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WEIGHTED OSTROWSKI INTEGRAL INEQUALITY FOR MAPPINGS OF BOUNDED VARIATION

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Abstract. In this paper, we establish some weighted Ostrowski integral inequality for mappings of bounded variation.

1. INTRODUCTION

Throughout in this paper, we define the following notations:

(a_1) $V_c^d(f)$ is the total variation of f on the interval $[c, d]$ where $f : [c, d] \to \mathbb{R}$ *is bounded variation on* $[a, b]$.

(a_2) $I_k : a = x_0 < x_1 < \cdots < x_k = b$ is a partition of the interval $[a, b]$.

(a_3) $l_i := x_{i+1} - x_i \ (i = 0, \cdots, k-1)$ where $x_i \ (i = 0, \cdots, k-1)$ is defined as in (a_2).

(a_4) $v(l) := \max_{i=0,\cdots,k-1} l_i$ where $l_i \ (i = 0, \cdots, k-1)$ is defined as in (a_3)

(a_5) $L_i := \int_{x_i}^{x_{i+1}} g(t)dt \ (i = 0, \cdots, k-1)$ where $x_i \ (i = 0, \cdots, k-1)$ is defined as in (a_2) and $g : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$.

(a_6) $v(L) := \max_{i=0,\cdots,k-1} L_i$ where $L_i \ (i = 0, \cdots, k-1)$ is defined as in (a_5).

(a_7) $\triangle_n : a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of partition of $[a, b]$.

(a_8) $l_i^{(n)} := x_{i+1}^{(n)} - x_i^{(n)} \ (i = 0, \cdots, n-1)$ where $x_i^{(n)} \ (i = 0, \cdots, n-1)$ is defined as in (a_7).

(a_9) $v(l^{(n)}) := \max_{i=0,\cdots,n-1} l_i^{(n)}$ where $l_i^{(n)} \ (i = 0, \cdots, n-1)$ is defined as in (a_8).

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(a_{10}) $L_i^{(n)} := \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} g(t)dt$ $(i = 0, \cdots, n-1)$ where $x_i^{(n)}$ $(i = 0, \cdots, n-1)$ is defined as in (a_7) and $g : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$.

(a_{11}) $v(L^{(n)}) := \max_{i=0,\cdots,n-1} L_i^{(n)}$ where $L_i^{(n)}$ $(i = 0, \cdots, n-1)$ is defined as in (a_{10}).

(a_{12}) $I_n(f, \triangle_n, w_n) := \sum_{j=0}^{n} w_j^{(n)} f(x_j^{(n)})$ where $f : [a,b] \to \mathbb{R}$, $w_j^{(n)}$ $(j = 0, \cdots, n)$ are the quadrature weights with $\sum_{j=0}^{n} w_j^{(n)} = b - a$ and $x_i^{(n)} - a \le \sum_{j=0}^{i} w_j^{(n)} \le x_{i+1}^{(n)} - a$ $(i = 0, \cdots, n-1)$.

(a_{13}) $I_n(f, h, \triangle_n, \rho_n) := \sum_{j=0}^{n} \rho_j^{(n)} f(x_j^{(n)})$ where $f : [a,b] \to \mathbb{R}$, $h : [a,b] \to \mathbb{R}$, $h(w_j^{(n)})$ $(j = 0, \cdots, n)$ are the quadrature weights with $\sum_{j=0}^{n} \rho_j^{(n)} = h(b) - h(a)$ and $h(x_i^{(n)}) - h(a) \le \sum_{j=0}^{i} \rho_j^{(n)} \le h(x_{i+1}^{(n)}) - h(a)$ $(i = 0, \cdots, n-1)$.

In [1], Ostrowski proved the following integral inequality.

Theorem A. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with derivative $f' : (a,b) \to \mathbb{R}$ bounded on (a,b), that is, $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$. Then the inequality*

$$\left| \int_a^b f(t)dt - (b-a) f(x) \right| \le \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty \tag{1}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

For some recent results which generalize, improve and extend the inequality (1), see the papers [2-6].

2. Dragomir and Tseng-hwang-dragomir's Inequalities

In [4], Dragomir pointed out the following natural generalisation of (1) for mappings of bounded variation.

