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EXISTENCE OF MULTIPLE POSITIVE RADIAL SOLUTIONS FOR p-LAPLACIAN PROBLEMS WITH AN L^1-INDEFINITE WEIGHT

Chan-Gyun Kim and Yong-Hoon Lee

Abstract. In this paper we study the existence, multiplicity and nonexistence of positive solutions for p-Laplacian problems with L^1-indefinite weight. As an application, we give some existence and multiplicity results for Emden-Fowler type p-Laplacian radial problems defined on an exterior domain depending on the boundary value which plays the role of a parameter.

1. Introduction

In this paper, we consider the existence, multiplicity, and nonexistence of positive solutions for the following p-Laplacian problems.

$$(P_\lambda) \qquad \begin{cases} \varphi_p(u'(t))' + \lambda h(t) f(u(t)) = 0, \ t \in (0,1), \\ u(0) = a > 0, \ u(1) = 0, \end{cases}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, λ is a nonnegative real parameter, $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $h \in C((0,1), \mathbb{R}^+)$ may be singular at $t = 0$ and/or 1 with $\mathbb{R}_+ = [0,\infty), \mathbb{R}^+ = (0,\infty)$. Throughout this paper, we assume $f(u) > 0$ for $u > 0$.

By a positive solution to this problem we understand a function $u \in C^1[0,1]$ with $\varphi_p(u') \in C^1[0,1]$ satisfying (P_λ) and $u \geq 0$ on $[0,1]$.

Recently, Kong-Wang [5] and Agarwal-Lü-O'Regan [1] proved that if f satisfies assumptions $f_0 \triangleq \lim_{u\to 0} \frac{f(u)}{u^{p-1}} = 0$ and $f_\infty \triangleq \lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = 0$, then the Dirichlet boundary value problem

$$(D_\lambda) \qquad \begin{cases} \varphi_p(u'(t))' + \lambda h(t) f(u(t)) = 0, \ t \in (0,1), \\ u(0) = 0, \ u(1) = 0, \end{cases}$$

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has at least two positive solutions for sufficiently large λ. Sánchez [7] proved a similar result for the case $f_0 = \infty$ and $f_\infty = \infty$. Wang [8] proved that if f satisfies assumptions $f_0 = 0$ and $f_\infty = \infty$, then problem (D_λ) has at least one positive solution for all λ.

By the effect of boundary condition we concern in (P_λ), do not giving any growth restriction on f near 0, we obtain the following main results. For this, we give the list of assumptions first.

(F_1) $h \in L^1(0,1)$,

(F_2) $f_\infty = \infty$,

(F_2') $f_\infty = 0$,

(F_3) f is nondecreasing.

Result 1. Assume (F_1) and (F_2'). Then (P_λ) has at least one positive solution for all $\lambda > 0$.

Result 2. Assume $(F_1), (F_2)$ and (F_3). Then, there exists $\lambda^* > 0$ such that (P_λ) has at least two positive solutions for $\lambda \in (0, \lambda^*)$, at least one positive solution for $\lambda = \lambda^*$ and no solution for $\lambda > \lambda^*$.

As an application, let us consider the following p-Laplacian radial problems depending on the boundary value μ as a parameter

$$(P) \qquad \operatorname{div}(|\nabla u|^{p-2}\nabla u) + K(|x|)u^q = 0 \text{ in } \Omega,$$

$$(D_1) \qquad u|_{\partial\Omega} = 0 \text{ and } u \to \mu > 0 \text{ as } |x| \to \infty,$$

$$(D_2) \qquad u|_{\partial\Omega} = \mu \text{ and } u \to 0 \text{ as } |x| \to \infty,$$

where $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\},\ r_0 > 0,\ N > p > 1,\ \mu$ a positive real parameter, $K \in C(\Omega, (0,\infty))$.

Deng and Li ([3]) considered a semilinear problem of the form

$$(DL) \qquad \begin{cases} \Delta u + K(x)u^q = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega,\ u \in H^1_{\text{loc}}(\Omega) \cap C(\bar{\Omega}), \\ u|_{\partial\Omega} = 0,\ u \to \mu > 0 \text{ as } |x| \to \infty, \end{cases}$$

where $\Omega = \mathbb{R}^N \backslash \omega$ is an exterior domain in $\mathbb{R}^N$, $\omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N > 2,\ q > 1$. Consider the following hypotheses;

(K_1) $K \in C^\alpha_{\text{loc}}(\Omega),\ K \geq 0, K \not\equiv 0$ and there exist $C,\ \epsilon,\ M > 0$ such that $|K(x)| \leq C|x|^{-l}$ for $|x| \geq M$ with $l \geq 2 + \epsilon$.

