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CLASSIFICATION THEOREMS FOR SPACE-LIKE SURFACES IN 4-DIMENSIONAL INDEFINITE SPACE FORMS WITH INDEX 2

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Abstract. Surfaces in 4D Riemannian space forms have been investigated extensively. In contrast, only few results are known for surfaces in 4D neutral indefinite space forms $R^4_2(c)$. Thus, in this paper we study space-like surfaces in $R^4_2(c)$ satisfying certain simple geometric properties. In particular, we classify space-like surfaces in $\mathbb{E}^4_2$ with constant mean and Gauss curvatures and null normal curvature. We also classify Wintgen ideal surfaces in $R^4_2(c)$ whose Gauss and normal curvatures satisfy $K^D = 2K$.

1. INTRODUCTION

Let $\mathbb{E}^m_t$ denote the pseudo-Euclidean m-space equipped with pseudo-Euclidean metric of index t given by

$$g_t = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{n} dx_j^2, \tag{1.1}$$

where $(x_1, \ldots, x_m)$ is a rectangular coordinate system of $\mathbb{E}^m_t$. We put

$$S^k_s(c) = \left\{ x \in \mathbb{E}^{k+1}_s : \langle x, x \rangle = c^{-1} > 0 \right\}, \tag{1.2}$$

$$H^k_s(c) = \left\{ x \in \mathbb{E}^{k+1}_{s+1} : \langle x, x \rangle = c^{-1} < 0 \right\}, \tag{1.3}$$

where $\langle\ ,\ \rangle$ is the associated inner product. Then $S^k_s(c)$ and $H^k_s(c)$ are pseudo-Riemannian manifolds of constant curvature c and with index s, which are known as *pseudo-Riemannian k-sphere* and the *pseudo-hyperbolic k-space*, respectively. The pseudo-Riemannian manifolds $\mathbb{E}^k_s, S^k_s(c)$ and $H^k_s(-c)$ are called *indefinite space forms*, denoted by R^k_s.

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Surfaces in 4-dimensional Riemannian space forms have been investigated very extensively (see, for instance, [1, 2, 3]). In contrast, only few results are known for surfaces in 4-dimensional neutral indefinite space forms $R^4_2(c)$ of constant curvature c and index 2. Thus, we study in this paper space-like surfaces in $R^4_2(c)$ satisfying some simple geometric properties.

In Section 2 of this paper we provide basic definitions and formulas. In Section 3 we completely classify space-like surfaces in $\mathbb{E}^4_2$ with constant mean and Gauss curvatures and null normal curvature. In Section 4, we present a result of Sasaki and the precise expression of a minimal immersion $\psi_{\mathcal{B}}$ of the hyperbolic plane $H^2(-\frac{1}{3})$ of curvature $-\frac{1}{3}$ into the unit pseudo-hyperbolic 4-space $H^4_2(-1)$ discovered by the first author in [4]. It is known that the immersion $\psi_{\mathcal{B}}$ is a Wintgen ideal surfaces in $H^4_2(-1)$ whose Gauss and normal curvatures satisfy $K^D = 2K$. In the last section, we classify Wintgen ideal surfaces in $R^4_2(c)$ whose Gauss and normal curvatures satisfy the condition $K^D = 2K$. The later result provides us another simple geometric characterization of the minimal immersion $\psi_{\mathcal{B}} : H^2(-\frac{1}{3}) \to H^4_2(-1)$.

2. Preliminaries

A vector v is called *space-like* (resp., *time-like*) if $\langle v, v\rangle > 0$ (resp., $\langle v, v\rangle < 0$). A surface M in a pseudo-Riemannian manifold is called *space-like* if each nonzero tangent vector is space-like.

Let $R^4_2(c)$ denote an indefinite space form of constant curvature c and with index 2. The curvature tensor $\tilde{R}$ of $R^4_2(c)$ is given by

$$\tilde{R}(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} \tag{2.1}$$

for vectors X, Y, Z tangent to $R^4_2(c)$. Let $\psi : M \to R^4_2(c)$ be an isometric immersion of a space-like surface M into $R^4_2(c)$. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M and $R^4_2(c)$, respectively. For vector fields X, Y tangent to M and ξ normal to M, the formulas of Gauss and Weingarten are given respectively by (cf. [1, 2, 10]):

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.2}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.3}$$

where $\nabla_X Y$ and $A_\xi X$ are the tangential components and $h(X,Y)$ and $D_X\xi$ the normal components of $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X \xi$, respectively.

The shape operator A and the second fundamental form h are related by

$$\langle h(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle . \tag{2.4}$$

The mean curvature vector H of M in $H^4_2(-1)$ is defined by $H = \frac{1}{2}\operatorname{trace} h$.

The equations of Gauss, Codazzi and Ricci are given respectively by

$$R(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} + A_{h(Y,Z)}X - A_{h(X,Z)}Y, \tag{2.5}$$

$$(\bar\nabla_X h)(Y,Z) = (\bar\nabla_Y h)(X,Z), \tag{2.6}$$

$$\left\langle R^D(X,Y)\xi,\eta\right\rangle = \langle [A_\xi, A_\eta]X, Y\rangle \tag{2.7}$$

for vector fields X, Y, Z tangent to M, and ξ,η normal to M, where $\bar\nabla h$ is defined by

$$(\bar\nabla_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

and R^D is the curvature tensor associated with the normal connection D, i.e.,

$$R^D(X,Y)\xi = D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]}\xi. \tag{2.8}$$

The normal curvature K^D is given by

$$K^D = \left\langle R^D(e_1,e_2)e_3, e_4\right\rangle. \tag{2.9}$$

A surface M in $R^4_2(c)$ is called *parallel* (resp., *minimal*) if $\bar\nabla h = 0$ (resp., $H = 0$) holds identically. An immersion $\psi : M \to R^4_2(c)$ is called *full* if the image $\psi(M)$ does not lies in any totally geodesic submanifold of $R^4_2(c)$. A surface M in $R^4_2(c)$ is called *isotropic* if, at each point $p \in M$, $|h(u,u)|$ is independent of the choice of the unit vector $u \in T_pM$.

For an immersion $\psi : M \to H^4_2(-1)$, let $L = \iota \circ \psi : M \to \mathbb{E}^5_3$ be the composition of ψ with the standard inclusion $\iota : H^4_2(-1) \to \mathbb{E}^5_3$ via (1.2). Since $H^4_2(-1)$ is totally umbilical with mean curvature one in $\mathbb{E}^5_3$, we have

$$\hat\nabla_X Y = \nabla_X Y + h(X,Y) + \langle X,Y\rangle L \tag{2.10}$$

for X, Y tangent to M, where h is the second fundamental form of ψ and $\hat\nabla$ denotes the Levi-Civita connection of $\mathbb{E}^5_3$.

