)}{2\Gamma(q+1)} + \frac{1+7p}{4\Gamma(q)}\}]^{-1}$. *Then the impulsive anti-periodic boundary value problem (1.1) has a unique solution on J.*

Proof. Define an operator $\Theta : PC^1(J,\mathbb{R}) \to PC^1(J,\mathbb{R})$ by

$$\begin{aligned}
(\Theta x)(t) \ =& \int_{t_p}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \frac{1}{2}\int_{t_p}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
&+ \frac{(T-2t)}{4}\int_{t_p}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds \\
&- \frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \mathcal{I}_k(x(t_k^-))\Big) \\
&- \frac{1}{4}\sum_{0<t_k<T}\big(T+2(t-t_k)\big)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds + \mathcal{J}_k(x(t_k^-))\Big) \\
&+ \sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \mathcal{I}_k(x(t_k^-))\Big) \\
&+ \sum_{0<t_k<t}(t-t_k)\Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds + \mathcal{J}_k(x(t_k^-))\Big).
\end{aligned}$$

Setting $\sup_{t\in[0,T]}|f(t,0)| = M_1$ and choosing

$$r \ge 2\Big[M_1T^q\Big(\frac{3(1+p)}{2\Gamma(q+1)} + \frac{1+7p}{4\Gamma(q)}\Big) + \frac{p}{4}\Big(6M_2 + 7TM_3\Big)\Big],$$

we show that $\Theta B_r \subset B_r$, where $B_r = \{x \in PC^1(J,\mathbb{R}) : \|x\| \le r\}$. For $x \in B_r$, we have

$$
\begin{aligned}
&\|(\Theta x)(t)\| \\
\leq\ & \int_{t_p}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(s,x(s))\| ds + \frac{1}{2}\int_{t_p}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} \|f(s,x(s))\| ds \\
& + \frac{|T-2t|}{4} \int_{t_p}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} \|f(s,x(s))\| ds \\
& + \frac{1}{2} \sum_{0<t_k<T} \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} \|f(s,x(s))\| ds + \|\mathcal{I}_k(x(t_k^-))\| \Big) \\
& + \frac{1}{4} \sum_{0<t_k<T} \big| T + 2(t-t_k) \big| \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} \|f(s,x(s))\| ds + \|\mathcal{J}_k(x(t_k^-))\| \Big) \\
& + \sum_{0<t_k<t} \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} \|f(s,x(s))\| ds + \|\mathcal{I}_k(x(t_k^-))\| \Big) \\
& + \sum_{0<t_k<t} |t-t_k| \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} \|f(s,x(s))\| ds + \|\mathcal{J}_k(x(t_k^-))\| \Big) \\
\leq\ & \int_{t_p}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \Big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\| \Big) ds \\
& + \frac{1}{2} \int_{t_p}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} \Big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\| \Big) ds \\
& + \frac{|T-2t|}{4} \int_{t_p}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} \Big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\| \Big) ds \\
& + \frac{1}{2} \sum_{0<t_k<T} \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} \Big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\| \Big) ds + \|\mathcal{I}_k(x(t_k^-))\| \Big) \\
& + \frac{1}{4} \sum_{0<t_k<T} \big| T + 2(t-t_k) \big| \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} \Big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\| \Big) ds \\
& + \|\mathcal{J}_k(x(t_k^-))\| \Big) \\
& + \sum_{0<t_k<t} \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} \Big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\| \Big) ds + \|\mathcal{I}_k(x(t_k^-))\| \Big) \\
& + \sum_{0<t_k<t} |t-t_k| \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} \Big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\| \Big) ds \\
& + \|\mathcal{J}_k(x(t_k^-))\| \Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq (L_1 r+M_1)\Big[\int_{t_p}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}ds+\frac{1}{2}\int_{t_p}^{T}\frac{(T-s)^{q-1}}{\Gamma(q)}ds+\frac{|T-2t|}{4}\int_{t_p}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}ds\\
&\quad+\frac{1}{2}\sum_{0<t_k<T}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}ds+\frac{1}{4}\sum_{0<t_k<T}\big|T+2(t-t_k)\big|\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}ds\\
&\quad+\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}ds+\sum_{0<t_k<t}|t-t_k|\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}ds\Big]\\
&\quad+\frac{1}{2}\sum_{0<t_k<T}\|\mathcal{I}_k(x(t_k^-))\|+\frac{1}{4}\sum_{0<t_k<T}\big|T+2(t-t_k)\big|\|\mathcal{J}_k(x(t_k^-))\|\\
&\quad+\sum_{0<t_k<t}\|\mathcal{I}_k(x(t_k^-))\|+\sum_{0<t_k<t}|t-t_k|\|\mathcal{J}_k(x(t_k^-))\|\\
&\leq L_1T^q\Big(\frac{3(1+p)}{2\Gamma(q+1)}+\frac{1+7p}{4\Gamma(q)}\Big)r+\Big[M_1T^q\Big(\frac{3(1+p)}{2\Gamma(q+1)}+\frac{1+7p}{4\Gamma(q)}\Big)\\
&\quad+\frac{p}{4}\Big(6M_2+7TM_3\Big)\Big]\leq r.
\end{aligned}
$$