Theorem B. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt - (b-a)f(x)\right| \le \left[\frac{(b-a)}{2} + \left(x - \frac{a+b}{2}\right)\right] V_a^b(f) \tag{2}$$

holds for all $x \in [a,b]$. *The constant* $\frac{1}{2}$ *is the best possible.*

In [2], Dragomir proved the following two theorems concerning Ostrowski type inequalities.

Theorem C. *Let* $x_i\,(i = 0, \cdots, k)$, $v(l)$, $V_a^b(f)$ be as above and let $\alpha_i\,(i = 0, \cdots, k+1)$ *be "*$k+2$*" points so that* $\alpha_0 = a, \alpha_i \in [x_{i-1}, x_i] (i = 1, \cdots, k)$ *and* $\alpha_{k+1} = b$. *If* $f : [a,b] \to \mathbb{R}$ *is a mapping of bounded variation on* $[a,b]$*, then we have the inequality*

$$\begin{aligned} &\left|\int_a^b f(t)dt - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i)\right| \\ &\le \left[\frac{1}{2}v(l) + \max_{i=0,\cdots,k-1}\left|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\right|\right] V_a^b(f) \\ &\le v(l) V_a^b(f). \end{aligned} \tag{3}$$

Theorem D. *Let* $x_i^{(n)}$, $w_j^{(n)}(i = 0, \cdots, n)$, $I_n(f, \triangle_n, w_n)$, $v(l^{(n)})$ *be as above and let* $f : [a,b] \to \mathbb{R}$ *be a mapping of bounded variation on* $[a,b]$ *and* $r_i = a + \sum_{j=0}^{i} w_j^{(n)}\,(i = 0, \cdots, n-1)$. *Then we have the estimate*

$$\begin{aligned} &\left|I_n(f, \triangle_n, w_n) - \int_a^b f(t)dt\right| \\ &\le \left[\frac{1}{2}v(l^{(n)}) + \max_{i=0,\cdots,n-1}\left|r_i - \frac{x_i^{(n)} + x_{i+1}^{(n)}}{2}\right|\right] V_a^b(f) \\ &\le v(l^{(n)}) V_a^b(f). \end{aligned} \tag{4}$$

In particular

$$\lim_{v(l^{(n)}) \to 0} I_n(f, \triangle_n, w_n) = \int_a^b f(x)dx$$

uniformly related as the w_n.

In [6], Tseng, Hwang and Dragomir established the following weighted Ostrowski type inequality.

Theorem E. *Let* $0 \le \alpha \le 1$, $g : [a,b] \to (0, \infty)$ *be continuous on* $[a,b]$ *and let* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h'(t) = g(t)$ *on* $[a,b]$. *Let*

$c = h^{-1}((1-\frac{\alpha}{2})h(a)+\frac{\alpha}{2}h(b))$ *and* $d = h^{-1}(\frac{\alpha}{2}h(a)+(1-\frac{\alpha}{2})h(b))$. *Suppose that* f *and* $V_a^b(f)$ *are defined as in Theorem B. Then, for all* $x \in [c,d]$, *we have*

$$\left|\int_a^b f(t)g(t)dt - \left[(1-\alpha)f(x)+\alpha\cdot\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)dt\right| \le M\cdot V_a^b(f) \tag{5}$$

where

$$M := \begin{cases} \dfrac{1-\alpha}{2}\displaystyle\int_a^b g(t)dt + \left|h(x)-\dfrac{h(a)+h(b)}{2}\right|, & if\ 0\le\alpha\le\dfrac{1}{2} \\ \max\left\{\dfrac{1-\alpha}{2}\displaystyle\int_a^b g(t)dt + \left|h(x)-\dfrac{h(a)+h(b)}{2}\right|, \dfrac{\alpha}{2}\displaystyle\int_a^b g(t)dt\right\}, & if\ \frac{1}{2}<\alpha<\frac{2}{3} \\ \frac{\alpha}{2}\displaystyle\int_a^b g(t)dt, & if\ \dfrac{2}{3}\le\alpha\le 1. \end{cases}$$