(K_2) $K(x) > 0$ in a neighborhood V of some point $x_0 \in \Omega$ such that

$$K(x_0) = \sup_{x\in\Omega} K(x) \text{ and } K(x) = K(x_0) + O(|x - x_0|^2) \text{ near } x_0.$$

Assuming (K_1), they proved that there exists $\mu^* > 0$ such that (DL) has at least one solution for $\mu \in (0, \mu^*)$ and no solution for $\mu \in (\mu^*, \infty)$. Furthermore, if $K \in L^1(\Omega)$, then the solution at $\mu = \mu^*$ exists and is unique. On the other hand, when $q = \frac{N+2}{N-2}$, assuming $(K_1), (K_2)$ and $0 \le K(x) \in L^1(\Omega)$, they also proved that there exists $\mu^* > 0$ such that (DL) has at least two solutions for $\mu \in (0, \mu^*)$, (unique) solution for $\mu = \mu^*$ and no solution for $\mu \in (\mu^*, \infty)$.

As a corollary of Result 2, we see that for radial problem (P), the second result is true without the restriction on the exponent q. More precisely, assume

(K) $K \in L^1(\Omega)$ with $K > 0$ in Ω,

and $q > p-1$. Then there exists $\mu^* > 0$ such that $(P) + (D_i),\ i = 1, 2$ has at least two positive radial solutions for $\mu \in (0, \mu^*)$, at least one positive radial solution for $\mu = \mu^*$ and no positive radial solution for $\mu > \mu^*$.

This paper is organized as follows. In Section 2, we introduce well-known theorems such as Global Continuation Theorem, the generalized Picone identity and a fixed point index Theorem for the index computation. In Section 3, we state and prove the main results. In section 4, introducing several transformations to obtain equivalent one-dimensional p-Laplacian problems, we give the existence, multiplicity or nonexistence of positive radial solutions for problems $(P)+(D_i),\ i = 1, 2$.

2. Preliminaries

In this section, We introduce some known theorems which will be used in the following sections.

Theorem 2.1. ([9], Global Continuation Theorem). *Let X be a Banach space and $\mathcal{K}$ an order cone in X. Consider*

$$x = H(\mu, x), \tag{2.1}$$

where $\mu \in \mathbb{R}_+$ and $x \in \mathcal{K}$. If $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ is completely continuous and $H(0, x) = 0$ for all $x \in \mathcal{K}$. Then $\mathcal{C}_+(\mathcal{K})$, the component of the solution set of (2.1) containing $(0, 0)$ is unbounded.

Theorem 2.2. ([6], Generalized Picone Identity). *Let us define*

$$l_p[y] = (\varphi_p(y'))' + b_1(t)\varphi_p(y),$$

$$L_p[z] = (\varphi_p(z'))' + b_2(t)\varphi_p(z).$$

If y and z are any functions such that $y,\ z,\ b_1\varphi_p(y'),\ b_2\varphi_p(z')$ are differentiable on I and $z(t)\neq 0$ for $t\in I$, then the generalized Picone identity can be written as

$$\frac{d}{dt}\left\{\frac{|y|^p\varphi_p(z')}{\varphi_p(z)}-y\varphi_p(y')\right\} \tag{2.2}$$

$$=(b_1-b_2)|y|^p \tag{2.3}$$

$$-\left[|y'|^p+(p-1)\left|\frac{yz'}{z}\right|^p-p\varphi_p(y)y'\varphi_p\left(\frac{z'}{z}\right)\right] \tag{2.4}$$

$$-yl_p(y)+\frac{|y|^p}{\varphi_p(z)}L_p(z). \tag{2.5}$$

Remark 2.3. By Young's inequality, we get

$$|y'|^p+(p-1)\left|\frac{yz'}{z}\right|^p-p\varphi_p(y)\varphi_p\left(\frac{z'}{z}\right)\geq 0,$$

and the equality holds if and only if $sgn\ y'=sgn\ z'$ and $|\frac{y'}{y}|^p=|\frac{z'}{z}|^P$.

Theorem 2.4. ([4]). *Let X be a Banach space, $\mathcal{K}$ a cone in X and $\mathcal{O}$ bounded open in X. Let $0\in\mathcal{O}$ and $A:\mathcal{K}\cap\bar{\mathcal{O}}\to\mathcal{K}$ be condensing. Suppose that $Ax\neq\nu x$ for all $x\in\mathcal{K}\cap\partial\mathcal{O}$ and all $\nu\geq 1$. Then $i(A,\mathcal{K}\cap\mathcal{O},\mathcal{K})=1$.*

3. Main Result

In this section, we state and prove the main results for problem (P_λ).