3. Surfaces with Null Normal Curvature in $\mathbb{E}^4_2$

Theorem 3.1. *Let M be a space-like surface in the pseudo-Euclidean 4-space $\mathbb{E}^4_2$. If M has constant mean and Gauss curvatures and null normal curvature, then M is congruent to an open part of one of the following six types of surfaces:*

(1) *A totally geodesic plane in $\mathbb{E}^4_2$ defined by $(0,0,x,y)$;*

(2) *a totally umbilical hyperbolic plane $H^2(-\frac{1}{a^2}) \subset \mathbb{E}^3_1 \subset \mathbb{E}^4_2$ given by*

$$\big(0, a\cosh u, a\sinh u\cos v, a\sinh u \sin v\big),$$

where a is a positive number;

(3) *A flat surface in* $\mathbb{E}^4_2$ *defined by*

$$\frac{1}{\sqrt{2}m}\Big(\cosh(\sqrt{2}mx), \cosh(\sqrt{2}my), \sinh(\sqrt{2}mx), \sinh(\sqrt{2}my)\Big),$$

where m *is a positive number;*

(4) *A flat surface in* $\mathbb{E}^4_2$ *defined by*

$$\Big(0, \frac{1}{a}\cosh(ax), \frac{1}{a}\sinh(ax), y\Big),$$

where a *is a positive number;*

(5) *A flat surface in* $\mathbb{E}^4_2$ *defined by*

$$\left(\frac{\cosh(\sqrt{2}x)}{\sqrt{2mr}}, \frac{\cosh(\sqrt{2}y)}{\sqrt{2m(2m-r)}}, \frac{\sinh(\sqrt{2}x)}{\sqrt{2mr}}, \frac{\sinh(\sqrt{2}y)}{\sqrt{2m(2m-r)}}\right),$$

where m *and* r *are positive numbers satisfying* $2m > r > 0$*;*

(6) *A surface of negative curvature* $-b^2$ *in* $\mathbb{E}^4_2$ *defined by*

$$\left(\frac{1}{b}\cosh(bx)\cosh(by), \int_0^y \cosh(by)\sinh\left(\frac{4\sqrt{m^2-b^2}}{b}\tan^{-1}\Big(\tanh\frac{by}{2}\Big)\right)dy,\right.$$
$$\left.\frac{1}{b}\sinh(bx)\cosh(by), \int_0^y \cosh(by)\cosh\left(\frac{4\sqrt{m^2-b^2}}{b}\tan^{-1}\Big(\tanh\frac{by}{2}\Big)\right)dy\right),$$

where b *and* m *are real numbers satisfying* $0 < b < m$.

Proof. Assume that $L : M \to \mathbb{E}^4_2$ is an isometric immersion of a space-like surface M into $\mathbb{E}^4_2$. If M is totally geodesic in $\mathbb{E}^4_2$, we obtain case (1). Thus, from now on, we assume that M is non-totally geodesic in $\mathbb{E}^4_2$.

Let us choose an orthonormal tangent frame $\{e_1, e_2\}$ of the tangent bundle and an orthonormal normal frame $\{e_3, e_4\}$ of the normal bundle of M which satisfy

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1,\ \langle e_1, e_2\rangle = 0, \tag{3.1}$$

$$\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = -1,\ \langle e_3, e_4\rangle = 0. \tag{3.2}$$

We may also choose e_1, e_2 which diagonalize A_{e_3} so that the shape operator satisfies

$$A_{e_3} = \begin{pmatrix}\alpha & 0\\ 0 & \mu\end{pmatrix},\ A_{e_4} = \begin{pmatrix}\delta & \gamma\\ \gamma & -\delta\end{pmatrix} \tag{3.3}$$

for some functions $\alpha, \gamma, \delta, \mu$.

By definition, the *normal curvature* K^D of M is defined by

$$K^D = \langle [A_{e_3}, A_{e_4}]e_1, e_2\rangle\,. \tag{3.4}$$

For the orthonormal frame $\{e_1, e_2, e_3, e_4\}$, we put

(3.5) $$\nabla_X e_1 = \omega_1^2(X)e_2, \ \ D_X e_3 = \omega_3^4(X)e_4.$$

From (2.3), (3.2) and (3.3) we have

(3.6) $$h(e_1, e_1) = -\alpha e_3 - \delta e_4, \ h(e_1, e_2) = -\gamma e_4, \ h(e_2, e_2) = -\mu e_3 + \delta e_4.$$

Thus, the mean curvature vector, the Gauss curvature and the normal curvature are given respectively by

(3.7) $$H = -\frac{\alpha+\mu}{2}e_3, \ K = \gamma^2 + \delta^2 - \alpha\mu, \ K^D = \gamma(\mu - \alpha).$$

It follows from (3.5), (3.6) and the equation of Codazzi that

(3.8) $$e_1\gamma - e_2\delta = \alpha\omega_3^4(e_2) - 2\gamma\omega_1^2(e_2) - 2\delta\omega_1^2(e_1),$$

(3.9) $$e_2\alpha = -\gamma\omega_3^4(e_1) + \delta\omega_3^4(e_2) + (\alpha - \mu)\omega_1^2(e_1),$$

(3.10) $$e_2\gamma + e_1\delta = \mu\omega_3^4(e_1) - 2\delta\omega_1^2(e_2) + 2\gamma\omega_1^2(e_1),$$

(3.11) $$e_1\mu = -\delta\omega_3^4(e_1) - \gamma\omega_3^4(e_2) + (\alpha - \mu)\omega_1^2(e_2).$$

Since M has null normal curvature, we may also assume that $\gamma = 0$. Thus, by the constancy of mean and Gauss curvatures, we obtain from (3.7) that

(3.12) $$\mu = 2m - \alpha, \ \ k = \delta^2 + \alpha^2 - 2m\alpha$$

for some constants k, m. Without loss of generality, we may assume $m \geq 0$.

Case (i). $\mu = \alpha$. In this case, $\mu = \alpha = m$ is a constant, which gives $A_{e_3} = mI$. Moreover, (3.12) gives

(3.13) $$\delta^2 = m^2 + k \geq 0.$$

Case (i.1). $m^2 = -k$. From (3.13), we get $\delta = 0$. Hence, M is a totally umbilical surfaces in $\mathbb{E}^4_2$. Such a surface has parallel second fundamental form. Therefore, after applying Proposition 4.3 of [5], we obtain case (2) of the theorem,

Case (i.2). $m^2 > -k$. Without loss of generality, we may put $\delta = \sqrt{m^2 + k}$, which is a nonzero constant. Thus, we find from (3.8)-(3.11) that $\omega_1^2 = \omega_3^4 = 0$. Hence, M must be flat. So, we have $k = 0$. Because $\omega_1^2 = 0$, we may choose coordinates $\{x, y\}$ such that $e_1 = \partial/\partial x, e_2 = \partial/\partial y$. The metric tensor is then given by $g = dx^2 + dy^2$. Moreover, we know that the second fundamental form satisfies

(3.14) $$h(e_1, e_1) = -me_3 - m\,e_4, \ h(e_1, e_2) = 0, \ h(e_2, e_2) = -me_3 + m\,e_4.$$

Now, it follows from (2.1), (3.14) that the immersion $L : M \to \mathbb{E}^4_2$ satisfies

$$\begin{aligned} &L_{xx} = -me_3 - me_4,\ L_{xy} = 0,\ L_{yy} = -me_3 + me_4, \\ &\tilde\nabla_{\frac{\partial}{\partial x}} e_3 = -mL_x,\ \tilde\nabla_{\frac{\partial}{\partial y}} e_3 = -mL_y,\ \tilde\nabla_{\frac{\partial}{\partial x}} e_4 = -mL_x,\ \tilde\nabla_{\frac{\partial}{\partial y}} e_4 = mL_y. \end{aligned} \tag{3.15}$$

After solving this system and choosing suitable initial conditions, we get case (3).

