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# PRINCIPAL QUASI-BAERNESS OF MODULES OF GENERALIZED POWER SERIES

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**Abstract.** Let $R$ be a ring, $M$ a right $R$-module and $(S, \leq)$ a strictly totally ordered monoid. It is shown that $[[M^{S,\leq}]]$, the module of generalized power series with coefficients in $M$ and exponents in $S$, is a p.q.Baer right $[[R^{S,\leq}]]$-module if and only if the right annihilator of any $S$-indexed family of cyclic submodules of $M$ in $R$ is generated by an idempotent of $R$. Furthermore, we will show that for a ring $R$ with all left semicentral idempotents are central, the ring $[[R^{S,\leq}]]$ consisting of generalized power series over $R$ is a right p.q.Baer ring if and only if $R$ is a right p.q.Baer ring and any $S$-indexed family of central idempotents of $R$ has a generalized join in $I(R)$, where $I(R)$ is the set of all idempotents of $R$.

## 1. Introduction

Throughout this paper all rings $R$ are associative with identity and modules are unital right $R$-modules. We write $M[x], M[[x]], M[x, x^{-1}]$ and $M[x^{-1}, x]]$ for the polynomial extension, the power series extension, the Laurent polynomial extension and the Laurent series extension of a module $M$, respectively. For a subset $X$ of a module $M_R$, let $r_R(X) = \{r \in R \mid Xr = 0\}$.

Recall that $R$ is (quasi-) Baer if the right annihilator of every nonempty subset (every right ideal) of $R$ is generated by an idempotent. A lot of works on Baer rings and quasi-Baer rings appears in [3–6, 9]. As a generalization of quasi-Baer rings, G.F. Birkenmeier, J.Y. Kim and J.K. Park in [7] introduced the concept of principally quasi-Baer rings. A ring $R$ is called right principally quasi-Baer (or simply right p.q.Baer) if the right annihilator of a principal right ideal of $R$ is generated by an idempotent. Similarly, left p.q.Baer rings can be defined. A ring is called p.q.Baer if it is both right and left p.q.Baer. Observe that every biregular ring and every

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quasi-Baer ring is a p.q.Baer ring. For more details and examples of right p.q.Baer rings, see [7].

It was proved in [6, Theorem 1.8] that a ring $R$ is quasi-Baer if and only if $R[X]$ is quasi-Baer if and only if $R[[X]]$ is quasi-Baer, where $X$ is an arbitrary nonempty set of not necessarily commuting indeterminates. If $R$ is a reduced ring, then $R$ is Baer if and only if $R[X]$ is Baer if and only if $R[[X]]$ is Baer [6, Corollary 1.10]. If $\alpha$ is an endomorphism and $\delta$ an $\alpha$-derivation of the ring $R$ such that $R$ is $\alpha$-rigid, then it is shown in [9, Theorem 11 and Theorem 21] that $R$ is Baer if and only if the Ore extension $R[x;\alpha,\delta]$ is Baer if and only if the skew power series ring $R[[x;\alpha]]$ is Baer. If $R$ is commutative and $(S,\leq)$ is a strictly totally ordered monoid, then it is shown in [12, Theorem 7] that $R$ is Baer if and only if $[[R^{S,\leq}]]$, the ring of generalized power series with coefficients in $R$ and exponents in $S$, is Baer. In [8, Theorem 2.1], the authors showed that $R$ is a right p.q.Baer ring if and only if $R[x]$ is a right p.q.Baer ring. Also an example was given in [8, Example 2.6] which shows that there exists a commutative von Neumann regular ring $R$ (hence p.q.Baer) such that the ring $R[[x]]$ is not p.q.Baer. Let $R$ be a ring such that all left semicentral idempotents are central. It is shown in [13] that $R[[x]]$ is right p.q.Baer if and only if $R$ is right p.q.Baer and any countable family of idempotents in $R$ has a generalized join in $I(R)$.

In [10] Lee-Zhou introduced Baer and quasi-Baer modules as follows: (a) $M_R$ is called Baer if, for any subset $X$ of $M$, $r_R(X)=eR$ where $e^2=e\in R$; (b) $M_R$ is called quasi-Baer if, for any submodule $N$ of $M$, $r_R(N)=eR$ where $e^2=e\in R$. Also, the Baerness and quasi-Baerness of the (Laurent) polynomial extension and the (Laurent) power series extension of rings were extended to modules, see [10], for more details.

Recently, in [2], the notion of principally quasi-Baer modules was introduced. A module $M_R$ is called principally quasi-Baer (p.q.-Baer for short) if, for any $m\in M$, $r_R(mR)=eR$ where $e^2=e\in R$. It is clear that $R$ is a right p.q.-Baer ring if and only if $R_R$ is a p.q.-Baer module. Some of the results related to this paper can be recalled as following.

**Theorem.** [2, Theorem 7]. *Let $\alpha : R \longrightarrow R$ be an endomorphism of $R$ and assume that, for $m\in M$ and $a\in R$, $ma=0 \iff m\alpha(a)=0$. Then the following hold:*

(1) $(a)$ *If $M[x;\alpha]_{R[x;\alpha]}$ is a p.q.-Baer module, then $M_R$ is a p.q.-Baer module. The converse holds if in addition $M_R$ is $\alpha$-reduced.*
$(b)$ *If $M[[x;\alpha]]_{R[[x;\alpha]]}$ is p.q.-Baer, then $M_R$ is p.q.-Baer.*

(2) *Let* $\alpha\in \mathrm{Aut}(R)$.