3. Main Results

Theorem 1. *Let* f, $x_i\,(i=0,\cdots,k+1)$ and $\alpha_i\,(i=0,\cdots,k+1)$ *be defined as in Theorem C and let* $h:[a,b]\to\mathbb{R}$, $g:[a,b]\to(0,\infty)$ *be continuous on* $[a,b]$ *with* $h'(t)=g(t)$ *on* $[a,b]$. *Then we have the inequality*

$$\begin{aligned} &\left|\int_a^b f(t)g(t)dt - \sum_{i=0}^{k} f(x_i)\int_{\alpha_i}^{\alpha_{i+1}} g(t)dt\right| \\ &\le \left[\max_{i=0,\cdots,k-1}\frac{L_i}{2} + \left|h(\alpha_{i+1}) - \frac{h(x_i)+h(x_{i+1})}{2}\right|\right] V_a^b(f) \\ &\le \left[\frac{1}{2}v(L) + \max_{i=0,\cdots,k-1}\left|h(\alpha_{i+1}) - \frac{h(x_i)+h(x_{i+1})}{2}\right|\right] V_a^b(f) \\ &\le v(L)\,V_a^b(f) \end{aligned} \tag{6}$$

where $L_i\ (i=0,\cdots,k-1)$, $v(L)$ *and* $V_a^b(f)$ *are as above.*

Proof. Define the kernel $K_h:[a,b]\to\mathbb{R}$ by

$$K_h(t) := \begin{cases} h(t)-h(\alpha_1), t\in[a,x_1) \\ h(t)-h(\alpha_2), t\in[x_1,x_2) \\ \quad\vdots \\ h(t)-h(\alpha_{k-1}), t\in[x_{k-2},x_{k-1}) \\ h(t)-h(\alpha_k), t\in[x_{k-1},b]. \end{cases}$$

Using the integrating by parts formula, we have the following identity

$$\int_a^b K_h(t)\ df(t) = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} [h(t) - h(\alpha_{i+1})]\ df(t)$$

$$= \sum_{i=0}^{k-1} \left[[h(t) - h(\alpha_{i+1})]\ f(t) \Big|_{t=x_i}^{t=x_{i+1}} - \int_{x_i}^{x_{i+1}} f(t)g(t)\ dt \right]$$

$$= \sum_{i=0}^{k-1} [(h(x_{i+1}) - h(\alpha_{i+1}))f(x_{i+1}) + (h(\alpha_{i+1}) - h(x_i))f(x_i)] - \int_a^b f(t)g(t)dt$$

$$= (h(b) - h(\alpha_k))f(b) + \sum_{i=0}^{k-2} (h(x_{i+1}) - h(\alpha_{i+1}))f(x_{i+1})$$

$$+ (h(\alpha_1) - h(a))f(a) + \sum_{i=1}^{k-1} (h(\alpha_{i+1}) - h(x_i))f(x_i) - \int_a^b f(t)g(t)dt$$

$$= (h(b) - h(\alpha_k))f(b) + \sum_{i=1}^{k-1} (h(x_i) - h(\alpha_i))f(x_i)$$

$$+ (h(\alpha_1) - h(a))f(a) + \sum_{i=1}^{k-1} (h(\alpha_{i+1}) - h(x_i))f(x_i) - \int_a^b f(t)g(t)dt$$

$$= (h(b) - h(\alpha_k))f(b) + (h(\alpha_1) - h(a))f(a)$$

$$+ \sum_{i=1}^{k-1} (h(\alpha_{i+1}) - h(\alpha_i))f(x_i) - \int_a^b f(t)g(t)dt$$

$$= \sum_{i=0}^{k} (h(\alpha_{i+1}) - h(\alpha_i))f(x_i) - \int_a^b f(t)g(t)dt$$

$$= \sum_{i=0}^{k} f(x_i) \int_{\alpha_i}^{\alpha_{i+1}} g(t)dt - \int_a^b f(t)g(t)dt$$

and then we have the inequality

$$\left| \int_a^b f(t)g(t)dt - \sum_{i=0}^{k} f(x_i) \int_{\alpha_i}^{\alpha_{i+1}} g(t)dt \right|$$

$$= \left| \int_a^b K_h(t)\ df(t) \right| = \left| \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} (h(t) - h(\alpha_{i+1}))\ df(t) \right|$$

$$\le \sum_{i=0}^{k-1} \left| \int_{x_i}^{x_{i+1}} (h(t) - h(\alpha_{i+1}))\ df(t) \right| := T.$$

It is well known [7, p.159] that if $\mu, \nu : [c,d] \to \mathbb{R}$ are such that μ is continuous on $[c,d]$ and ν is of bounded variation on $[c,d]$, then $\int_c^d \mu(t)\ d\nu(t)$ exists and [7, p. 177]