Theorem 3.1. *Assume (F_1) and (F_2'). Then (P_λ) has at least one positive solution for all $\lambda>0$.*

Theorem 3.2. *Assume $(F_1),(F_2)$ and (F_3). Then, there exists $\lambda^*>0$ such that (P_λ) has at least two positive solutions for $\lambda\in(0,\lambda^*)$, at least one positive solution for $\lambda=\lambda^*$ and no solution for $\lambda>\lambda^*$.*

To fulfil conditions in Global Continuation Theorem, we need to consider problems with Dirichlet boundary condition. For this, we substitute $v(t)=u(t)-a(1-t)$ in problem (P_λ) to get the following equivalent problem;

$$(\hat{P}_\lambda)\qquad \begin{cases}\varphi_p(v'(t)-a)'+\lambda h(t)f(v(t)+a(1-t))=0,\ t\in(0,1),\\ v(0)=0=v(1).\end{cases}$$

Denote $\mathcal{K} = \{w \in C_0^1[0,1] : w \text{ is concave }\}$. Then, it is easy to check that $\mathcal{K}$ is an ordered cone. For $u \in \mathcal{K}$ and $\lambda > 0$, define $x_{\lambda,u}$ by

$$x_{\lambda,u}(t) = \int_0^t \varphi_p^{-1}\left(\int_s^t \lambda h(\tau)f(u(\tau)+a(1-\tau))d\tau - \varphi_p(a)\right)ds + at$$
$$-\left[\int_t^1 \varphi_p^{-1}\left(\int_t^s \lambda h(\tau)f(u(\tau)+a(1-\tau))d\tau+\varphi_p(a)\right)ds - a(1-t)\right],$$

for $0 < t < 1$. Clearly, $x_{\lambda,u}$ is continuous. For $0 < s < t$, we have

$$\int_s^t \lambda h(\tau)f(u(\tau)+a(1-\tau))d\tau - \varphi_p(a) > -\varphi_p(a).$$

Since φ_p^{-1} is increasing, we get

$$\varphi_p^{-1}\left(\int_s^t \lambda h(\tau)f(u(\tau)+a(1-\tau))d\tau - \varphi_p(a)\right) > -\varphi_p^{-1}(\varphi_p(a)) = -a.$$

Therefore,

$$\varphi_p^{-1}\left(\int_s^t \lambda h(\tau)f(u(\tau)+a(1-\tau))d\tau - \varphi_p(a)\right) + a > 0. \tag{3.1}$$

Similarly, for $t < s < 1$, we have

$$\varphi_p^{-1}\left(\int_t^s \lambda h(\tau)f(u(\tau)+a(1-\tau))d\tau + \varphi_p(a)\right) - a > 0. \tag{3.2}$$

It follows from (3.1) and (3.2) that $x_{\lambda,u}$ is strictly increasing in $(0,1)$ and $x_{\lambda,u}(0^+) < 0 < x_{\lambda,u}(1^-)$. Thus $x_{\lambda,u}$ has a unique zero in $(0,1)$ so let $A_{\lambda,u}$ be the zero of $x_{\lambda,u}$ in $(0,1)$. Then

$$\int_0^{A_{\lambda,u}} \varphi_p^{-1}\left(\int_s^{A_{\lambda,u}} \lambda h(\tau)f(u(\tau)+a(1-\tau))d\tau - \varphi_p(a)\right)ds + aA_{\lambda,u}$$
$$= \int_{A_{\lambda,u}}^1 \varphi_p^{-1}\left(\int_{A_{\lambda,u}}^s \lambda h(\tau)f(u(\tau)+a(1-\tau))d\tau + \varphi_p(a)\right)ds - a(1-A_{\lambda,u}).$$

Let us define operator $H : \mathbb{R}_+ \times \mathcal{K} \to C_0^1[0,1]$ as follows.
For $\lambda > 0$,

$$H(\lambda,v)(t) = \begin{cases} \displaystyle\int_0^t \varphi_p^{-1}\left(\int_s^{A_{\lambda,v}} \lambda h(\tau)f(v(\tau)+a(1-\tau))d\tau - \varphi_p(a)\right)ds + at, \\ \quad \text{if } 0 \le t \le A_{\lambda,v}, \\ \displaystyle\int_t^1 \varphi_p^{-1}\left(\int_{A_{\lambda,v}}^s \lambda h(\tau)f(v(\tau)+a(1-\tau))d\tau + \varphi_p(a)\right)ds - a(1-t), \\ \quad \text{if } A_{\lambda,v} \le t \le 1, \end{cases}$$

where

$$\begin{aligned}(3.3)\quad &\int_0^{A_{\lambda,v}} \varphi_p^{-1}\left(\int_s^{A_{\lambda,v}} \lambda h(\tau)f(v(\tau)+a(1-\tau))d\tau-\varphi_p(a)\right)ds+aA_{\lambda,v}\\ &=\int_{A_{\lambda,v}}^1 \varphi_p^{-1}\left(\int_{A_{\lambda,v}}^s \lambda h(\tau)f(v(\tau)+a(1-\tau))d\tau+\varphi_p(a)\right)ds-a(1-A_{\lambda,v}),\end{aligned}$$

and for $\lambda = 0$, $H(\lambda, v) = 0$. Then by the definition of $A_{\lambda,v}$, we can easily see that H is well-defined and $H(\mathbb{R}_+ \times \mathcal{K}) \subset \mathcal{K}$. Furthermore, u is a positive solution of $(\hat{P}_\lambda)$ if and only if $u = H(\lambda, u)$ on $\mathcal{K}$.