Case (ii). $\mu \neq \alpha$. It follows from (3.12) that

$$\mu = 2m - \alpha,\ \ \delta = \sqrt{k + 2m\alpha - \alpha^2}. \tag{3.16}$$

Case (ii.1). $\delta = 0$. In this case, we have $k = \alpha^2 - 2m\alpha$ which is constant. Hence, α is also a constant. Thus, we derive from (3.8)-(3.11) that

$$\omega_1^2 = 0,\ \ \alpha\omega_3^4(e_2) = \mu\omega_3^4(e_1) = 0. \tag{3.17}$$

Therefore, M is flat and $\alpha\mu = 0$. Since M is non-totally geodesic, without loss of generality we may assume that $\alpha \neq 0$ and $\mu = 0$. Since $\omega_1^2 = 0$, we may choose coordinates $\{x, y\}$ such that $e_1 = \partial/\partial x, e_2 = \partial/\partial y$. So, we obtain

$$h(e_1, e_1) = -\alpha e_3,\ \ h(e_1, e_2) = h(e_2, e_2) = 0. \tag{3.18}$$

It follows from (3.17) and (3.18) that immersion $L : M \to \mathbb{E}^4_2$ satisfies

$$\begin{aligned} &L_{xx} = -\alpha e_3,\ \ L_{xy} = L_{yy} = 0, \\ &\tilde\nabla_{\frac{\partial}{\partial x}} e_3 = -\alpha L_x,\ \tilde\nabla_{\frac{\partial}{\partial y}} e_3 = 0. \end{aligned} \tag{3.19}$$

After solving this system and choosing suitable initial conditions, we get case (4).

Case (ii.2). $\delta \neq 0$. We have

$$\mu = 2m - \alpha,\ \ \gamma = 0,\ \ \delta = \sqrt{k + 2m\alpha - \alpha^2} \neq 0. \tag{3.20}$$

Case (ii.2.1). $m\alpha = -k$. In this case, α and δ are constant. Moreover, we have

$$\alpha = -\frac{k}{m},\ \ \delta = \frac{\sqrt{-k(k + m^2)}}{m},\ \ \mu = 2m + \frac{k}{m},\ \ \gamma = 0. \tag{3.21}$$

Because δ is a real nonzero number, we must have $-m^2 < k < 0$. Thus, we may put $k = -b^2$ with $0 < b < m$. Substituting (3.21) into (3.8)-(3.11) yields

$$\omega_1^2(e_2) = \omega_3^4(e_1) = 0,\ \ \omega_3^4(e_2) = \frac{2\sqrt{m^2 - b^2}}{b}\omega_1^2(e_1) \tag{3.22}$$

Thus, if f is a function satisfying $e_2(\ln f) = \omega_1^2(e_1)$, then we get $[fe_1, e_2] = 0$, which implies that there exist coordinates $\{x, y\}$ such that

$$\frac{\partial}{\partial x} = fe_1, \ \ \frac{\partial}{\partial y} = e_2. \tag{3.23}$$

Therefore, the metric tensor is given by

$$g = f^2 dx^2 + dy^2. \tag{3.24}$$

Consequently, the Levi-Civita connection satisfies

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \frac{f_x}{f}\frac{\partial}{\partial x} - ff_y\frac{\partial}{\partial y}, \ \ \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = \frac{f_y}{f}\frac{\partial}{\partial x}, \ \ \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = 0. \tag{3.25}$$

From (3.22), (3.23) and (3.25), we derive that

$$\omega_1^2(e_1) = -\frac{f_y}{f}, \ \omega_1^2(e_2) = \omega_3^4(e_1) = 0, \ \omega_3^4(e_2) = -\frac{2f_y\sqrt{m^2 - b^2}}{bf}. \tag{3.26}$$

Moreover, it follow from (3.24) and $K = -b^2$ that f satisfies

$$f_{yy} = b^2 f. \tag{3.27}$$

By solving (3.27) we obtain $f = u(x)\cosh(by+v(x))$ for some functions $u(x), v(x)$. After replacing x by an anti-derivative of $u(x)$, we find from (3.24) and (3.25) that

$$g = \cosh^2(by + v(x))dx^2 + dy^2, \tag{3.28}$$

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} &= v'\tanh(by+v)\frac{\partial}{\partial x} - \frac{b}{2}\sinh(2by+2v)\frac{\partial}{\partial y}, \\ \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} &= b\tanh(by+v)\frac{\partial}{\partial x}, \ \ \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = 0. \end{aligned} \tag{3.29}$$

Also, it follows from (3.6), (3.21), and (3.28) that

$$\begin{aligned} h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) &= -\frac{\cosh^2(by+v)}{m}\{b^2 e_3 + b\sqrt{m^2 - b^2}e_4\}, \\ h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= 0, \\ h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) &= \frac{(b^2 - 2m^2)e_3 + b\sqrt{m^2 - b^2}e_4}{m}. \end{aligned} \tag{3.30}$$

Therefore, the immersion $L : M \to \mathbb{E}^4_2$ satisfies

$$
\begin{aligned}
L_{xx} &= v' \tan(by+v) L_x - \frac{b}{2}\sinh(2by+2v) L_y \\
&\quad - \frac{\cosh^2(by+v)}{m}\{b^2 e_3 + b\sqrt{m^2-b^2}e_4\}, \\
L_{xy} &= b\tanh(by+v)L_x, \\
L_{yy} &= \frac{(b^2-2m^2)e_3 + b\sqrt{m^2-b^2}e_4}{m}, \\
\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 &= -\frac{b^2}{m}L_x, \quad \tilde{\nabla}_{\frac{\partial}{\partial x}} e_4 = -\frac{b\sqrt{m^2-b^2}}{m}L_x, \\
\tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 &= \frac{b^2-2m^2}{m}L_y - 2\sqrt{m^2-b^2}\tanh(by+v)e_4, \\
\tilde{\nabla}_{\frac{\partial}{\partial y}} e_4 &= \frac{b\sqrt{m^2-b^2}}{m}L_y + 2\sqrt{m^2-b^2}\tanh(by+v)e_3.
\end{aligned} \tag{3.31}
$$