(7) $$\left|\int_c^d \mu(t)d\nu(t)\right| \le \sup_{t\in[c,d]} |\mu(t)|\, V_c^d(\nu).$$

Using (7), we have

$$\begin{aligned}
&\left|\int_{x_i}^{x_{i+1}} (h(t) - h(\alpha_{i+1}))\ df(t)\right| \\
&\le \sup_{t\in[x_i,x_{i+1}]} |(h(t) - h(\alpha_{i+1})|\, V_{x_i}^{x_{i+1}}(f) \\
&= \max\{h(\alpha_{i+1}) - h(x_i), h(x_{i+1}) - h(\alpha_{i+1})\}\, V_{x_i}^{x_{i+1}}(f) \\
&= \left[\frac{h(x_{i+1}) - h(x_i)}{2} + \left|h(\alpha_{i+1}) - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right] V_{x_i}^{x_{i+1}}(f) \\
&= \left[\frac{1}{2}\int_{x_i}^{x_{i+1}} g(t)dt + \left|h(\alpha_{i+1}) - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right] V_{x_i}^{x_{i+1}}(f) \\
&= \left[\frac{L_i}{2} + \left|h(\alpha_{i+1}) - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right] V_{x_i}^{x_{i+1}}(f).
\end{aligned}$$

Then

(8) $$\begin{aligned}
T &\le \sum_{i=0}^{k-1} \left[\frac{L_i}{2} + \left|h(\alpha_{i+1}) - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right] V_{x_i}^{x_{i+1}}(f) \\
&\le \max_{i=0,\cdots,k-1} \left[\frac{L_i}{2} + \left|h(\alpha_{i+1}) - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right] \sum_{i=0}^{k-1} V_{x_i}^{x_{i+1}}(f) \\
&\le \max_{i=0,\cdots,k-1} \left[\frac{L_i}{2} + \left|h(\alpha_{i+1}) - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right] V_a^b(f) \\
&\le \left[\frac{1}{2}v(L) + \max_{i=0,\cdots,k-1} \left|h(\alpha_{i+1}) - \frac{h(x_i) + h(x_{i+1})}{2}\right|\right] V_a^b(f) := S.
\end{aligned}$$

Now, as

$$\begin{aligned}
&\left|h(\alpha_{i+1}) - \frac{h(x_i)+h(x_{i+1})}{2}\right| \\
&\le \frac{h(x_{i+1})-h(x_i)}{2} \\
&= \frac{1}{2}\int_{x_i}^{x_{i+1}} g(t)dt = \frac{L_i}{2} \ \ (i=0,\cdots,k-1)\,, \\
&\max_{i=0,\cdots,k-1}\left|h(\alpha_{i+1}) - \frac{h(x_i)+h(x_{i+1})}{2}\right| \le \frac{1}{2}v\left(L\right)
\end{aligned}$$

and we have

$$S \le v\left(L\right) V_a^b\left(f\right). \tag{9}$$

Then, by (8) and (9), we obtain (6).
This completes the proof.

Remark 1. Let $g(t)=1$ and $h(t)=t$ $(t \in [a,b])$ in Theorem 1. Then the second and third inequalities of (6) reduce to the inequality (3).

Remark 2. Let c, d, M, be defined as in Theorem E and let $k=2$, $x_0=a$, $\alpha_1=c$, $x_1=x$, $\alpha_2=d$, $x_2=b$ in Theorem 1. Since

$$\begin{aligned}
&\frac{L_0}{2} + \left|h(c) - \frac{h(a)+h(x)}{2}\right| \\
&= \frac{h(x)-h(a)}{2} + \left|h(c) - \frac{h(a)+h(x)}{2}\right| \\
&= \max\left\{h(x)-h(c),\ \ (h(c)-h(a))\right\} \\
&= \max\left\{h(x)-h(c),\ \frac{\alpha}{2}\left(h(b)-h(a)\right)\right\} \\
&= \max\left\{h(x)-h(c),\ \frac{\alpha}{2}\int_a^b g(t)dt\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{L_1}{2} + \left|h(d) - \frac{h(x)+h(b)}{2}\right| \\
&= \frac{h(b)-h(x)}{2} + \left|h(d) - \frac{h(x)+h(b)}{2}\right| \\
&= \max\left\{h(d)-h(x),\ \ (h(b)-h(d))\right\} \\
&= \max\left\{h(d)-h(x),\ \frac{\alpha}{2}\left(h(b)-h(a)\right)\right\} \\
&= \max\left\{h(d)-h(x),\ \frac{\alpha}{2}\int_a^b g(t)dt\right\},
\end{aligned}$$