To apply Global Continuation Theorem, we need to guarantee the compactness of H on $\mathbb{R}_+ \times \mathcal{K}$. The proof basically follows on the lines of Lemmas 2 and 3 in [1] or in [7].

Lemma 3.3. *$H : [0,\infty) \times K \to \mathcal{K}$ is completely continuous.*

Since $H(0, u) = 0$, for all $u \in K$, by Lemma 3.3 and Global Continuation Theorem (Theorem 2.1), we know that there exists an unbounded continuum $\mathcal{C}$ of positive solutions of $(\hat{P}_\lambda)$ emanating from $(0, 0)$. Equivalently, there exists an unbounded continuum $\mathcal{C}'$ of positive solutions of (P_λ) emanating from $(0, a(1-t))$. We now give *a priori* estimate for problem (P_λ).

Lemma 3.4. *Assume $(F_1), (F_2')$ and let $J = [0, l]$ with $l > 0$. Then there exists $M_J > 0$ such that for all possible positive solution u of (P_λ) with $\lambda \in J$, we have*

$$\|u\| \le M_J.$$

Proof. Suppose on the contrary that there exists a sequence $\{u_n\}$ of positive solutions of (P_{λ_n}) with $\{\lambda_n\} \subset J \triangleq [0, l]$ and $\|u_n\| \to \infty$ as $n \to \infty$. Then, we can easily see that $\|u_n\|_\infty \to \infty$. Let $\alpha \in \left(0, \frac{1}{l\varphi_p(4\gamma_p Q)}\right)$, where $\gamma_p = \max\left\{1, 2^{\frac{-p+2}{p-1}}\right\}, Q = \varphi_p^{-1}\left(\int_0^1 h(s)ds\right)$. Then by (F_2'), there exists $u_\alpha > 0$ such that $u > u_\alpha$ implies $f(u) < \alpha u^{p-1}$. Let $m_\alpha \triangleq \max_{u\in[0,u_\alpha]} f(u)$ and let $A_n \triangleq \{t \in [0, 1] : u_n(t) \le u_\alpha\}$ and $B_n \triangleq \{t \in [0, 1] : u_n(t) > u_\alpha\}$. Put $u_n(\delta_n) = \|u_n\|_\infty$. By the facts $u_n(0) = a$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$, we may assume $\delta_n > 0$ and $u_n(\delta_n) > 2a$ for all n. By simple calculation, we know that

$$u_n(\delta_n) = \int_0^{\delta_n} \varphi_p^{-1}\left(\lambda_n \int_s^{\delta_n} h(\tau)f(u_n(\tau))d\tau\right) ds + a.$$

Then we have

$$\frac{1}{2}u_n(\delta_n)$$

$$\leq \int_0^{\delta_n} \varphi_p^{-1}\left(\lambda_n \int_0^{\delta_n} h(\tau)f(u_n(\tau))d\tau\right) ds$$

$$\leq \varphi_p^{-1}(\lambda_n) \int_0^{\delta_n} \varphi_p^{-1}\left(\int_{A_n} h(\tau)f(u_n(\tau))d\tau + \int_{B_n} h(\tau)f(u_n(\tau))d\tau\right) ds$$

$$\leq \varphi_p^{-1}(\lambda_n) \int_0^{\delta_n} \varphi_p^{-1}\left(m_\alpha \int_{A_n} h(\tau)d\tau + \int_{B_n} h(\tau)f(u_n(\tau))d\tau\right) ds$$

$$\leq \varphi_p^{-1}(\lambda_n) \int_0^{\delta_n} \gamma_p \left[\varphi_p^{-1}\left(m_\alpha \int_{A_n} h(\tau)d\tau\right) + \varphi_p^{-1}\left(\int_{B_n} h(\tau)f(u_n(\tau))d\tau\right)\right] ds.$$

Thus

$$\frac{1}{2\varphi_p^{-1}(\lambda_n)} \leq \gamma_p \int_0^{\delta_n} \left[\frac{\varphi_p^{-1}(m_\alpha)Q}{||u_n||_\infty} + \varphi_p^{-1}\left(\int_{B_n} \frac{h(\tau)f(u_n(\tau))}{||u_n||_\infty^{p-1}} d\tau\right)\right] ds.$$