The compatibility condition of (3.31) is given by $v'(x) = 0$. Then, after applying a suitable translation in y, we may put $v = 0$. Therefore, system (3.31) reduces to

$$
\begin{aligned}
L_{xx} &= -\frac{b}{2}\sinh(2by) L_y - \frac{\cosh^2(by)}{m}\{b^2 e_3 + b\sqrt{m^2-b^2}e_4\}, \\
L_{xy} &= b\tanh(by)L_x, \\
L_{yy} &= \frac{(b^2-2m^2)e_3 + b\sqrt{m^2-b^2}e_4}{m}, \\
\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 &= -\frac{b^2}{m}L_x, \quad \tilde{\nabla}_{\frac{\partial}{\partial x}} e_4 = -\frac{b\sqrt{m^2-b^2}}{m}L_x, \\
\tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 &= \frac{b^2-2m^2}{m}L_y - 2\sqrt{m^2-b^2}\tanh(by)e_4, \\
\tilde{\nabla}_{\frac{\partial}{\partial y}} e_4 &= \frac{b\sqrt{m^2-b^2}}{m}L_y + 2\sqrt{m^2-b^2}\tanh(by)e_3.
\end{aligned} \tag{3.32}
$$

Solving the second equation in (3.32) gives

$$L = A(x)\cosh by + B(y) \tag{3.33}$$

for some vector-valued functions $A(x), B(y)$. Substituting this into the first, third and fourth equations in (3.32) gives $A'''(x) = b^2 A'(x)$. Thus, we get

$$A(x) = c_5 + c_1\cosh(bx) + c_2\sinh(bx) \tag{3.34}$$

for some vectors c_5, c_1, c_2. Combining this with (3.33) gives

$$L = (c_5 + c_1\cosh(bx) + c_2\sinh(bx))\cosh by + B(y). \tag{3.35}$$

By substituting (3.35) into the first, third and fifth equations in (3.32), we find

$$\begin{aligned} &\cosh^2(by)B''' - \frac{b}{2}\sinh(2by)B'' + (3b^2-4m^2)B' \\ &= c_5 b(3b^2-4m^2)\sinh(by). \end{aligned} \tag{3.36}$$

A direct computation shows that $B_p = -c_5\cosh(by)$ is a particular solution of (3.36). Thus, it follows from (3.35) and (3.36) that

$$L = (c_1\cosh(bx) + c_2\sinh(bx))\cosh by + C(y), \tag{3.37}$$

where $C(y)$ satisfies the homogeneous differential equation:

$$\cosh^2(by)C'''(y) - \frac{b}{2}\sinh(2by)C''(y) + (3b^2-4m^2)C'(y) = 0. \tag{3.38}$$

After solving this differential equation, we have

$$\begin{aligned} C(y) = c_3 &\int_0^y \cosh(by)\cosh\left(\frac{4\sqrt{m^2-b^2}}{b}\tan^{-1}\left(\tanh\frac{by}{2}\right)\right)dy \\ &+ c_4\int_0^y \cosh(by)\sinh\left(\frac{4\sqrt{m^2-b^2}}{b}\tan^{-1}\left(\tanh\frac{by}{2}\right)\right)dy + c_0 \end{aligned} \tag{3.39}$$

for some vectors $c_3, c_4, c_5 \in \mathbb{E}^4_2$. Combining this with (3.37) yields

$$\begin{aligned} L = c_0 &+ (c_1\cosh(bx) + c_2\sinh(bx))\cosh by \\ &+ c_3\int_0^y \cosh(by)\cosh\left(\frac{4\sqrt{m^2-b^2}}{b}\tan^{-1}\left(\tanh\frac{by}{2}\right)\right)dy \\ &+ c_4\int_0^y \cosh(by)\sinh\left(\frac{4\sqrt{m^2-b^2}}{b}\tan^{-1}\left(\tanh\frac{by}{2}\right)\right)dy. \end{aligned}$$

Therefore, after choosing suitable initial conditions, we obtain case (6).

Case (ii.2.2). $m\alpha \neq -k$. By substituting (3.20) into (3.8)-(3.11) we obtain

$$\omega_3^4(e_1) = \frac{2(k+m^2)\omega_1^2(e_2)}{m\sqrt{k+2m\alpha-\alpha^2}},\ \ \omega_3^4(e_2) = \frac{2(k+m^2)\omega_1^2(e_1)}{m\sqrt{k+2m\alpha-\alpha^2}}, \tag{3.40}$$

$$\omega_1^2(e_1) = e_2(\ln\sqrt{k+m\alpha}),\ \ \omega_2^1(e_2) = e_1(\ln\sqrt{k+2m^2-m\alpha}). \tag{3.41}$$

It follows from (3.41) that $\left[e_1/\sqrt{k+m\alpha}, e_2/\sqrt{k+2m^2-m\alpha}\right] = 0$. Thus, there exist coordinates $\{x, y\}$ such that

$$\frac{\partial}{\partial x} = \frac{e_1}{\sqrt{k+m\alpha}},\ \ \frac{\partial}{\partial y} = \frac{e_2}{\sqrt{k+2m^2-m\alpha}}. \tag{3.42}$$

Hence, the metric tensor is given by

$$g = \frac{dx^2}{k+m\alpha} + \frac{dy^2}{k+2m^2-m\alpha}. \tag{3.43}$$

From (3.43) we have

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} &= \frac{-m\alpha_x}{2(k+m\alpha)} \frac{\partial}{\partial x} + \frac{m(k+2m^2-m\alpha)\alpha_y}{2(k+m\alpha)^2} \frac{\partial}{\partial y}, \\ \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} &= \frac{-m\alpha_y}{2(k+m\alpha)} \frac{\partial}{\partial x} + \frac{m\alpha_x}{2(k+2m^2-m\alpha)} \frac{\partial}{\partial y}, \\ \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} &= \frac{-m(k+m\alpha)\alpha_x}{2(k+2m^2-m\alpha)^2} \frac{\partial}{\partial x} + \frac{m\alpha_y}{2(k+2m^2-m\alpha)} \frac{\partial}{\partial y}. \end{aligned} \tag{3.44}$$

It follows from (3.6), (3.20) and (3.23) that the second fundamental form satisfies

$$\begin{aligned} & h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \frac{-\alpha e_3 - \sqrt{k+2m\alpha-\alpha^2}\, e_4}{k+m\alpha}, \quad h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0, \\ & h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \frac{(\alpha-2m)e_3 + \sqrt{k+2m\alpha-\alpha^2}\, e_4}{k+2m^2-m\alpha}. \end{aligned} \tag{3.45}$$