$(a)$ *If $M[x,x^{-1};\alpha]_{R[x,x^{-1};\alpha]}$ is a p.q.-Baer module, then $M_R$ is a p.q.-Baer module. The converse holds if in addition $M_R$ is $\alpha$-reduced.*

$(b)$ *If* $M[x^{-1},x;\alpha]]_{R[x^{-1},x;\alpha]]}$ *is a p.q.-Baer module, then* $M_R$ *is a p.q.-Baer module.*

*where* $M_R$ *is an* $\alpha$*-reduced module if, for any* $m\in M$ *and* $a\in R$*,* $ma=0$ *implies* $mR\cap Ma=0,$ *and* $ma=0\Longleftrightarrow m\alpha(a)=0.$

Thus, a natural question of sufficient conditions for modules under which the skew power series extension and the skew Laurent series extension of a module are p.q.Baer modules arisen. In this paper, we investigate the necessary and sufficient conditions under which the skew power series modules $M[[x;\alpha]]$ and the skew Laurent series modules $M[x^{-1},x;\alpha]]$ are p.q.Baer modules. In fact, we worked for a more general module extension which is called skew generalized power series modules and introduced in Section 2.

## 2. Skew Generalized Power Series Modules

Let $(S,\leq)$ be an ordered set. Recalled that $(S,\leq)$ is artinian if every strictly decreasing sequence of elements of $S$ is finite, and that $(S,\leq)$ is narrow if every subset of pairwise order-incomparable elements of $S$ is finite. Let $S$ be a commutative monoid. Unless stated otherwise, the operation of $S$ shall be denoted additively, and the neutral element by 0. The following definition is due to [15].

Let $(S,\leq)$ be a strictly ordered monoid (that is, $(S,\leq)$ is an ordered monoid satisfying the condition that, if $s,s',t\in S$ and $s<s'$, then $s+t<s'+t$), $R$ a ring and $\lambda:S\longrightarrow \mathrm{End}(R)$ be a monoid homomorphism. Consider the set $A$ of all maps $f:S\longrightarrow R$ whose support $\mathrm{supp}(f)=\{s\in S\mid f(s)\neq 0\}$ is artinian and narrow. With pointwise addition, $A$ is an abelian additive group. For every $s\in S$ and $f,g\in A$, let $X_s(f,g)=\{(u,v)\in S\times S\mid u+v=s, f(u)\neq 0, g(v)\neq 0\}$. It follows from [17, 4.1] that $X_s(f,g)$ is finite. This fact allows to define the operation of convolution as follows:

$$(fg)(s)=\sum_{(u,v)\in X_s(f,g)} f(u)\lambda(u)\big(g(v)\big)$$

and $(fg)(s)=0$ if $X_s(f,g)=\emptyset$. With this operation, and pointwise addition, $A$ becomes a ring, which is called the ring of skew generalized power series with coefficients in $R$ and exponents in $S$, and we denote by $[[R^{S,\leq},\lambda]]$.

Let $M$ be a right $R$-module, we let $B$ be the set of all maps $\phi:S\longrightarrow M$ such that the set $\mathrm{supp}(\phi)=\{s\in S\mid \phi(s)\neq 0\}$ is artinian and narrow. With pointwise addition, $B$ is an abelian additive group. For each $f\in[[R^{S,\leq},\lambda]]$ and each $\phi\in B$, by [11, Lemma 1], the set $X_s(\phi,f)=\{(u,v)\in S\times S\mid u+v=s,\phi(u)\neq 0, f(v)\neq 0\}$ is finite. This allows to define the scalar multiplication as following:

$$(\phi f)(s)=\sum_{(u,v)\in X_s(\phi,f)} \phi(u)\lambda(u)\big(f(v)\big)$$

and $(\phi f)(s) = 0$ if $X_s(\phi, f) = \emptyset$. With this operation and pointwise addition, by analogy with the discussion of [15], one can easily to show that $B$ is a right $[[R^{S,\leq}, \lambda]]$-module, which is called the module of skew generalized power series with coefficients in $M$ and exponents in $S$, and we denote by $[[M^{S,\leq}, \lambda]]$.