On $B_n, u_n(s) > u_\alpha$ implies $\frac{f(u_n(s))}{||u_n||_\infty^{p-1}} \leq \frac{f(u_n(s))}{u_n^{p-1}(s)} \leq \alpha$. Thus

$$\frac{1}{2\varphi_p^{-1}(\lambda_n)} \leq \gamma_p \left[\frac{\varphi_p^{-1}(m_\alpha)Q}{||u_n||_\infty} + \varphi_p^{-1}(\alpha)Q\right].$$

Since $\lambda_n \leq l$ for all n, we have $\frac{1}{\varphi_p^{-1}(\lambda_n)} \geq \frac{1}{\varphi_p^{-1}(l)}$ for all n and thus

$$\frac{1}{2\varphi_p^{-1}(l)} \leq \gamma_p \left[\frac{\varphi_p^{-1}(m_\alpha)Q}{||u_n||_\infty} + \varphi_p^{-1}(\alpha)Q\right].$$

By the fact $||u_n||_\infty \to \infty$ as $n \to \infty$, we get

$$\frac{1}{2\varphi_p^{-1}(l)} \leq \gamma_p \varphi_p^{-1}(\alpha)Q \leq \gamma_p \varphi_p^{-1}\left(\frac{1}{l\varphi_p(4\gamma_p Q)}\right) Q = \frac{1}{4\varphi_p^{-1}(l)}.$$

This contradiction completes the proof. ∎

The proof of Theorem 3.1 is straightforward from Lemma 3.4 and the existence of unbounded continuum $\mathcal{C}'$. We now prove the second main theorem. Using the generalized Picone identity and the properties of the p-sine function ([2], [10]), we obtain the following lemmas.

Lemma 3.5. *Assume* $(F_1), (F_2)$. *Then there exists* $\bar{\lambda} > 0$ *such that if* (P_λ) *has a positive solution* u_λ, *then* $\lambda \leq \bar{\lambda}$.

Proof. Let problem (P_λ) have a positive solution u_λ, then u_λ is concave and $u_\lambda(0) = a, u_\lambda(t) \geq \frac{1}{4}a$ for all $t \in (0, \frac{3}{4})$. It follows from (F_2) that there exists $A > 0$ such that $f(u) > Au^{p-1}$ for $u \geq \frac{1}{4}a$. This implies

$$\varphi_p(u_\lambda'(t))' + \lambda A h(t)\varphi_p(u_\lambda(t)) < \varphi_p(u_\lambda'(t))' + \lambda h(t) f(u_\lambda(t)) = 0,\ t \in (0, \frac{3}{4}).$$

Putting $m := \min_{t\in[\frac{1}{4},\frac{3}{4}]} h(t) > 0$, we have

$$\varphi_p(u_\lambda'(t))' + \lambda A m \varphi_p(u_\lambda(t)) < 0,\ t \in (\frac{1}{4}, \frac{3}{4}).$$

It is easy to check that $w(t) = S_q\left(2\pi_p(t - \frac{1}{4})\right)$ is a solution of

$$\begin{cases} \varphi_p(w'(t))' + (2\pi_p)^p \varphi_p(w(t)) = 0,\ t \in (\frac{1}{4}, \frac{3}{4}) \\ w(\frac{1}{4}) = 0 = w(\frac{3}{4}), \end{cases}$$

where S_q is the q-sine function with $\frac{1}{p} + \frac{1}{q} = 1$ and $\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$. Taking $y = w$ and $z = u_\lambda$ in (2.2)-(2.5) and integrating from $1/4$ to $3/4$, we have

$$\int_{1/4}^{3/4} ((2\pi_p)^p - \lambda A m)|w|^p dt \geq 0.$$

This implies

$$\lambda \leq \frac{(2\pi_p)^p}{Am} \triangleq \overline{\lambda}$$

and the proof is complete. ∎

Lemma 3.6. *Assume* $(F_1), (F_2)$. *Let* I *be a compact interval in* $(0, \infty)$. *Then there exists* $b_I > 0$ *such that for all possible positive solution* u *of* (P_λ) *with* $\lambda \in I$, *we have*

$$\|u\| \leq b_I.$$

Proof. Suppose on the contrary that there exists a sequence (u_n) of positive solutions of (P_{λ_n}) with $(\lambda_n) \subset J = [\alpha, \beta]$ and $||u_n|| \to \infty$ as $n \to \infty$. Then, we can easily see that $\|u_n\|_\infty \to \infty$. It follows from the concavity of u_n,

$$u_n(t) \geq \frac{1}{4}\|u_n\|_\infty,$$

for all n and $t \in (\frac{1}{4}, \frac{3}{4})$. Take $M = 2\frac{(2\pi_p)^p}{\alpha m}$, where $m := \min_{t\in[\frac{1}{4},\frac{3}{4}]} h(t) > 0$. By (F_2), there exists $K > 0$ such that $f(u) > M\varphi_p(u)$, for all $u > K$. From the assumption, we get $||u_N||_\infty > 4K$, for sufficiently large N. Therefore, we have

$$f(u_N(t)) > M\varphi_p(u_N(t)),\ t \in (\frac{1}{4}, \frac{3}{4}).$$

This implies

$$\varphi_p(u_N'(t))' + \alpha Mm\varphi_p(u_N(t)) < 0,\ t \in (\frac{1}{4}, \frac{3}{4}).$$