Now, for $x, y\in PC^1(J,\mathbb{R})$ and for each $t\in[0,T]$, we obtain

$$
\begin{aligned}
&\|(\Theta x)(t)-(\Theta y)(t)\|\\
&\leq \int_{t_p}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s,x(s))-f(s,y(s))\|ds\\
&\quad+\frac{1}{2}\int_{t_p}^{T}\frac{(T-s)^{q-1}}{\Gamma(q)}\|f(s,x(s))-f(s,y(s))\|ds\\
&\quad+\frac{|T-2t|}{4}\int_{t_p}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}\|f(s,x(s))-f(s,y(s))\|ds\\
&\quad+\frac{1}{2}\sum_{0<t_k<T}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\|f(s,x(s))-f(s,y(s))\|ds\\
&\qquad+\|\mathcal{I}_k(x(t_k^-))-\mathcal{I}_k(y(t_k^-))\|\Big)\\
&\quad+\frac{1}{4}\sum_{0<t_k<T}\big|T+2(t-t_k)\big|\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\|f(s,x(s))-f(s,y(s))\|ds\\
&\qquad+\|\mathcal{J}_k(x(t_k^-))-\mathcal{J}_k(y(t_k^-))\|\Big)\\
&\quad+\sum_{0<t_k<t}\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\|f(s,x(s))-f(s,y(s))\|ds\\
&\qquad+\|\mathcal{I}_k(x(t_k^-))-\mathcal{I}_k(y(t_k^-))\|\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{0<t_k<t}|t-t_k|\Big(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\|f(s,x(s))-f(s,y(s))\|ds \\
&+\|\mathcal{J}_k(x(t_k^-))-\mathcal{J}_k(y(t_k^-))\|\Big) \\
\leq\ & L_1\Big[\int_{t_p}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\|x(s)-y(s)\|ds+\frac{1}{2}\int_{t_p}^{T}\frac{(T-s)^{q-1}}{\Gamma(q)}\|x(s)-y(s)\|ds \\
&+\frac{|T-2t|}{4}\int_{t_p}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}\|x(s)-y(s)\|ds \\
&+\frac{1}{2}\sum_{0<t_k<T}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\|x(s)-y(s)\|ds \\
&+\frac{1}{4}\sum_{0<t_k<T}\big|T+2(t-t_k)\big|\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\|x(s)-y(s)\|ds \\
&+\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}\|x(s)-y(s)\|ds \\
&+\sum_{0<t_k<t}|t-t_k|\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}\|x(s)-y(s)\|ds\Big] \\
&+L_2\Big(\frac{1}{2}\sum_{0<t_k<T}\|x(t_k^-))-y(t_k^-)\|+\sum_{0<t_k<t}\|x(t_k^-))-y(t_k^-)\|\Big) \\
&+L_3\Big(\frac{1}{4}\sum_{0<t_k<T}\big|T+2(t-t_k)\big|\|x(t_k^-))-y(t_k^-)\| \\
&+\sum_{0<t_k<t}|t-t_k|\|x(t_k^-))-y(t_k^-)\|\Big) \\
\leq\ & \Lambda_{p,q,T,L_1,L_2,L_3}\|x-y\|,
\end{aligned}
$$

where

$$\Lambda_{p,q,T,L_1,L_2,L_3}=L_1T^q\Big(\frac{3(1+p)}{2\Gamma(q+1)}+\frac{1+7p}{4\Gamma(q)}\Big)+\frac{p}{4}(6L_2+7TL_3),$$

which depends only on the parameters involved in the problem. As $\Lambda_{p,q,T,L_1,L_2,L_3}<1$, therefore Θ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle.

Theorem 3.2. Assume that $(A_1)-(A_2)$ hold with $\frac{p}{4}(6L_2+7TL_3)<1$ and $\|f(t,x)\|\leq\mu(t),\ \forall(t,x)\in[0,T]\times\mathbb{R}$, where $\mu\in L^1([0,T],\mathbb{R}^+)$. Then the boundary value problem (1.1) has at least one solution on $[0,T]$.

Proof. Let us fix

$$r \geq \mu\|_{L^1} T^q \Big(\frac{3(1+p)}{2\Gamma(q+1)} + \frac{1+7p}{4\Gamma(q)} \Big) + \frac{p}{4}(6M_2 + 7TM_3),$$

and consider $B_r = \{x \in PC^1(J, \mathbb{R}) : \|x\| \leq r\}$. We define the operators Φ and Ψ on B_r as

$$\begin{aligned}
(\Phi x)(t) \,=\, & \int_{t_p}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds - \frac{1}{2} \int_{t_p}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds \\
& + \frac{(T-2t)}{4} \int_{t_p}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s, x(s))ds \\
& - \frac{1}{2} \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds \\
& - \frac{1}{4} \sum_{0<t_k<T} \big(T + 2(t-t_k)\big) \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f(s, x(s))ds \\
& + \sum_{0<t_k<t} \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds + (t-t_k) \\
& \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{q-2}}{\Gamma(q-1)} f(s, x(s))ds \Big), \\
(\Psi x)(t) \,=\, & -\frac{1}{4} \sum_{0<t_k<T} \Big(2\mathcal{I}_k(x(t_k^-)) + \big(T + 2(t-t_k)\big) \mathcal{J}_k(x(t_k^-)) \Big) \\
& + \sum_{0<t_k<t} \Big(\mathcal{I}_k(x(t_k^-)) + (t-t_k) \mathcal{J}_k(x(t_k^-)) \Big).
\end{aligned}$$