By applying (2.1), (3.24), (3.25) and (3.26) we obtain

$$\begin{aligned} L_{xx} &= \frac{-m\alpha_x L_x}{2(k+m\alpha)} + \frac{m(k+2m^2-m\alpha)\alpha_y L_y}{2(k+m\alpha)^2} - \frac{\alpha e_3 + \sqrt{k+2m\alpha-\alpha^2}\, e_4}{k+m\alpha}, \\ L_{xy} &= \frac{-m\alpha_y L_x}{2(k+m\alpha)} + \frac{m\alpha_x L_y}{2(k+2m^2-m\alpha)}, \\ L_{yy} &= \frac{-m(k+m\alpha)\alpha_x L_x}{2(k+2m^2-m\alpha)^2} + \frac{m\alpha_y L_y}{2(k+2m^2-m\alpha)} \\ &\quad + \frac{(\alpha-2m)e_3 + \sqrt{k+2m\alpha-\alpha^2}\, e_4}{k+2m^2-m\alpha}, \\ \tilde\nabla_{\frac{\partial}{\partial x}} e_3 &= -\alpha L_x + \frac{(k+m^2)\alpha_x}{(k+2m^2-m\alpha)\sqrt{k+2m\alpha-\alpha^2}} e_4, \\ \tilde\nabla_{\frac{\partial}{\partial y}} e_3 &= (\alpha-2m) L_y + \frac{(k+m^2)\alpha_y}{(k+m\alpha)\sqrt{k+2m\alpha-\alpha^2}} e_4, \\ \tilde\nabla_{\frac{\partial}{\partial x}} e_4 &= -\sqrt{k+2m\alpha-\alpha^2}\, L_x - \frac{(k+m^2)\alpha_x}{(k+2m^2-m\alpha)\sqrt{k+2m\alpha-\alpha^2}} e_3, \\ \tilde\nabla_{\frac{\partial}{\partial y}} e_4 &= \sqrt{k+2m\alpha-\alpha^2}\, L_y - \frac{(k+m^2)\alpha_y}{(k+m\alpha)\sqrt{k+2m\alpha-\alpha^2}} e_3. \end{aligned} \tag{3.46}$$

After applying (3.46) and a long computation, we find from $\langle L_{xxy}, L_y\rangle = \langle L_{xyx}, L_y\rangle$ and from $\langle L_{xyy}, L_x\rangle = \langle L_{yyx}, L_x\rangle$ that

$$\alpha_y\{(k+m\alpha)\alpha_x + (k+2m^2-m\alpha)\alpha_y\} = 0. \tag{3.47}$$

Hence, we have either

(1) $\alpha_y = 0$ or
(2) $(k+m\alpha)\alpha_x + (k+2m^2-m\alpha)\alpha_y = 0$.

Case (ii.2.2.a). $\alpha_y = 0$. In this case, system (3.46) reduces to

$$\begin{aligned}
&L_{xx} = \frac{-m\alpha_x L_x}{2(k+m\alpha)} - \frac{\alpha e_3 + \sqrt{k+2m\alpha-\alpha^2}e_4}{k+m\alpha},\\
&L_{xy} = \frac{m\alpha_x L_y}{2(k+2m^2-m\alpha)},\\
&L_{yy} = \frac{-m(k+m\alpha)\alpha_x L_x}{2(k+2m^2-m\alpha)^2} + \frac{(\alpha-2m)e_3 + \sqrt{k+2m\alpha-\alpha^2}e_4}{k+2m^2-m\alpha},\\
&\tilde\nabla_{\frac{\partial}{\partial x}} e_3 = -\alpha L_x + \frac{(k+m^2)\alpha_x}{(k+2m^2-m\alpha)\sqrt{k+2m\alpha-\alpha^2}}e_4,\\
&\tilde\nabla_{\frac{\partial}{\partial y}} e_3 = (\alpha-2m)L_y,\\
&\tilde\nabla_{\frac{\partial}{\partial x}} e_4 = -\sqrt{k+2m\alpha-\alpha^2}L_x - \frac{(k+m^2)\alpha_x}{(k+2m^2-m\alpha)\sqrt{k+2m\alpha-\alpha^2}}e_3,\\
&\tilde\nabla_{\frac{\partial}{\partial y}} e_4 = \sqrt{k+2m\alpha-\alpha^2}L_y.
\end{aligned} \tag{3.48}$$

Now, after applying (3.48), $\langle L_{xxy}, L_y\rangle = \langle L_{xyx}, L_y\rangle$ and $\langle L_{xyy}, L_y\rangle = \langle L_{yyx}, L_y\rangle$, we obtain that

$$a_{xx} = -\frac{2k(k+2m^2-m\alpha)^2 - m^2(2k+m^2+m\alpha)\alpha_x^2}{m(k+2m^2-m\alpha)(k+m\alpha)} \tag{3.49}$$

$$\alpha_x^2 = \frac{2k(k+2m^2-m\alpha)^2}{m^2(k+m^2)}. \tag{3.50}$$

Next, by differentiating (3.50) and by applying (3.49), we find $\alpha_x = 0$. Thus, α is a constant, say $\alpha = r$. Because $\delta \neq 0$, (3.50) gives $k = 0$. Therefore, system (3.48) becomes

$$\begin{aligned}
L_{xx} &= -\frac{re_3 + \sqrt{2mr-r^2}e_4}{mr},\\
L_{xy} &= 0,\\
L_{yy} &= \frac{(r-2m)e_3 + \sqrt{2mr-r^2}e_4}{2m^2-mr},
\end{aligned}$$

$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -\alpha L_x, \ \ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = (\alpha - 2m) L_y,$$
$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_4 = -\sqrt{2mr - r^2} L_x, \ \ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_4 = \sqrt{2mr - r^2} L_y.$$

After solving this system and choosing suitable initial conditions, we have case (5) of the theorem.

Case (ii.2.2.b). $(k + m\alpha)\alpha_x + (k + 2m^2 - m\alpha)\alpha_y = 0$. In this case, we have

$$\alpha_y = \frac{(k + m\alpha)\alpha_x}{m\alpha - k - 2m^2}. \tag{3.51}$$

Thus, system (3.46) becomes

$$\begin{aligned}
&L_{xx} = -\frac{m\alpha_x(L_x + L_y)}{2(k+m\alpha)} - \frac{\alpha e_3 + \sqrt{k+2m\alpha-\alpha^2} e_4}{k + m\alpha},\\
&L_{xy} = \frac{m\alpha_x(L_x + L_y)}{2(k + 2m^2 - m\alpha)},\\
&L_{yy} = -\frac{m(k + m\alpha)\alpha_x(L_x + L_y)}{2(k+2m^2-m\alpha)^2} + \frac{(\alpha-2m)e_3 + \sqrt{k+2m\alpha-\alpha^2} e_4}{k + 2m^2 - m\alpha},\\
&\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -\alpha L_x + \frac{(k + m^2)\alpha_x e_4}{(k + 2m^2 - m\alpha)\sqrt{k + 2m\alpha - \alpha^2}},\\
&\tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = (\alpha - 2m) L_y - \frac{(k + m^2)\alpha_x e_4}{(k + 2m^2 - m\alpha)\sqrt{k + 2m\alpha - \alpha^2}},\\
&\tilde{\nabla}_{\frac{\partial}{\partial x}} e_4 = -\sqrt{k + 2m\alpha - \alpha^2} L_x - \frac{(k + m^2)\alpha_x e_3}{(k + 2m^2 - m\alpha)\sqrt{k + 2m\alpha - \alpha^2}},\\
&\tilde{\nabla}_{\frac{\partial}{\partial y}} e_4 = \sqrt{k + 2m\alpha - \alpha^2} L_y + \frac{(k + m^2)\alpha_x e_4}{(k + 2m^2 - m\alpha)\sqrt{k + 2m\alpha - \alpha^2}}.
\end{aligned} \tag{3.52}$$

Now, from $L_{xxy} = L_{xyx}$ we find $(k+m\alpha)\alpha_x = 0$. Also, we find from $L_{xyy} = L_{yyx}$ that $k=0$. Thus, α is a constant and $k=0$. Hence, this case reduces to (ii.2.a). ∎

4. Spacelike Minimal Surfaces with Constant Gauss Curvature

From the equation of Gauss, we have

Lemma 4.1. *Let M be a space-like minimal surface in $R^4_2(c)$. Then $K \geq c$. In particular, if $K = c$ holds identically, then M is totally geodesic.*

For space-like *minimal surfaces* in $R^4_2(c)$, Theorem 1 of [12] implies that M has constant Gauss curvature if and only if it has constant normal curvature.