we have

$$\begin{aligned}
&\max\left\{\frac{L_0}{2}+\left|h(c)-\frac{h(a)+h(x)}{2}\right|,\ \frac{L_1}{2}+\left|h(d)-\frac{h(x)+h(b)}{2}\right|\right\}\\
&=\max\left\{h(x)-h(c),\ h(d)-h(x),\frac{\alpha}{2}\int_a^b g(t)dt\right\}\\
&=\max\left\{\max\left\{h(x)-h(c),\ h(d)-h(x)\right\},\frac{\alpha}{2}\int_a^b g(t)dt\right\}\\
&=\max\left\{\frac{h(d)-h(c)}{2}+\left|h(x)-\frac{h(c)+h(d)}{2}\right|,\ \frac{\alpha}{2}\int_a^b g(t)dt\right\}\\
&=\max\left\{\frac{1-\alpha}{2}\int_a^b g(t)dt+\left|h(x)-\frac{h(a)+h(b)}{2}\right|,\ \frac{\alpha}{2}\int_a^b g(t)dt\right\}=M
\end{aligned}$$

and then the first inequality of (6) reduces to the inequality (5).

Theorem 2. *Let* $x_i^{(n)}$ $(i=0,\cdots,n)$, f, h, g *be defined as in Theorem 1 and let* $s_i=h(a)+\sum_{j=0}^{i}\rho_j^{(n)}$ $(i=0,\cdots,n-1)$. *Then we have the estimate*

$$\begin{aligned}
&\left|I_n\left(f,h,\triangle_n,\rho_n\right)-\int_a^b f(t)g(t)dt\right|\\
&\le\left[\frac{1}{2}v(L^{(n)})+\max_{i=0,\cdots,n-1}\left|s_i-\frac{h(x_i^{(n)})+h(x_{i+1}^{(n)})}{2}\right|\right]V_a^b\left(f\right)\\
&\le v(L^{(n)})V_a^b\left(f\right)
\end{aligned}\tag{10}$$

where $v(L^{(n)})$ *and* $V_a^b\left(f\right)$ *are as above.*

Proof. Define the sequence

$$h(\alpha_{i+1}^{(n)}):=h(a)+\sum_{j=0}^{i}\rho_j^{(n)},\ i=0,\cdots,n,$$

then we have

$$h(\alpha_{n+1}^{(n)})=h(a)+\sum_{j=0}^{n}\rho_j^{(n)}=h(b)$$

and we observe also that $\alpha_{n+1}^{(n)}=b$ and $\alpha_{i+1}^{(n)}\in\left[x_i^{(n)},x_{i+1}^{(n)}\right]$ $(i=0,\cdots,n-1)$.

Define $h(\alpha_0^{(n)}):=h(a)$ and compute for $(i=1,\cdots,n-1)$

$$h(\alpha_1^{(n)}) - h(\alpha_0^{(n)}) = \rho_0^{(n)},$$

$$h(\alpha_{i+1}^{(n)}) - h(\alpha_i^{(n)}) = \rho_i^{(n)},$$

and

$$h(\alpha_{n+1}^{(n)}) - h(\alpha_n^{(n)}) = \rho_n^{(n)},$$

then

$$\sum_{i=0}^{n}(h(\alpha_{i+1}^{(n)}) - h(\alpha_i^{(n)}))f(x_i^{(n)}) = \sum_{i=0}^{n}\rho_i^{(n)}f(x_i^{(n)}) = I_n\left(f, h, \triangle_n, \rho_n\right).$$

Applying the second and third inequalities of (6), we get the inequality (10). This completes the proof.

Remark 3. Let $g(t) = 1$ and $h(t) = t$ $(t \in [a, b])$ in Theorem 2. Then the inequality (10) reduces to the inequality (4).

4. Some Particular Integral Inequalities

Proposition 1. *Let f, h, g be defined as in Theorem 1. Then the following inequality*

$$\begin{aligned} &\left|\int_a^b f(t)g(t)dt - \left[f(a)\int_a^\alpha g(t)dt + f\left(b\right)\int_\alpha^b g(t)dt\right]\right| \\ (11)\qquad &\leq \left[\frac{1}{2}\int_a^b g(t)dt + \left|h(\alpha) - \frac{h(a)+h(b)}{2}\right|\right] V_a^b\left(f\right) \end{aligned}$$

holds for all $\alpha \in [a, b]$.