**Example 2.1.**

1. If $\lambda(s) = 1$, the identity map of $R$ for every $s \in S$, then $[[R^{S,\leq}, \lambda]] = [[R^{S,\leq}]]$ is the ring of generalized power series in the sense of Ribenboim [17] and $[[M^{S,\leq}, \lambda]] = [[M^{S,\leq}]]$ is the untwisted module of generalized power series in the sense of [11] or [18]. Thus, the following modules are skew generalized power series modules: the module $M[[x_1, x_2, \ldots, x_n]]$ of formal power series extension of $M$ with $n$ indeterminates; the module $M[x^{-1}, x]]$ of Laurent series extension of $M$. Further example and work on the modules of generalized power series appear in [11,18].
2. Let $\alpha$ be a ring endomorphism of $R$. Let $S = \mathbb{N} \cup \{0\}$ be endowed with the usual order and define $\lambda : S \longrightarrow \mathrm{End}(R)$ via $\lambda(k) = \alpha^k$ for every $k \in \mathbb{N} \cup \{0\}$ (where $\alpha^0 = 1$, the identity map of $R$). Then $[[M^{S,\leq}, \lambda]]_{[[R^{S,\leq},\lambda]]} = M[[x; \alpha]]_{R[[x;\alpha]]}$, the usual skew power series extension of $M_R$.
3. Let $\alpha$ be a ring automorphism of $R$. Let $S = \mathbb{Z}$ be endowed with the usual order and define $\lambda : S \longrightarrow \mathrm{End}(R)$ via $\lambda(k) = \alpha^k$ for every $k \in \mathbb{Z}$ (where $\alpha^0 = 1$, the identity map of $R$). Then $[[M^{S,\leq}, \lambda]]_{[[R^{S,\leq},\lambda]]} = M[x^{-1}, x; \alpha]]_{R[x^{-1},x;\alpha]]}$, the usual skew Laurent series extension of $M_R$.
4. Let $\alpha$ be a ring endomorphism of $R$. Set $S = \mathbb{N} \cup \{0\}$ endowed with the trivial order. Define $\lambda : S \longrightarrow \mathrm{End}(R)$ via $\lambda(k) = \alpha^k$ for every $k \in \mathbb{N} \cup \{0\}$. Then $[[M^{S,\leq}, \lambda]]_{[[R^{S,\leq},\lambda]]} = M[x; \alpha]_{R[x;\alpha]}$, the usual skew polynomial extension of $M_R$.
5. Let $G$ be an abelian group acting on $R$ as a group of automorphisms. Define $\lambda : G \longrightarrow \mathrm{End}(R)$ via $\lambda(g) = g$ for every $g \in G$. Let $\leq$ be the trivial order of $G$. Then it is easy to see that $[[M^{G,\leq}, \lambda]]_{[[R^{G,\leq},\lambda]]} = M * G_{R*G}$, the usual skew group ring extension of $M_R$. If $G$ is an infinite cyclic group generated by $\sigma$ where $\sigma$ acts on $R$ as a ring automorphism, then $[[M^{G,\leq}, \lambda]]_{[[R^{G,\leq},\lambda]]} \cong M[x^{-1}, x; \sigma]_{R[x^{-1},x;\sigma]}$, the usual skew Laurent polynomial extension of $M_R$.

Before starting the main results, we explain some notations involved.

To any $r \in R$ and any $s \in S$ we associated the maps $c_r \in [[R^{S,\leq}, \lambda]]$ defined by

$$c_r(x) = \begin{cases} r, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases}$$

For any $m \in M$ and any $s \in S$, we define $d_m^s \in [[M^{S,\leq}, \lambda]]$ via

$$d_m^s(x) = \begin{cases} m, & \text{if } x = s, \\ 0, & \text{if } x \neq s. \end{cases}$$

## 3. Main Results

Let $\alpha : R \longrightarrow R$ be a ring endomorphism, according to [1], a module $M_R$ is called $\alpha$-compatible if, for any $m \in M$ and $a \in R$, $ma = 0 \Longleftrightarrow m\alpha(a) = 0$. Similarly, we give the following definition.

**Definition 3.1.** *Given $M_R$ and $\lambda : S \longrightarrow \mathrm{End}(R)$ as above. We say $M_R$ is $\lambda$-compatible, if for any $s \in S$, any $m \in M$ and any $a \in R$, $ma = 0 \Longleftrightarrow m\lambda(s)(a) = 0$.*

Clearly, if $\lambda(s) = 1$, the identity map of $R$ for every $s \in S$, then any module is $\lambda$-compatible. Given a ring endomorphism $\alpha : R \longrightarrow R$, define $\lambda : \mathbb{N} \cup \{0\} \longrightarrow \mathrm{End}(R) : \lambda(k) = \alpha^k$ for every $k \in \mathbb{N} \cup \{0\}$, then $M_R$ is $\lambda$-compatible if and only if $M$ is $\alpha$-compatible. In particular, $R_R$ is $\alpha$-compatible if and only if (1) $\alpha$ is a monomorphism, and (2) for any $a, b \in R$, $ab = 0$ implies that $a\alpha(b) = \alpha(a)b = 0$. In this situation, $\alpha$ is called a weakly rigid endomorphism in [14].

For every $0 \neq \phi \in [[M^{S,\leq}, \lambda]]$ (resp. $0 \neq f \in [[R^{S,\leq}, \lambda]]$), denote by $\pi(\phi)$ (resp. $\pi(f)$) the set of minimal elements of supp$(\phi)$ (resp. supp$(f)$). Then $\pi(\phi)$ (resp. $\pi(f)$) is a nonempty finite set, consisting of pairwise order incomparable elements. If $\pi(\phi)$ (resp. $\pi(f)$) consists only of one element $s$, we write $\pi(\phi) = s$ (resp. $\pi(f) = s$).

**Lemma 3.2.** *Let $(S, \leq)$ be a strictly totally ordered monoid and $M_R$ a $\lambda$-compatible $p.q.Baer$ module. If $\phi \in [[M^{S,\leq}, \lambda]]$ and $f \in [[R^{S,\leq}, \lambda]]$ are such that $\phi[[R^{S,\leq}, \lambda]]f = 0$, then $\phi(u)Rf(v) = 0$ for all $u, v \in S$.*

*Proof.* Let $0 \neq \phi \in [[M^{S,\leq}, \lambda]]$ and $0 \neq f \in [[R^{S,\leq}, \lambda]]$ be such that $\phi[[R^{S,\leq}, \lambda]]f = 0$. Assume that $\pi(\phi) = u_0$, $\pi(f) = v_0$. Then for any $(u, v) \in X_{u_0+v_0}(\phi, f)$, $u_0 \leq u$, $v_0 \leq v$. If $u_0 < u$, since $\leq$ is a strict order, $u_0 + v_0 < u + v_0 \leq u + v = u_0 + v_0$, a contradiction. Thus $u = u_0$. Similarly, $v = v_0$. Hence, for any $r \in R$,

$$0 = (\phi c_r f)(u_0+v_0) = \sum_{(u,v)\in X_{u_0+v_0}(\phi,c_r f)} \phi(u)\lambda(u)\big(rf(v)\big) = \phi(u_0)\lambda(u_0)\big(rf(v_0)\big).$$

Then $\phi(u_0)Rf(v_0) = 0$ by the $\lambda$-compatibility of $M_R$.