We recall the following result of Sasaki from [12].

Theorem 4.2. *Let M be a space-like minimal surface in $R^4_2(c)$. If M has constant Gauss curvature, then either*

(1) $K = c$ *and* M *is a totally geodesic surface in* $R^4_2(c)$*;*

(2) $c < 0$*,* $K = 0$ *and* M *is congruent to an open part of the minimal surface defined by* $\frac{1}{\sqrt{2}}(\cosh u, \cosh v, 0, \sinh u, \sinh v)$*, or*

(3) $c < 0$*,* $K = c/3$ *and* M *is isotropic.*

Let $\mathbf{R}^2$ be a plane with coordinates s, t. Consider a map $\mathcal{B} : \mathbf{R}^2 \to \mathbb{E}^5_3$ given by

$$\begin{aligned}\mathcal{B}(s,t) = \Bigg(&\sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{7}{8} + \frac{t^4}{18}\right) e^{\frac{2s}{\sqrt{3}}}, t + \left(\frac{t^3}{3} - \frac{t}{4}\right) e^{\frac{2s}{\sqrt{3}}}, \\ &\frac{1}{2} + \frac{t^2}{2} e^{\frac{2s}{\sqrt{3}}}, t + \left(\frac{t^3}{3} + \frac{t}{4}\right) e^{\frac{2s}{\sqrt{3}}}, \sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{1}{8} + \frac{t^4}{18}\right) e^{\frac{2s}{\sqrt{3}}}\Bigg).\end{aligned} \tag{4.1}$$

The first author proved in [4] that $\mathcal{B}$ defines a full isometric parallel immersion

$$\psi_{\mathcal{B}} : H^2(-\tfrac{1}{3}) \to H^4_2(-1) \tag{4.2}$$

of the hyperbolic plane $H^2(-\frac{1}{3})$ of curvature $-\frac{1}{3}$ into $H^4_2(-1)$.

The following result was also obtained in [4].

Theorem 4.3. *Let $\psi : M \to H^4_2(-1)$ be a parallel full immersion of a space-like surface M into $H^4_2(-1)$. Then M is minimal in $H^4_2(-1)$ if and only if M is congruent to an open part of the surface defined by*

$$\begin{aligned}\Bigg(&\sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{7}{8} + \frac{t^4}{18}\right) e^{\frac{2s}{\sqrt{3}}}, t + \left(\frac{t^3}{3} - \frac{t}{4}\right) e^{\frac{2s}{\sqrt{3}}}, \\ &\frac{1}{2} + \frac{t^2}{2} e^{\frac{2s}{\sqrt{3}}}, t + \left(\frac{t^3}{3} + \frac{t}{4}\right) e^{\frac{2s}{\sqrt{3}}}, \sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{1}{8} + \frac{t^4}{18}\right) e^{\frac{2s}{\sqrt{3}}}\Bigg).\end{aligned}$$

Combining Theorem 4.2 and Theorem 4.3, we obtain the following.

Theorem 4.4. *Let M be a non-totally geodesic space-like minimal surface in $H^4_2(-1)$. If M has constant Gauss curvature K, then either*

(1) $K = 0$ *and* M *is congruent to an open part of the surface defined by*

$$\frac{1}{\sqrt{2}}(\cosh u, \cosh v, 0, \sinh u, \sinh v),$$

or

(2) $K = -\frac{1}{3}$ *and* M *is is congruent to an open part of the surface defined by*

$$\begin{aligned}\Bigg(&\sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{7}{8} + \frac{t^4}{18}\right) e^{\frac{2s}{\sqrt{3}}}, t + \left(\frac{t^3}{3} - \frac{t}{4}\right) e^{\frac{2s}{\sqrt{3}}}, \\ &\frac{1}{2} + \frac{t^2}{2} e^{\frac{2s}{\sqrt{3}}}, t + \left(\frac{t^3}{3} + \frac{t}{4}\right) e^{\frac{2s}{\sqrt{3}}}, \sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{1}{8} + \frac{t^4}{18}\right) e^{\frac{2s}{\sqrt{3}}}\Bigg).\end{aligned}$$

5. Wintgen Ideal Surfaces Satisfying $K^D = -2K$

In 1979, P. Wintgen [13] proved a basic relationship between Gauss curvature K, normal curvature K^D, and mean curvature vector H of a surface M in a Euclidean 4-space $\mathbb{E}^4$; namely,

$$K + |K^D| \leq \langle H, H \rangle , \tag{5.1}$$

with the equality holding if and only if the curvature ellipse is a circle.

The following Wintgen type inequality for space-like surfaces in $R^4_2(c)$ can be found in [7].

Theorem 5.1. *Let M be a space-like surface in a 4-dimensional indefinite space form $R^4_2(c)$ of constant sectional curvature c and index two. Then we have*

$$K + K^D \geq \langle H, H \rangle + c \tag{5.2}$$

at every point. Moreover, the equality sign of (5.2) *holds at a point $p \in M$ if and only if, with respect to some suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operator at p satisfies*

$$A_{e_3} = \begin{pmatrix} \mu + 2\gamma & 0 \\ 0 & \mu \end{pmatrix}, \; A_{e_4} = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}. \tag{5.3}$$

Following [6, 9, 11], we call a surface in $R^4_2(c)$ *Wintgen ideal* if it satisfies the equality case of (5.2) identically. Wintgen ideal surfaces in $\mathbb{E}^4_2$ satisfying $|K| = |K^D|$ are classified by the first author in [7] (see [6] for the classification of Wintgen ideal surfaces in $\mathbb{E}^4$ satisfying $|K| = |K^D|$).

We need the following existence result.

Theorem 5.2. *Let c be a real number and γ with $3\gamma^2 > -c$ be a positive solution of the second order elliptic differential equation*

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{(3\gamma\sqrt{c+3\gamma^2}-c)(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}}\gamma_x}{2\gamma(c+3\gamma^2)}\right) \\ &\quad - \frac{\partial}{\partial y}\left(\frac{(3\gamma\sqrt{c+3\gamma^2}-c)\gamma_y}{2\gamma(c+3\gamma^2)(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}}}\right) = \gamma\sqrt{c+3\gamma^2} \end{aligned} \tag{5.4}$$

defined on a simply-connected domain $D \subset \mathbf{R}^2$. Then $M_\gamma = (D, g_\gamma)$ with the metric

$$g_\gamma = \frac{\sqrt{c+3\gamma^2}}{\gamma(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}}}\left(dx^2 + (6\gamma+2\sqrt{3c+9\gamma^2}\,)^{2\sqrt{3}}dy^2\right) \tag{5.5}$$

admits a non-minimal Wintgen ideal immersion $\psi_\gamma : M_\gamma \to R^4_2(c)$ into a complete simply-connected indefinite space form $R^4_2(c)$ satisfying $K^D = 2K$ identically.