As in the proof of Lemma 3.5, if we take $w(t) = S_q\left(2\pi_p(t - \frac{1}{4})\right)$, we obtain

$$M \leq \frac{(2\pi_p)^p}{\alpha m}.$$

This is a contradiction. ■

Let us assume that problem $(\hat{P}_\lambda)$ has a positive solution say, u_* at $\lambda_* > 0$ i.e., u_* satisfies

$$\varphi_p(u_*'(t) - a)' + \lambda_* h(t) f(u_*(t) + a(1-t)) = 0,\quad t \in (0,1). \tag{3.4}$$

Consider a fixed parameter $\lambda \in (0, \lambda_*)$. For $N > 0$, put

$$\Sigma_N = \{u \in C_0^1[0,1] \mid 0 < u(t) < u_*(t),\ t \in (0,1),\ 0 < u'(0) < u_*'(0),$$
$$u_*'(1) < u'(1) < 0 \text{ and } \|u'\|_\infty < N\}.$$

Then, Σ_N is bounded and open in $C_0^1[0,1]$. Consider the following modified problem

$$(M_\lambda)\qquad \begin{cases} \varphi_p(u'(t) - a)' + \lambda h(t) f(\gamma(t, u(t)) + a(1-t)) = 0,\ t \in (0,1) \\ u(0) = 0 = u(1), \end{cases}$$

where $\gamma : (0,1) \times \mathbb{R} \to \mathbb{R}_+$ by $\gamma(t,u) = \begin{cases} u_*(t) & \text{if } u > u_*(t) \\ u & \text{if } 0 \leq u \leq u_*(t) \\ 0 & \text{if } u < 0. \end{cases}$

Lemma 3.7. *Assume* $(F_1), (F_2), (F_3)$ *and let* $\lambda \in (0, \lambda_*)$. *Then, there exists* $N > 0$ *such that* $u \in \Sigma_N \cap \mathcal{K}$, *for all positive solution* u *of* (M_λ).

Proof. Let u be a positive solution of (M_λ). Clearly, $u(t) > 0$, $t \in (0,1)$. First, we claim $u(t) \leq u_*(t),\ t \in (0,1)$. If not, there exists $t_1 \in (0,1)$ such that $u(t_1) > u_*(t_1)$. Since $u - u_* \in C_0[0,1]$, there exists $A \in (0,1)$ such that

$$u'(A) = u_*'(A) \text{ and } u(A) > u_*(A). \tag{3.5}$$

Since $\lambda < \lambda_*$ and f is nondecreasing,

$$\lambda_* f(u_*(t) + a(1-t)) > \lambda f(\gamma(t, u(t)) + a(1-t)),\ t \in (0,1).$$

This implies

$$\varphi_p(u'(t)-a)'+\lambda_* h(t)f(u_*(t)+a(1-t))>0,\ t\in(0,1). \tag{3.6}$$

By (3.4) and (3.6), we have

$$\varphi_p(u_*'(t)-a)'-\varphi_p(u'(t)-a)'<0,\ t\in(0,1). \tag{3.7}$$

For $t\in(A,1)$, integrating (3.7) from A to t, we have $u_*'(t)\le u'(t)$. Again, integrating this from A to 1, we get

$$u_*(A)\ge u(A).$$

This contradicts (3.5).
Second, we claim $u(t)<u_*(t),\ t\in(0,1)$. If not, by (3.7), we have only one case; there exist $t_2\in(0,1)$ and $\delta_1>0$ such that

$$u(t_2)=u_*(t_2), \tag{3.8}$$

$$u(t)<u_*(t),\ t\in(t_2-\delta_1,t_2+\delta_1)\backslash\{t_2\} \tag{3.9}$$

and

$$u'(t_2)=u_*'(t_2). \tag{3.10}$$

For $t\in(t_2-\delta_1,t_2)$, integrating (3.7) from t to t_2, by (3.10) we have

$$u_*'(t)\ge u'(t),\ t\in(t_2-\delta_1,t_2).$$

Again, integrating this from $t_2-\frac{\delta_1}{2}$ to t_2, by (3.8) we get

$$u_*(t_2-\frac{\delta_1}{2})\le u(t_2-\frac{\delta_1}{2}).$$

This contradicts (3.9).
Third, we claim $0>u'(1)>u_*'(1)$. We first show that there exists $c\in(0,1)$ such that $u'(c)>u_*'(c)$. If not, for all $t\in(0,1)$, $u'(t)\le u_*'(t)$. Integrating this from t to 1, we have $u(t)\ge u_*(t)$, $t\in(0,1)$ and this is a contradiction. Integrating (3.7) from c to 1, we get

$$u_*'(1)<u'(1).$$

Since u is a positive solution of (M_λ), clearly, $u'(1)<0$ and the claim is valid. Similarly, we can prove $0<u'(0)<u_*'(0)$.