For $x, y \in B_r$, we find that

$$\|\Phi x + \Psi y\| \leq \mu\|_{L^1} T^q \Big(\frac{3(1+p)}{2\Gamma(q+1)} + \frac{1+7p}{4\Gamma(q)} \Big) + \frac{p}{4}(6M_2 + 7TM_3) \leq r.$$

Thus, $\Phi x + \Psi y \in B_r$. It follows from the assumption (A_2) that Ψ is a contraction mapping for $\frac{p}{4}(6L_2 + 7TL_3) < 1$. Continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as

$$\|\Phi x\| \leq \mu\|_{L^1} T^q \Big(\frac{3(1+p)}{2\Gamma(q+1)} + \frac{1+7p}{4\Gamma(q)} \Big).$$

Now we prove the compactness of the operator Φ. Letting $\Omega = [0, T] \times B_r$, we define $\sup_{(t,x)\in\Omega} \|f(t, x)\| = f_1 < \infty$, and consequently we have

$$
\begin{aligned}
&\|(\Phi x)(\tau_2)-(\Phi x)(\tau_1)\| \\
=& \Big\| \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \int_{t_p}^{\tau_1} \frac{(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
&-\frac{(\tau_2-\tau_1)}{2}\int_{t_p}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds \\
&-\frac{1}{2}\sum_{0<t_k<T}(\tau_2-\tau_1)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds \\
&+\sum_{0<t_k<\tau_2-\tau_1}\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds \\
&+\sum_{0<t_k<\tau_2-\tau_1}(\tau_2-t_k)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds \\
&+\sum_{0<t_k<\tau_1}(\tau_2-\tau_1)\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds\Big\| \\
\le& f_1\Big[\frac{1}{\Gamma(q+1)}\Big(|(\tau_2-\tau_1)^q|+|-(\tau_2-\tau_1)^q+(\tau_2-t_p)^q-(\tau_1-t_p)^q|\Big) \\
&+\frac{|(\tau_2-\tau_1)(T-t_p)^{q-1}|}{2\Gamma(q)}+\sum_{0<t_k<T}\frac{|(\tau_2-\tau_1)(t_k-t_{k-1})^{q-1}|}{2\Gamma(q)} \\
&+\sum_{0<t_k<\tau_2-\tau_1}\frac{|(t_k-t_{k-1})^q|}{\Gamma(q+1)} \\
&+\frac{1}{\Gamma(q)}\Big(\sum_{0<t_k<\tau_2-\tau_1}|(t_k-t_{k-1})^{q-1}(\tau_2-t_k)|+\sum_{0<t_k<\tau_1}|(\tau_2-\tau_1)(t_k-t_{k-1})^{q-1}|\Big)\Big],
\end{aligned}
$$

which is independent of x. So Φ is relatively compact on B_r. Hence, By Arzela Ascoli Theorem, Φ is compact on B_r. Thus all the assumptions of Theorem 2.1 are satisfied and the conclusion of Theorem 2.1 implies that the boundary value problem (1.1) has at least one solution on $[0,T]$.

Example. Consider the following impulsive fractional boundary value problem

$$
(3.1)\qquad \begin{cases}
{}^cD^{\frac{3}{2}}x(t)=\frac{1}{(t+6)^2}\dfrac{|x(t)|}{(1+|x(t)|)},\ t\in[0,1],\ t\neq\dfrac{1}{3}, \\
\Delta x(\dfrac{1}{3})=\dfrac{|x(\frac{1}{3}^-)|}{(5+|x(\frac{1}{3}^-)|)},\ \Delta x'(\dfrac{1}{3})=\dfrac{|x(\frac{1}{3}^-)|}{(7+|x(\frac{1}{3}^-)|)}, \\
x(0)=-x(1),\qquad x'(0)=-x'(1).
\end{cases}
$$

Clearly $L_1=\frac{1}{36},\ L_2=\frac{1}{5},\ L_3=\frac{1}{7},\ q=\frac{3}{2}$ and $p=1$. Further,

$$L_1T^q\Big(\frac{3(1+p)}{2\Gamma(q+1)}+\frac{1+7p}{4\Gamma(q)}\Big)+\frac{p}{4}(6L_2+7TL_3)=(\frac{2}{9\sqrt{\pi}}+\frac{11}{20})<1.$$

Thus, all the assumptions of Theorem 3.1 are satisfied. Hence, by the conclusion of Theorem 3.1, the impulsive fractional boundary value problem (3.1) has a unique solution on $[0,1]$.

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