Now let $w \in S$ with $u_0 + v_0 \leq w$. Assume that for any $u \in$ supp$(\phi)$ and any $v \in$ supp$(f)$, if $u+v < w$, then $\phi(u)Rf(v) = 0$. We will show that $\phi(u)Rf(v) = 0$

for each $u \in \text{supp}(\phi)$ and each $v \in \text{supp}(f)$ with $u+v=w$. For convenience, we write

$$\begin{aligned} X_w(\phi, f) &= \{(u,v) \mid u+v=w,\ u \in \text{supp}(\phi),\ v \in \text{supp}(f)\} \\ &= \{(u_i, v_i) \mid i=1,2,\dots,n\} \end{aligned}$$

with $u_1 < u_2 < \cdots < u_n$ (Note that if $u_1 = u_2$, then from $u_1+v_1 = u_2+v_2$ it follows that $v_1 = v_2$, and thus $(u_1,v_1) = (u_2,v_2)$). Then for any $r \in R$,

$$(1) \quad 0 = (\phi c_r f)(w) = \sum_{(u,v)\in X_w(\phi, c_r f)} \phi(u)\lambda(u)\big(rf(v)\big) = \sum_{i=1}^{n} \phi(u_i)\lambda(u_i)\big(rf(v_i)\big).$$

For each $i = 1,2,\dots,n$, since $M_R$ is a p.q.Baer module, there exists an $e_{u_i}^2 = e_{u_i} \in R$ such that $r_R(\phi(u_i)R) = e_{u_i}R$. Let $r' \in R$, take $r = r'e_{u_1}$ in (1). Then, by $\phi(u_1)r'e_{u_1} = 0$ and the $\lambda$-compatibility of $M_R$, we have $\phi(u_1)\lambda(u_1)\big(r'e_{u_1}f(v_1)\big) = 0$. Thus

$$\sum_{i=2}^{n} \phi(u_i)\lambda(u_i)\big(r'e_{u_1}f(v_i)\big) = 0.$$

Note that $u_1 + v_i < u_i + v_i = w$ for any $i \geq 2$, so by induction hypothesis, $\phi(u_1)Rf(v_i) = 0$. Thus $f(v_i) = e_{u_1}f(v_i)$ for each $i \geq 2$. Thus

$$(2) \qquad \sum_{i=2}^{n} \phi(u_i)\lambda(u_i)\big(r'f(v_i)\big) = 0.$$

Let $p \in R$ and take $r' = pe_{u_2}$ in (2). Then since $\phi(u_2)pe_{u_2} = 0$, we have $\phi(u_2)\lambda(u_2)\big(pe_{u_2}f(v_2)\big) = 0$. Thus

$$\sum_{i=3}^{n} \phi(u_i)\lambda(u_i)\big(pe_{u_2}f(v_i)\big) = \sum_{i=3}^{n} \phi(u_i)\lambda(u_i)\big(pf(v_i)\big) = 0.$$

Continuing in this manner, we have $\phi(u_n)\lambda(u_n)\big(qf(v_n)\big) = 0$, where $q$ is an arbitrary element of $R$. Thus $\phi(u_n)qf(v_n) = 0$ since $M_R$ is a $\lambda$-compatible module. Hence

$$\phi(u_{n-1})qf(v_{n-1}) = 0,\ \dots,\ \phi(u_2)qf(v_2) = 0,\ \phi(u_1)qf(v_1) = 0.$$

Therefore, by transfinite induction, we have shown that $\phi(u)Rf(v) = 0$ for any $u, v \in S$. ■

**Lemma 3.3.** *Let $(S, \leq)$ be a strictly ordered monoid and $M_R$ a $\lambda$-compatible module. Then the following conditions are equivalent:*

(1) *For any* $\phi \in [[M^{S,\leq}, \lambda]]$ *and any* $f \in [[R^{S,\leq}, \lambda]]$, $\phi[[R^{S,\leq}, \lambda]]f = 0$ *implies* $\phi(u)Rf(v) = 0$ *for all* $u, v \in S$.

(2) *For any* $\phi \in [[M^{S,\leq}, \lambda]]$, $r_{[[R^{S,\leq},\lambda]]}(\phi[[R^{S,\leq}, \lambda]]) = [[r_R(X)^{S,\leq}, \lambda]]$, *where* $X = \{\phi(u)R \mid u \in S\}$.