Proof. Let $k = 1$, $x_0 = a$, $x_1 = b$, $\alpha_0 = a$, $\alpha_1 = \alpha \in [a, b]$ and $\alpha_2 = b$ in Theorem1. Then we get the inequality (11).

Remark 4. In Proposition 1, we get a weighted generalization of Proposition 1 in [2].

Remark 5. If we choose $\alpha = h^{-1}(\frac{h(a)+h(b)}{2})$ in Proposition 1, then we have $\int_a^\alpha g(t)dt = h(\alpha) - h(a) = \dfrac{h(b)-h(a)}{2} = \dfrac{1}{2}\int_a^b g(t)dt$, $\int_\alpha^b g(t)dt = h(b) - h(\alpha) = \dfrac{h(b)-h(a)}{2} = \dfrac{1}{2}\int_a^b g(t)dt$ and the inequality (11) reduces to the following inequality:

$$\left|\int_a^b f(t)g(t)dt - \left[\frac{f(a)+f\left(b\right)}{2}\int_a^b g(t)dt\right]\right|$$

$$\leq \frac{1}{2}\int_a^b g(t)dt \cdot V_a^b(f)$$

which is the "weighted trapezoid" inequality for mappings of bounded variation.

Remark 6. Let $g(t) = 1$ and $h(t) = t$ $(t \in [a,b])$ in Remark 5. Then we get a result of Remark 1 in [2].

Proposition 2. *Let f, h, g be as above and $a \leq x_1 \leq b$, $a \leq \alpha_1 \leq x_1 \leq \alpha_2 \leq b$. Then we have*

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)dt - \left[f(a)\int_a^{\alpha_1} g(t)dt + f(x_1)\int_{\alpha_1}^{\alpha_2} g(t)dt \right.\right. \\
&\left.\left. + f(b)\int_{\alpha_2}^b g(t)dt\right]\right| \\
&\leq \frac{1}{2}\left[\frac{1}{2}\int_a^b g(t)dt + \left|h(x_1) - \frac{h(a)+h(b)}{2}\right|\right. \\
&\quad + \left|h(\alpha_1) - \frac{h(a)+h(x_1)}{2}\right| + \left|h(\alpha_2) - \frac{h(x_1)+h(b)}{2}\right| \qquad (12) \\
&\quad \left. + \left|\left|h(\alpha_1) - \frac{h(a)+h(x_1)}{2}\right| - \left|h(\alpha_2) - \frac{h(x_1)+h(b)}{2}\right|\right|\right] V_a^b(f) \\
&\leq \left[\frac{1}{2}\int_a^b g(t)dt + \left|h(x_1) - \frac{h(a)+h(b)}{2}\right|\right] V_a^b(f) \\
&\leq \int_a^b g(t)dt V_a^b(f).
\end{aligned}$$

Proof. In Theorem 1, we choose $k = 2$ and the partition $a = x_0 \leq x_1 \leq x_2 = b$ and the number $\alpha_0 = a$, $\alpha_1 \in [a, x_1]$, $\alpha_2 \in [x_1, b]$ and $\alpha_3 = b$. Using the second and third inequalities of (6), we get