Finally, we claim $\|u'\|_\infty < N$ for some $N > 0$. Since $0 \le u(t) \le u_*(t),\ t \in [0,1]$, we can easily obtain

$$|u'(t)| \ \le \ \varphi_p^{-1}\left(\int_0^1 f_* h(\tau)d\tau + \varphi_p(a)\right) ds + a \triangleq N$$

where $f_* =: \lambda \sup_{v\in[0,\|u_*\|+a]} f(v)$ and this completes the proof. ∎

Now, we give the proof of the second main theorem.

Proof of Theorem 3.2. Let $\lambda^* = \sup\{\hat{\lambda} |\ (\hat{P}_\lambda)$ has at least two positive solutions for $\lambda \in (0,\hat{\lambda})\}$. Then, by Lemma 3.5 and Lemma 3.6, λ^* is well defined in $(0,\bar{\lambda}]$. By the choice of λ^*, $(\hat{P}_\lambda)$ has at least two positive solutions for $\lambda \in (0,\lambda^*)$, at least one positive solution for $\lambda = \lambda^*$. We will show that $(\hat{P}_\lambda)$ has no positive solution for all $\lambda > \lambda^*$. On the contrary, suppose that there exists $\lambda_* > \lambda^*$ such that $(\hat{P}_{\lambda_*})$ has a positive solution. If we show that $(\hat{P}_\lambda)$ has at least two positive solutions for $\lambda \in [\lambda^*,\lambda_*)$, then the contradiction to the choice of λ^* completes the proof. Define $M : \mathcal{K} \to \mathcal{K}$ by

$$Mu(t) = \begin{cases} \displaystyle\int_0^t \varphi_p^{-1}\left(\int_s^{A_u} \lambda h(\tau) f(\gamma(\tau,u(\tau)) + a(1-\tau))d\tau - \varphi_p(a)\right) ds + at, \\ \quad 0 \le t \le A_u, \\ \displaystyle\int_t^1 \varphi_p^{-1}\left(\int_{A_u}^s \lambda h(\tau) f(\gamma(\tau,u(\tau)) + a(1-\tau))d\tau + \varphi_p(a)\right) ds - a(1-t), \\ \quad A_u \le t \le 1, \end{cases}$$

where

$$\begin{aligned} &\int_0^{A_u} \varphi_p^{-1}\left(\int_s^{A_u} \lambda h(\tau) f(\gamma(u(\tau)) + a(1-\tau))d\tau - \varphi_p(a)\right) ds + aA_u \\ = &\int_{A_u}^1 \varphi_p^{-1}\left(\int_{A_u}^s \lambda h(\tau) f(\gamma(u(\tau)) + a(1-\tau))d\tau + \varphi_p(a)\right) ds - a(1-A_u). \end{aligned}$$

Then $M : \mathcal{K} \to \mathcal{K}$ is completely continuous and u is a solution of (M_λ) if and only if $u = Mu$ on $\mathcal{K}$. By simple calculations, we see that there exists $R_1 > 0$ such that $\|Mu\| < R_1$, for all $u \in \mathcal{K}$. Since a completely continuous operator is condensing, applying Theorem 2.4 with $O = B_{R_1}$, we get

$$i(M, B_{R_1} \cap \mathcal{K}, \mathcal{K}) = 1.$$

By Lemma 3.7 and excision property, we get

$$i(M, \Sigma_N \cap \mathcal{K}, \mathcal{K}) = i(M, B_{R_1} \cap \mathcal{K}, \mathcal{K}) = 1.$$

Since problem $(\hat{P}_\lambda)$ is equivalent to the problem (M_λ) on $\Sigma_N \cap \mathcal{K}$, we conclude that $(\hat{P}_\lambda)$ has a positive solution in $\Sigma_N \cap \mathcal{K}$. Assume $H(\lambda, \cdot)$ has no fixed point in $\partial\Sigma_N \cap \mathcal{K}$ (otherwise, the proof is done!). Then by Lemma 3.5, $(P_{\lambda_{N_0}})$ has no solution in $\mathcal{K}$ for $\lambda_{N_0} > \overline{\lambda}$. By *a priori* estimate (Lemma 3.6) with $I = [\lambda, \lambda_{N_0}]$, there exists $R_2(> R_1) > 0$ such that for all possible positive solution u of $(\hat{P}_\mu)$ with $\mu \in [\lambda, \lambda_{N_0}]$, we have

$$\|u\| < R_2.$$

Define $h : [0,1] \times (\overline{B}_{R_2} \cap \mathcal{K}) \to \mathcal{K}$ by

$$h(\tau, u) = H(\tau\lambda_{N_0} + (1-\tau)\lambda, u).$$

Then, h is completely continuous on $[0,1] \times \mathcal{K}$, $h(\tau, u) \neq u$, for all $(\tau, u) \in [0,1] \times (\partial B_{R_2} \cap \mathcal{K})$. By the property of homotopy invariance,

$$i(H(\lambda, \cdot), B_{R_2} \cap \mathcal{K}, \mathcal{K}) = i(H(\lambda_{N_0}, \cdot), B_{R_2} \cap \mathcal{K}, \mathcal{K}) = 0.$$