*Proof.* $(1) \Longrightarrow (2)$. Assume that $f \in r_{[[R^{S,\leq},\lambda]]}(\phi[[R^{S,\leq}, \lambda]])$ with $\phi \in [[M^{S,\leq}, \lambda]]$. By (1), $\phi(u)Rf(v) = 0$ for any $u, v \in S$. Thus $f(v) \in r_R(X)$ for any $v \in S$. Hence $f \in [[r_R(X)^{S,\leq}, \lambda]]$. Conversely, suppose that $f \in [[r_R(X)^{S,\leq}, \lambda]]$. Then $f(v) \in r_R(X)$ for each $v \in S$. Thus $\phi(u)Rf(v) = 0$ for all $u, v \in S$. Then, for any $g \in [[R^{S,\leq}, \lambda]]$, by the $\lambda$-compatibility of $M_R$, $\phi(u)\lambda(u)\big(g(w)\lambda(w)(f(v))\big) = 0$ for any $u, v, w \in S$. Thus, for any $s \in S$,

$$(\phi g f)(s) = \sum_{(u,w,v)\in X_s(\phi,g,f)} \phi(u)\lambda(u)\big(g(w)\lambda(w)(f(v))\big) = 0.$$

This means that $f \in r_{[[R^{S,\leq},\lambda]]}(\phi[[R^{S,\leq}, \lambda]])$. Therefore, (2) holds.

$(2) \Longrightarrow (1)$. Suppose that $\phi \in [[M^{S,\leq}, \lambda]]$ and $f \in [[R^{S,\leq}, \lambda]]$ are such that $\phi[[R^{S,\leq}, \lambda]]f = 0$. Then, by (2), $f \in [[r_R(X)^{S,\leq}, \lambda]]$, where $X = \{\phi(u)R \mid u \in S\}$. Thus $\phi(u)Rf(v) = 0$ for every $u, v \in S$. ∎

**Lemma 3.4.** *Let* $(S, \leq)$ *be a strictly ordered monoid and* $M_R$ *a* $\lambda$*-compatible module. Then for any* $m \in M$,

$$[[r_R(mR)^{S,\leq}, \lambda]] = r_{[[R^{S,\leq},\lambda]]}(d_m^0[[R^{S,\leq}, \lambda]]).$$

*Proof.* Let $f \in r_{[[R^{S,\leq},\lambda]]}(d_m^0[[R^{S,\leq}, \lambda]])$. Then for any $r \in R$ and any $s \in S$,

$$0 = (d_m^0 c_r f)(s) = \sum_{(u,v)\in X_s(d_m^0,c_r f)} d_m^0(u)\lambda(u)\big(rf(v)\big) = mrf(s),$$

which implies that $f(s) \in r_R(mR)$ and so $f \in [[r_R(mR)^{S,\leq}, \lambda]]$. Conversely, suppose that $f \in [[r_R(mR)^{S,\leq}, \lambda]]$. Then $mRf(v) = 0$ for any $v \in S$. Now, for any $g \in [[R^{S,\leq}, \lambda]]$, by the $\lambda$-compatibility of $M_R$, $mg(u)\lambda(u)\big(f(v)\big) = 0$ for any $u, v \in S$. Thus, for any $s \in S$,

$$\begin{aligned}(d_m^0 g f)(s) &= \sum_{(w,u,v)\in X_s(d_m^0,g,f)} d_m^0(w)\lambda(w)\big(g(u)\lambda(u)(f(v))\big)\\ &= \sum_{(u,v)\in X_s(g,f)} mg(u)\lambda(u)\big(f(v)\big) = 0.\end{aligned}$$

This implies that $f \in r_{[[R^{S,\leq},\lambda]]}(d_m^0[[R^{S,\leq}, \lambda]])$. Now the result follows. ∎

In order to prove the main result, we first give the necessity of the module $[[M^{S,\leq},\lambda]]_{[[R^{S,\leq},\lambda]]}$ to be a p.q.Baer module.

**Proposition 3.5.** *Let $(S,\leq)$ be a strictly ordered monoid and $M_R$ a $\lambda$-compatible module. If $[[M^{S,\leq},\lambda]]_{[[R^{S,\leq},\lambda]]}$ is a p.q.Baer module, then $M_R$ is a p.q.Baer module.*

*Proof.* Let $m\in M$. Then, by Lemma 3.4, $[[r_R(mR)^{S,\leq},\lambda]] = r_{[[R^{S,\leq},\lambda]]}(d_m^0[[R^{S,\leq},\lambda]])$. On the other hand, since $[[M^{S,\leq},\lambda]]_{[[R^{S,\leq},\lambda]]}$ is a p.q.Baer module, there exists an $f^2=f\in[[R^{S,\leq},\lambda]]$ such that $r_{[[R^{S,\leq},\lambda]]}(d_m^0[[R^{S,\leq},\lambda]]) = f[[R^{S,\leq},\lambda]]$. We will show that $r_R(mR)=f(0)R$ with $f(0)^2=f(0)$, which will imply that $M_R$ is a p.q.Baer module. Let $b\in r_R(mR)$. Then $c_b\in[[r_R(mR)^{S,\leq},\lambda]] = f[[R^{S,\leq},\lambda]]$, and so $c_b=fc_b$. Thus $b=f(0)b\in f(0)R$. Hence $r_R(mR)\subseteq f(0)R$. Note that $d_m^0[[R^{S,\leq},\lambda]]f=0$, so for any $r\in R$, $d_m^0c_rf=0$. Thus $mRf(0)=0$. Hence $f(0)\in r_R(mR)$. Therefore, $r_R(mR)=f(0)R$. From $f(0)\in r_R(mR)$ it follows that $f(0)=f(0)^2$. Now the result follows. ■

Let $(S,\leq)$ be a strictly ordered monoid and $X$ a non-empty set. We will say $X$ is $S$-indexed, if there exists an artinian and narrow subset $I$ of $S$ such that $X$ is indexed by $I$.