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)dt - \left[f(a)\int_a^{\alpha_1} g(t)dt \right.\right. \\
&\left.\left. + f(x_1)\int_{\alpha_1}^{\alpha 2} g(t)dt + f(b)\int_{\alpha_2}^b g(t)dt\right]\right| \\
&\leq \frac{1}{2}\left[\max\left\{\int_a^{x_1} g(t)dt, \int_{x_1}^b g(t)dt\right\}\right. \qquad (13) \\
&\quad \left. + \max\left\{\left|h(\alpha_1) - \frac{h(a)+h(x_1)}{2}\right|, \left|h(\alpha_2) - \frac{h(x_1)+h(b)}{2}\right|\right\}\right] V_a^b(f) \\
&= \frac{1}{2}\left[\max\left\{h(x_1) - h(a), h(b) - h(x_1)\right\}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\max\left\{\left|h(\alpha_1)-\frac{h(a)+h(x_1)}{2}\right|,\left|h(\alpha_2)-\frac{h(x_1)+h(b)}{2}\right|\right\}\Bigg]V_a^b(f)\\
=&\left[\frac{1}{4}\left(h(b)-h(a)\right)+\frac{1}{2}\left|h(x_1)-\frac{h(a)+h(b)}{2}\right|\right.\\
&+\frac{1}{2}\left|h(\alpha_1)-\frac{h(a)+h(x_1)}{2}\right|+\frac{1}{2}\left|h(\alpha_2)-\frac{h(x_1)+h(b)}{2}\right|\\
&\left.+\frac{1}{2}\left|\left|h(\alpha_1)-\frac{h(a)+h(x_1)}{2}\right|-\left|h(\alpha_2)-\frac{h(x_1)+h(b)}{2}\right|\right|\right]V_a^b(f)\\
=&\frac{1}{2}\left[\frac{1}{2}\int_a^b g(t)dt+\left|h(x_1)-\frac{h(a)+h(b)}{2}\right|\right.\\
&+\left|h(\alpha_1)-\frac{h(a)+h(x_1)}{2}\right|+\left|h(\alpha_2)-\frac{h(x_1)+h(b)}{2}\right|\\
&\left.+\left|\left|h(\alpha_1)-\frac{h(a)+h(x_1)}{2}\right|-\left|h(\alpha_2)-\frac{h(x_1)+h(b)}{2}\right|\right|\right]V_a^b(f)
\end{aligned}
$$

and the first inequality in (12) is proved.

Now, observe that

$$
\left|h(\alpha_1)-\frac{h(a)+h(x_1)}{2}\right|\le\frac{h(x_1)-h(a)}{2}
$$

and

$$
\left|h(\alpha_2)-\frac{h(x_1)+h(b)}{2}\right|\le\frac{h(b)-h(x_1)}{2}.
$$

Consequently,

$$
\begin{aligned}
&\max\left\{\left|h(\alpha_1)-\frac{h(a)+h(x_1)}{2}\right|,\left|h(\alpha_2)-\frac{h(x_1)+h(b)}{2}\right|\right\}\\
\le&\frac{1}{2}\max\left\{h(x_1)-h(a),h(b)-h(x_1)\right\}\\
=&\frac{1}{2}\left[\frac{h(b)-h(a)}{2}+\left|h(x_1)-\frac{h(a)+h(b)}{2}\right|\right]\\
=&\frac{1}{4}\int_a^b g(t)dt+\frac{1}{2}\left|h(x_1)-\frac{h(a)+h(b)}{2}\right|.
\end{aligned}
\tag{14}
$$

By (13) and (14), the second inequality in (12) is proved.

The last inequality in (12) is obvious.

Remark 7. In Proposition 2, we get a weighted generalization of Proposition 2 in [2].

Remark 8. If we choose $\alpha_1 = a, \alpha_2 = b$, $x_1 = x \in [a, b]$ in Proposition 2 then, the inequality (12) reduces to the following inequality:

$$\left| \int_a^b f(t)g(t)dt - f(x) \int_a^b g(t)dt \right|$$
$$\leq \left[\frac{1}{2} \int_a^b g(t)dt + \left| h(x) - \frac{h(a) + h(b)}{2} \right| \right] V_a^b(f)$$

which is the "weighted Ostrowski" inequality for mappings of bounded variation.

Remark 9. Let $g(t) = 1$ and $h(t) = t$ $(t \in [a, b])$ in Remark 8. Then we get theTheorem B.

Remark 10. If we choose $\alpha_1 = h^{-1}(\frac{5h(a)+h(b)}{6}), \alpha_2 = h^{-1}(\frac{h(a)+5h(b)}{6})$ and $x = h^{-1}(\frac{h(a)+h(b)}{2})$, then the inequality (12) reduces to the following inequality:

$$\left| \int_a^b f(t)g(t)dt - \frac{1}{3} \int_a^b g(t)dt \cdot \left[\frac{f(a) + f(b)}{2} + 2f(x) \right] \right|$$
$$\leq \frac{1}{3} \int_a^b g(t)dt \cdot V_a^b(f)$$

which is the "weighted Simpson" inequality for mappings of bounded variation.

Remark 11. Let $g(t) = 1$ and $h(t) = t$ $(t \in [a, b])$ in Remark 10. Then we get the inequality (4.9) in [2].

Remark 12. Similarly we can get some weighted inequalities related to the composite quadrature formula given in [2].

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