By additivity property,

$$i(H(\lambda, \cdot), (B_{R_2} \setminus \overline{\Sigma}_N) \cap \mathcal{K}, \mathcal{K}) = -1.$$

Therefore, $(\hat{P}_\lambda)$ has another positive solution in $(B_{R_2} \setminus \overline{\Sigma}_N) \cap \mathcal{K}$ and the proof is complete. ■

4. An Application

In this section, we introduce several transformations to obtain equivalent one-dimensional p-Laplacian problems which we mainly analyzed in the previous section and give the existence, multiplicity or nonexistence of positive radial solutions for problems $(P) + (D_i),\ i = 1, 2$. Let us consider problems $(P) + (D_i), i = 1, 2$

$$\text{div}(|\nabla u|^{p-2}\nabla u) + K(|x|)u^q = 0 \text{ in } \Omega, \tag{P}$$

$$u|_{\partial\Omega} = 0 \text{ and } u \to \mu > 0 \text{ as } |x| \to \infty, \tag{D_1}$$

$$u|_{\partial\Omega} = \mu \text{ and } u \to 0 \text{ as } |x| \to \infty, \tag{D_2}$$

where μ a positive real parameter, $N > p$ and $K \in C(\Omega, (0, \infty))$.

By applying consecutive changes of variables, $r = |x|,\ u(r) = u(|x|)$ and $t = \left(\frac{r}{r_0}\right)^{\frac{-N+p}{p-1}},\ z(t) = u(r)$, problem $(P) + (D_1)$ is equivalently written as

$$\begin{cases} \varphi_p(z'(t))' + h(t)z(t)^q = 0,\ t \in (0,1), \\ z(0) = \mu > 0, z(1) = 0, \end{cases} \tag{4.1}$$

where h is given by

$$h(t) = \left(\frac{p-1}{N-p}\right)^p r_0^p t^{\frac{-p(N-1)}{N-p}} K\left(r_0 t^{\frac{-(p-1)}{N-p}}\right).$$

We notice that h is singular at $t = 0$ and $h \in L^1(0,1]$ by the fact $K \in L^1(\Omega)$. Introducing $u(t) = \frac{z(t)}{\mu}$, we can rewrite problem (4.1) as

$$\begin{cases} \varphi_p(u'(t))' + \lambda h(t) u(t)^q = 0, \ t \in (0,1), \\ u(0) = 1, u(1) = 0, \end{cases} \tag{4.2}$$

where $\lambda = \mu^{q-p+1}$. Problems (4.1) and (4.2) share the same bifurcation phenomena with respect to μ and λ respectively. Similarly, if we use transformation $t = 1 - \left(\frac{r}{r_0}\right)^{\frac{-(N-p)}{p-1}}$, then h in (4.1) is given by

$$h(t) = \left(\frac{p-1}{N-p}\right)^p r_0^p (1-t)^{\frac{-p(N-1)}{N-p}} K\left(r_0 (1-t)^{\frac{-(p-1)}{N-p}}\right).$$

Notice that h is singular at $t = 1$ and $h \in L^1[0,1)$. Consequently, for radial problems $(P) + (D_i),\ i = 1,2$ it is enough to consider problem (4.2) with $h \in L^1(0,1)$.

Direct applications of Theorems 3.1 and 3.2 lead to the following corollaries for problems $(P) + (D_i),\ i = 1,2$.

Corollary 4.1. *Assume* $0 < q < p-1$ *and* $K \in L^1(\Omega)$ *with* $K > 0$ *in* Ω. *Then* $(P) + (D_i),\ i = 1,2$ *has at least one positive radial solutions for all* $\mu > 0$.

Corollary 4.2. *Assume* $q > p-1$ *and* $K \in L^1(\Omega)$ *with* $K > 0$ *in* Ω. *Then there exists* $\mu^* > 0$ *such that* $(P) + (D_i),\ i = 1,2$ *has at least two positive radial solutions for* $\mu \in (0, \mu^*)$, *at least one positive radial solution for* $\mu = \mu^*$ *and no positive radial solution for* $\mu > \mu^*$.

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Chan-Gyun Kim and Yong-Hoon Lee
Department of Mathematics
Pusan National University
Busan 609-735
Korea