**Theorem 3.6.** *Let $(S,\leq)$ be a strictly totally ordered monoid and $M_R$ a $\lambda$-compatible module. Then the following conditions are equivalent:*

(1) *$[[M^{S,\leq},\lambda]]_{[[R^{S,\leq},\lambda]]}$ is p.q.Baer.*
(2) *For any $S$-indexed set $X$ consisting of cyclic submodules of $M_R$, there exists an $e^2=e\in R$ such that $r_R(X)=eR$.*

*Proof.* $(1)\Longrightarrow(2)$. Let $X=\{m_sR\mid m_s\in M, s\in I\}$ be an $S$-indexed family of cyclic submodules of $M_R$. Define $\phi: S\to M$ via:

$$\phi(s)=\begin{cases} m_s, & s\in I,\\ 0, & s\notin I.\end{cases}$$

Then supp$(\phi)\subseteq I$, and so $\phi\in[[M^{S,\leq},\lambda]]$. Thus, by (1), there exists an $f^2=f\in[[R^{S,\leq},\lambda]]$ such that $r_{[[R^{S,\leq},\lambda]]}(\phi[[R^{S,\leq},\lambda]])=f[[R^{S,\leq},\lambda]]$. On the other hand, by Proposition 3.5, $M_R$ is a p.q.Baer module. Thus, by Lemma 3.2 and Lemma 3.3, $r_{[[R^{S,\leq},\lambda]]}(\phi[[R^{S,\leq},\lambda]])=[[r_R(X)^{S,\leq},\lambda]]$. Hence $[[r_R(X)^{S,\leq},\lambda]]=f[[R^{S,\leq},\lambda]]$. Then, by analogy with the proof of Proposition 3.5, we can conclude that $r_R(X)=f(0)R$ with $f(0)^2=f(0)$. Now (2) follows.

(2)$\Longrightarrow$(1). Suppose that $\phi\in[[M^{S,\leq},\lambda]]$. Set $X=\{\phi(s)R\mid s\in \text{supp}(\phi)\}$. Then $X$ is an $S$-indexed family of cyclic submodules of $M_R$. Thus, by (2), $r_R(X)=$

$eR$ for some $e^2 = e \in R$. Also by (2), $M$ is a p.q.Baer module. Thus, by Lemma 3.2 and Lemma 3.3,

$$r_{[[R^{S,\leq},\lambda]]}(\phi[[R^{S,\leq},\lambda]]) = [[r_R(X)^{S,\leq},\lambda]] = [[(eR)^{S,\leq},\lambda]] = c_e[[R^{S,\leq},\lambda]].$$

Clearly $c_e$ is an idempotent of $[[R^{S,\leq},\lambda]]$. Hence $[[M^{S,\leq},\lambda]]_{[[R^{S,\leq},\lambda]]}$ is a p.q.Baer module. ∎

**Corollary 3.7.** *Let $\alpha : R \longrightarrow R$ be an endomorphism of R and $M_R$ be an $\alpha$-compatible module. Then $M[[x;\alpha]]_{R[[x;\alpha]]}$ is a p.q.Baer module if and only if the right annihilator of any countable family of cyclic submodules of $M_R$ in $R$ is generated by an idempotent of $R$.*

**Corollary 3.8.** *Let $\alpha : R \longrightarrow R$ be an automorphism of R and $M_R$ be an $\alpha$-compatible module. Then $M[x^{-1},x;\alpha]]_{R[x^{-1},x;\alpha]]}$ is a p.q.Baer module if and only if the right annihilator of any countable family of cyclic submodules of $M_R$ in $R$ is generated by an idempotent of $R$.*

In the rest of this paper, we will work with the special module $R_R$, which will give more interesting results.

Recall from [7], an idempotent $e \in R$ is left (resp. right) semicentral in $R$ if $exe = xe$ (resp. $exe = ex$), for all $x \in R$. Equivalently, $e^2 = e \in R$ is left (resp. right) semicentral if $eR$ (resp. $Re$) is an ideal of $R$. If $R$ is a right p.q.Baer ring and $a \in R$, then $r_R(aR)$ is generated by a left semicentral idempotent since $r_R(aR)$ is an ideal. We use $I(R)$ for the set of all idempotents of $R$, use $C(R)$ for the set of all central idempotents of $R$ and use $\mathcal{S}_l(R)$ for the set of all left semicentral idempotents of $R$.

Let $\{e_s \mid s \in I\}$ be an $S$-indexed subset of $I(R)$. We say $\{e_s \mid s \in I\}$ has a generalized join in $I(R)$, if there exists an $e \in I(R)$ such that

(1) $e_sR(1-e) = 0$ for all $s \in I$, and

(2) if $f \in I(R)$ is such that $e_sR(1-f) = 0$ for all $s \in I$, then $eR(1-f) = 0$.