Proof. Let c be a real number and γ be positive solution of (5.4) with $3\gamma^2 > -c$ defined on a simply-connected domain D. Consider the surface $M_\gamma = (D, g_\gamma)$ with metric g_γ given by (5.5). Then the Levi-Civita connection of g_γ satisfies

$$
(5.6)\quad
\begin{aligned}
\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} &= -\frac{(3\gamma\sqrt{c+3\gamma^2}+c)\gamma_x}{2\gamma(c+3\gamma^2)}\frac{\partial}{\partial x} + \frac{(3\gamma\sqrt{c+3\gamma^2}+c)\gamma_y}{2\gamma(c+3\gamma^2)(6\gamma+2\sqrt{3c+9\gamma^2})^{2\sqrt{3}}}\frac{\partial}{\partial y},\\
\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} &= -\frac{(3\gamma\sqrt{c+3\gamma^2}+c)\gamma_x}{2\gamma(c+3\gamma^2)}\frac{\partial}{\partial x} + \frac{(3\gamma\sqrt{c+3\gamma^2}-c)\gamma_x}{2\gamma(c+3\gamma^2)}\frac{\partial}{\partial y},\\
\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} &= \frac{(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{2\sqrt{3}}(c-3\gamma\sqrt{c+3\gamma^2})\gamma_x}{2\gamma(c+3\gamma^2)}\frac{\partial}{\partial x}\\
&\quad + \frac{(3\gamma\sqrt{c+3\gamma^2}-c)\gamma_y}{2\gamma(c+3\gamma^2)}\frac{\partial}{\partial y}.
\end{aligned}
$$

Let us define a bilinear map: $h : TM \to NM$ by

$$
(5.7)\quad
\begin{aligned}
h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right) &= -\frac{(\gamma+\sqrt{c+3\gamma^2}\,)\sqrt{c+3\gamma^2}}{\gamma(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}}}e_3,\\
h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) &= -\sqrt{c+3\gamma^2}e_4,\\
h\left(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right) &= \frac{(\gamma-\sqrt{c+3\gamma^2}\,)\sqrt{c+3\gamma^2}(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}}}{\gamma}e_3,
\end{aligned}
$$

where NM is the plane bundle over M spanned by an orthonormal time-like frame $\{e_3, e_4\}$. Define a linear metric connection D on NM by

$$
(5.8)\quad
\begin{aligned}
D_{\frac{\partial}{\partial x}}e_3 &= \frac{-3\gamma\gamma_y e_4}{(c+3\gamma^2)(6\gamma+2\sqrt{3c+9\gamma^2})^{\sqrt{3}}},\\
D_{\frac{\partial}{\partial y}}e_3 &= \frac{3\gamma(6\gamma+2\sqrt{3c+9\gamma^2})^{\sqrt{3}}\gamma_x}{c+3\gamma^2}e_4,\\
D_{\frac{\partial}{\partial x}}e_4 &= \frac{3\gamma\gamma_y e_3}{(c+3\gamma^2)(6\gamma+2\sqrt{3c+9\gamma^2})^{\sqrt{3}}},\\
D_{\frac{\partial}{\partial y}}e_3 &= -\frac{3\gamma(6\gamma+2\sqrt{3c+9\gamma^2})^{\sqrt{3}}\gamma_x}{c+3\gamma^2}e_3.
\end{aligned}
$$

Then it follows from a very long direct computation that $(M_\gamma, g_\gamma, D, h)$ satisfies the equations of Gauss, Codazzi and Ricci. Hence, the fundamental existence and uniqueness theorem of submanifolds implies that, up to rigid motions, there exists a unique isometric immersion from M_γ into $R^4_2(c)$ whose second fundamental form and normal connection are given by h and D, respectively. By applying (5.5), (5.7) and $c + 3\gamma^2 > 0$ we see that M is a non-minimal Wintgen ideal surface in $R^4_2(c)$. ■

Now, we classify Wintgen ideal surfaces in $R^4_2(c)$ which satisfy $K^D = 2K$.

Theorem 5.3. *Let M be a Wintgen ideal surface in a complete simply-connected indefinite space form $R^4_2(c)$ with $c = 1, 0$ or -1. If M satisfies $K^D = 2K$ identically, then one of following three cases occurs:*

(1) $c = 0$ *and M is a totally geodesic surface in $\mathbb{E}^4_2$;*

(2) $c = -1$ *and M is a minimal surface in $H^4_2(-1)$ congruent to an open part of $\psi_{\mathcal{B}} : H^2(-\frac{1}{3}) \to H^4_2(-1) \subset \mathbb{E}^5_3$ defined by*

$$\left(\sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{7}{8} + \frac{t^4}{18}\right) e^{\frac{2s}{\sqrt{3}}}, t + \left(\frac{t^3}{3} - \frac{t}{4}\right) e^{\frac{2s}{\sqrt{3}}},\right.$$
$$\left.\frac{1}{2} + \frac{t^2}{2} e^{\frac{2s}{\sqrt{3}}}, t + \left(\frac{t^3}{3} + \frac{t}{4}\right) e^{\frac{2s}{\sqrt{3}}}, \sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{1}{8} + \frac{t^4}{18}\right) e^{\frac{2s}{\sqrt{3}}}\right);$$

(3) *M is a non-minimal surface in $R^4_2(c)$ which is congruent to an open part of $\psi_\gamma : M_\gamma \to R^4_2(c)$ associated with a positive solution γ of the elliptic differential equation* (5.4) *as described in Theorem 5.2.*

Proof. Let M be a Wintgen surface in $R^4_2(c)$. Then, according to Theorem 5.1, there exist an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that shape operator satisfies (5.3) for some functions γ, μ. Thus, the Gauss and normal curvatures are given by

$$K = c + \gamma^2 - \mu^2 - 2\gamma\mu, \quad K^D = -2\gamma^2. \tag{5.9}$$

It follows from the condition $K^D = 2K$ that $\mu = -\gamma \pm \sqrt{c + 3\gamma^2}$. Without loss of generality, we may assume $\gamma \geq 0$.

Case (i). $\mu = -\gamma + \sqrt{c + 3\gamma^2}$. We divide this into two subcases.

Case (i.1). $c + 3\gamma^2 = 0$. We have $\mu = -\gamma$ and $c \leq 0$. Thus, M is a minimal surface.

If $c = 0$, we get $\gamma = \mu = 0$, which implies that M is totally geodesic. So, we get case (1) of the theorem.

If $c = -1$, we have $\gamma = -\mu = \frac{1}{\sqrt{3}}$. Thus, by (5.9) M is a minimal surface with curvature $-\frac{1}{3}$. Hence, we obtain case (2) of the theorem according to Theorem 4.4.