Let $(S,\leq)$ be a strictly totally monoid satisfying the condition that $0 \leq s$ for all $s \in S$. In [16], it was shown that if $\mathcal{S}_l(R) \subseteq C(R)$, then $[[R^{S,\leq}]]$ is a right p.q.Baer ring if and only if $R$ is a right p.q.Baer ring and any $S$-indexed subset of $I(R)$ has a generalized join in $I(R)$. Here we have

**Corollary 3.9.** *Let $(S,\leq)$ be a strictly totally monoid and $R_R$ a $\lambda$-compatible module. Then the following conditions are equivalent:*

(1) $[[R^{S,\leq},\lambda]]$ *is a right p.q.Baer ring.*

(2) *The right annihilator of any $S$-indexed family of principally right ideals of $R$ in $R$ is generated by an idempotent of $R$.*

*If $\mathcal{S}_l(R) \subseteq C(R)$, then the following conditions are equivalent to the conditions above:*

(3) *$R$ is a right p.q.Baer ring and for any $S$-indexed subset $\{e_s \mid s \in I\}$ of $I(R)$, $\cap_{s\in I} r_R(e_sR) = eR$ for some $e \in I(R)$.*

(4) *$R$ is a right p.q.Baer ring and for any $S$-indexed subset $\{e_s \mid s \in I\}$ of $C(R)$, $\cap_{s\in I} r_R(e_sR) = eR$ for some $e \in I(R)$.*

(5) *$R$ is a right p.q.Baer ring and any $S$-indexed subset of $C(R)$ has a generalized join in $I(R)$.*

(6) *$R$ is a right p.q.Baer ring and any $S$-indexed subset of $I(R)$ has a generalized join in $I(R)$.*

*Proof.* (1)$\Longleftrightarrow$(2) follows from Theorem 3.6.

(2)$\Longrightarrow$(3). Note that for any $a \in R$, $\{aR\}$ is $S$-indexed. Thus (2)$\Longrightarrow$(3) is obviously.

(3)$\Longrightarrow$(4). It is directly verified.

(4)$\Longrightarrow$(5). Let $\{e_s \mid s \in I\}$ be an $S$-indexed subset of $C(R)$. By (4), there exists an $e \in I(R)$ such that $\cap_{s\in I} r_R(e_sR) = eR$. We will show that $1-e$ is a generalized join of the set $\{e_s \mid s \in I\}$. It is clearly that $e_sR(1-(1-e)) = e_sRe = 0$ for any $s \in I$. Assume that $f^2 = f \in R$ is such that $e_sR(1-f) = 0$ for any $s \in I$. Then $1-f \in \cap_{s\in I} r_R(e_sR) = eR$. So $(1-f) = e(1-f)$. Since $e \in \mathcal{S}_l(R)$, $(1-e)R(1-f) = 0$. Hence $1-e$ is a generalized join of $\{e_s \mid s \in I\}$ in $I(R)$.

(5)$\Longrightarrow$(6). Let $\{e_s \mid s \in I\}$ be an $S$-indexed subset of $I(R)$. Since $R$ is a right p.q.Baer ring, there exist $f_s \in \mathcal{S}_l(R) \subseteq C(R)$ such that $r_R(e_sR) = f_sR$ for all $s \in I$. By (5), $\{1-f_s \mid s \in I\}$ has a generalized join in $I(R)$, we say $e$. Then $(1-f_s)R(1-e) = 0$ for any $s \in I$. Thus, for any $r \in R$ and any $s \in I$, $r(1-e) = f_sr(1-e)$. Hence $e_sr(1-e) = e_sf_sr(1-e) = 0$ for any $s \in I$. This means that $e_sR(1-e) = 0$ for any $s \in I$. Suppose that $f \in I(R)$ is such that $e_sR(1-f) = 0$ for each $s \in I$. Then $1-f \in r_R(e_sR) = f_sR$, and so $(1-f) = f_s(1-f)$. Thus $(1-f_s)(1-f) = 0$. Hence $(1-f_s)R(1-f) = 0$. Since $e$ is a generalized join of $\{1-f_s \mid s \in I\}$, it follows that $eR(1-f) = 0$. Hence $e$ is a generalized join of $\{e_s \mid s \in I\}$.

(6)$\Longrightarrow$(2). Assume that $X = \{a_sR \mid s \in I\}$ is an $S$-indexed family of principal right ideals of $R$. Since $R$ is a right p.q.Baer ring, there exists an $e_s \in \mathcal{S}_l(R)$ such that $r_R(a_sR) = e_sR$ for each $s \in I$. By (6), $\{1-e_s \mid s \in I\}$ has a generalized join in $I(R)$, say $e$. Then $(1-e_s)R(1-e) = 0$ for any $s \in I$. Thus $a_sr(1-e) = a_se_sr(1-e) = 0$ for any $r \in R$ and any $s \in S$. Hence $(1-e) \in r_R(X)$. Let $p \in r_R(X)$. Then, for any $s \in I$, $a_sRp = 0$. Thus $p \in r_R(a_sR) = e_sR$. Hence $p = e_sp$ for any $s \in I$. On the other hand, since $R$ is a right p.q.Baer ring, there exists an $f \in I(R)$ such that $r_R(pR) = fR$. Since $e_s$ is left semicentral, by the hypothesis, $e_s$ is central. Thus $pr = e_spr = pre_s$ for any $r \in R$, which implies that $1-e_s \in r_R(pR) = fR$. Thus $(1-e_s) = f(1-e_s)$, and so $(1-e_s)R(1-f) = 0$. Since $e$ is a generalized join of $\{1-e_s \mid s \in I\}$, it follows that $eR(1-f) = 0$.

Hence $p = p - pf = p(1-f) = (1-f)p = (1-e)(1-f)p \in (1-e)R$. So $r_R(X) \subseteq (1-e)R$. Hence $r_R(X) = (1-e)R$. ■

**Corollary 3.10.** *Let $R$ be a ring with $\mathcal{S}_l(R) \subseteq C(R)$ and $\alpha$ a weakly rigid endo- morphism of R. Then $R[[x;\alpha]]$ is a right p.q.Baer ring if and only if $R$ is a right p.q.Baer ring and any countable subset of $C(R)$ has a generalized join in $I(R)$.*

**Corollary 3.11.** *Let $R$ be a ring with $\mathcal{S}_l(R) \subseteq C(R)$ and $\alpha$ a weakly rigid automorphism of R. Then $R[x^{-1}, x;\alpha]]$ is a right p.q.Baer ring if and only if $R$ is a right p.q.Baer ring and any countable subset of $C(R)$ has a generalized join in $I(R)$.*

Let $\alpha$ and $\beta$ be ring endomorphisms (resp. ring automorphisms) of $R$ such that $\alpha\beta = \beta\alpha$. Let $S = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ (resp. $\mathbb{Z} \times \mathbb{Z}$) be endowed the lexicographic order, or the reverse lexicographic order, or the product order of the usual order of $\mathbb{N} \cup \{0\}$ (resp. $\mathbb{Z}$), and define $\lambda : S \longrightarrow \mathrm{End}(R)$ via $\lambda(m,n) = \alpha^m\beta^n$ for any $m,n \in \mathbb{N} \cup \{0\}$ (resp. $m,n \in \mathbb{Z}$). Then $[[R^{S,\leq}, \lambda]] = R[[x,y;\alpha,\beta]]$ (resp. $R[[x,y,x^{-1},y^{-1};\alpha,\beta]]$), in which $(ax^my^n)(bx^py^q) = a\alpha^m\beta^n(b)x^{m+p}y^{n+q}$ for any $m,n,p,q \in \mathbb{N} \cup \{0\}$ (resp. $m,n,p,q \in \mathbb{Z}$).

**Corollary 3.12.** *Let $R$ be a ring with $\mathcal{S}_l(R) \subseteq C(R)$, $\alpha$ and $\beta$ be weakly rigid ring endomorphisms (resp. ring automorphisms) of R such that $\alpha\beta = \beta\alpha$. Then $R[[x,y;\alpha,\beta]]$ (resp. $R[[x,y,x^{-1},y^{-1};\alpha,\beta]]$) is a right p.q.Baer ring if and only if $R$ is a right p.q.Baer ring and any countable subset of $C(R)$ has a generalized join in $I(R)$.*

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REFERENCES

1. S. Annin, Couniform dimension over skew polynomial rings, *Comm. Algebra*, **33** (2005), 1195-1204.
2. M. Başer and A. Harmanci, Reduced and p.q.-Baer modules, *Taiwanese J. Math.*, **11** (2007), 267-275.
3. G. F. Birkenmeier, Baer rings and quasi-continuous rings have a MDSN, *Pacific J. Math.*, **97** (1981), 283-292.
4. G. F. Birkenmeier, Decompositions of Baer-like rings, *Acta Math. Hung.*, **59** (1992), 319-326.
5. G. F. Birkenmeier, J. Y. Kim and J. K. Park, On quasi-Baer rings, *Contemp. Math.*, **259** (2000), 67-92.

6. G. F. Birkenmeier, J. Y. Kim and J. K. Park, Polynomial extensions of Baer and quasi-Baer rings, *J. Pure Appl. Algebra*, **159** (2001), 25-42.
7. G. F. Birkenmeier, J. Y. Kim and J. K. Park, Principally quasi-Baer rings, *Comm. Algebra*, **29** (2001), 639-660.
8. G. F. Birkenmeier, J. Y. Kim and J. K. Park, On polynomial extensions of principally quasi-Baer rings, *Kyungpook Math. J.*, **40** (2000), 247-253.
9. C. Y. Hong, N. K. Kim and T. K. Kwak, Ore extensions of Baer and p.p.-rings, *J. Pure Appl. Algebra*, **151** (2000), 215-226.
10. T. K. Lee and Y. Q. Zhou, Reduced modules, *Rings, Modules, Algebras and Abelian Groups*, 365-377, *Lecture Notes in Pure and Appl. Math.*, 236, Dekker, New York, 2004.
11. Z. K. Liu, A note on Hopfian modules, *Comm. Algebra*, **28** (2000), 3031-3040.
12. Z. K. Liu, Baer rings of generalized power series, *Glasgow Math. J.*, **44** (2002), 463-469.
13. Z. K. Liu, A note on principally quasi-Baer rings, *Comm. Algebra*, **30** (2002), 3885-3890.
14. Z. K. Liu and W. L. Fan, Principal quasi-Baerness of skew power series rings, *J. Math. Research* & *Exposition*, **25** (2005), 197-203.
15. Z. K. Liu, Triangular matrix representations of rings of generalized power series, *Acta Math. Sinica (English Series)*, **22** (2006), 989-998.
16. Z. K. Liu, Principal quasi-Baerness of generalized power series, *Northeastern Math. J.*, **23** (2007), 283-292.
17. P. Ribenboim, Semisimple rings and von Neumann regular rings of generalized power series, *J. Algebra*, **198** (1997), 327-338.
18. K. Varadarajan, Generalized power series modules, *Comm. Algebra*, **29** (2001), 1281-1294.



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