Case (i.2). $c + 3\gamma^2 \neq 0$. From (5.3) we obtain

$$\begin{aligned} h(e_1, e_1) &= -(\gamma + \sqrt{c + 3\gamma^2}\,)e_3, \\ h(e_1, e_2) &= -\gamma e_4, \\ h(e_2, e_2) &= (\gamma - \sqrt{c + 3\gamma^2}\,)e_3. \end{aligned} \tag{5.10}$$

Thus, it follows from Codazzi's equation that

$$\omega_1^2(e_1) = \frac{3\gamma\sqrt{c + 3\gamma^2} + c}{2\gamma(c + 3\gamma^2)} e_2\gamma, \ \ \omega_1^2(e_2) = \frac{3\gamma\sqrt{c + 3\gamma^2} - c}{2\gamma(c + 3\gamma^2)} e_1\gamma, \tag{5.11}$$

$$\omega_3^4(e_1) = -\frac{3\gamma e_2\gamma}{c+3\gamma^2}, \;\; \omega_3^4(e_2) = \frac{3\gamma e_1\gamma}{c+3\gamma^2}. \tag{5.12}$$

After applying (5.11) we derive that

$$\left[\frac{(c+3\gamma^2)^{1/4}}{\sqrt{\gamma}(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}/2}}e_1, \frac{(c+3\gamma^2)^{1/4}(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}/2}}{\sqrt{\gamma}}e_2\right] = 0.$$

Hence there exist coordinates $\{x, y\}$ such that

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{(c+3\gamma^2)^{1/4}}{\sqrt{\gamma}(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}/2}}e_1, \\ \frac{\partial}{\partial y} &= \frac{(c+3\gamma^2)^{1/4}(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}/2}}{\sqrt{\gamma}}e_2. \end{aligned} \tag{5.13}$$

By using (5.13) we know that the metric tensor is given by

$$g = \frac{\sqrt{c+3\gamma^2}}{\gamma(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}}}dx^2 + \frac{\sqrt{c+3\gamma^2}(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}}}{\gamma}dy^2, \tag{5.14}$$

which implies that the Levi-Civita connection satisfies

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} &= -\frac{(3\gamma\sqrt{c+3\gamma^2}+c)\gamma_x}{2\gamma(c+3\gamma^2)}\frac{\partial}{\partial x} + \frac{(3\gamma\sqrt{c+3\gamma^2}+c)\gamma_y}{2\gamma(c+3\gamma^2)(6\gamma+2\sqrt{3c+9\gamma^2})^{2\sqrt{3}}}\frac{\partial}{\partial y}, \\ \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} &= -\frac{(3\gamma\sqrt{c+3\gamma^2}+c)\gamma_x}{2\gamma(c+3\gamma^2)}\frac{\partial}{\partial x} + \frac{(3\gamma\sqrt{c+3\gamma^2}-c)\gamma_x}{2\gamma(c+3\gamma^2)}\frac{\partial}{\partial y}, \\ \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} &= \frac{(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{2\sqrt{3}}(c-3\gamma\sqrt{c+3\gamma^2})\gamma_x}{2\gamma(c+3\gamma^2)}\frac{\partial}{\partial x} \\ &\quad + \frac{(3\gamma\sqrt{c+3\gamma^2}-c)\gamma_y}{2\gamma(c+3\gamma^2)}\frac{\partial}{\partial y}. \end{aligned} \tag{5.15}$$

From (5.12) and (5.13) we find

$$\begin{aligned} \omega_3^4\left(\frac{\partial}{\partial x}\right) &= \frac{-3\gamma\gamma_y}{(c+3\gamma^2)(6\gamma+2\sqrt{3c+9\gamma^2})^{\sqrt{3}}}, \\ \omega_3^4\left(\frac{\partial}{\partial y}\right) &= \frac{3\gamma(6\gamma+2\sqrt{3c+9\gamma^2})^{\sqrt{3}}}{c+3\gamma^2}\gamma_x. \end{aligned} \tag{5.16}$$

Also, it follows from (5.10) and (5.13) that

$$
\begin{aligned}
&h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right)=-\frac{(\gamma+\sqrt{c+3\gamma^2}\,)\sqrt{c+3\gamma^2}}{\gamma(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}}}e_3,\\
&h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)=-\sqrt{c+3\gamma^2}e_4,\\
&h\left(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right)=\frac{(\gamma-\sqrt{c+3\gamma^2}\,)\sqrt{c+3\gamma^2}(6\gamma+2\sqrt{3c+9\gamma^2}\,)^{\sqrt{3}}}{\gamma}e_3.
\end{aligned}
\tag{5.17}
$$

Moreover, from (5.10), (5.15) and the equation of Gauss we know that γ satisfies the elliptic differential equation (5.4). Consequently, after applying Theorem 5.2 we obtain case (3) of the theorem.

Case (ii). $\mu=-\gamma-\sqrt{c+3\gamma^2}$. After replacing e_3,e_4 by $-e_3,-e_4$, respectively, this reduces to (i). ∎

References

1. B.-Y. Chen, *Geometry of Submanifolds*, Mercer Dekker, New York, 1973.
2. B.-Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, New Jersey, 1984.
3. B.-Y. Chen, *Riemannian submanifolds*, Handbook of Differential Geometry, Vol. I, North-Holland, pp. 187-418. Amsterdam, 2000.
4. B.-Y. Chen, A minimal immersion of hyperbolic plane in neutral pseudo-hyperbolic 4-space and its characterization, *Arch. Math.*, **94** (2010), 257-265.
5. B.-Y. Chen, Complete classification of parallel spatial surfaces in pseudo-Riemannian space forms with arbitrary index and dimension, *J. Geom. Phys.*, **60** (2010), 260-280.
6. B.-Y. Chen, Classification of Wintgen ideal surfaces in Euclidean 4-space with equal Gauss and normal curvatures, *Ann. Global Anal. Geom.*, **38** (2010), 145-160.
7. B.-Y. Chen, Wintgen ideal surfaces in 4-dimensional neutral indefinite space form $R^4_2(c)$, (submitted for publication).
8. B.-Y. Chen and J. Van der Veken, Complete classification of parallel surfaces in 4-dimensional Lorentzian space forms, *Tohoku Math. J.*, **61** (2009), 1-40.
9. S. Decu, M. Petrović-Torgašev and L. Verstraelen, On the intrinsic Deszcz symmetries and the extrinsic Chen character of Wintgen ideal submanifolds, *Tamkang J. Math.*, **41(2)** (2010), 109-116.
10. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1982.
11. M. Petrović-Torgašev and L. Verstraelen, On Deszcz symmetries of Wintgen ideal submanifolds, *Arch. Math.* (*Brno.*), **44** (2008), 57-67.

12. M. Sasaki, Spacelike maximal surfaces in 4-dimensional space forms of index 2, *Tokyo J. Math.*, **25** (2002), 295-306.

13. P. Wintgen, Sur l'inégalité de Chen-Willmore, *C. R. Acad. Sci. Paris*, **288** (1979), 993-995.

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