LOCAL CONDITION FOR PLANAR GRAPHS OF MAXIMUM DEGREE 6 TO BE TOTAL 8-COLORABLE

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Abstract. Recently Sun et al [X.-Y. Sun, J.-L. Wu, Y.-W. Wu, J.-F. Hou, Total colorings of planar graphs without adjacent triangles, *Discrete Math 309:202-206* (2009)] proved that planar graphs with maximum degree six and with no adjacent triangles are total 8-colorable. This results implies that if every vertex of a planar graph of maximum degree six is missing either a 3-cycle or a 4-cycle, then the graph is total 8-colorable. In this paper we strengthen that condition by showing that if every vertex of a planar graph of maximum degree six is missing some k_v-cycle for $k_v \in \{3, 4, 5, 6, 7, 8\}$, then the graph is total 8-colorable.

1. INTRODUCTION

A total k-coloring of a graph $G = (V, E)$ is a coloring of $V \cup E$ using at most k colors such that no two adjacent or incident elements get the same color. The total chromatic number of G, denoted by $\chi''(G)$ is the smallest integer k such that G has a total k-coloring. It is clear that $\chi''(G) \geq \Delta + 1$ where Δ is the maximum degree of G. Vizing conjectured [8] that $\chi''(G) \leq \Delta + 2$ for every graph G. This conjecture was verified for planar graphs with maximum degree $\Delta \leq 5$ [4], for planar graphs with $\Delta \geq 8$ [3], and for planar graphs with $\Delta = 7$ [6]. Therefore the only remaining case for planar graphs is when $\Delta = 6$. We are interested in finding sufficient conditions for a planar graph of maximum degree 6 to be total 8-colorable. All graphs are assumed to be simple and with no loop. Recently, Sun et al. [7] proved that every planar graph with maximum degree 6 and with no adjacent triangles is total 8-colorable. This result implies that if G is a planar graph of maximum degree 6 such that for every vertex v, there exists an integer $k_v \in \{3, 4\}$ so that v is not incident to any k_v-cycle, then G is total 8-colorable. In this paper we give a stronger statement:

Theorem 1. *Let G be a planar graph of maximum degree* 6. *If for every vertex v, there exists an integer $k_v \in \{3, 4, 5, 6, 7, 8\}$ so that v is not incident to any k_v-cycle, then G is total* 8-*colorable.*

2. PROPERTIES OF A MINIMAL COUNTEREXAMPLE

Assume to the contrary that Theorem 1 is false. Let H be a counterexample with minimum number of edges and vertices, that is H verifies the following four properties:

1. H is a planar graph of maximum degree 6;
2. For every vertex v of H, there exists an integer $k_v \in \{3, 4, 5, 6, 7, 8\}$ so that v is not incident to any k_v-cycle;
3. H is not total 8-colorable;
4. $|V(H)| + |E(H)|$ is minimum subject to (1), (2), (3).

In this section we will deduce structural properties of H. In the next section we will use a discharging procedure to find a contradiction and therefore prove the theorem.

Lemma 1. *H is connected.*

Assume to the contrary that H is not connected. All connected components of H are planar. If all connected components are total 8-colorable, then so is H. Therefore H must have a connected component, say G, that is not total 8-colorable. If G has maximum degree at most 5, then by [4], G is total 7-colorable, and hence total 8-colorable. So G must have maximum degree 6. Therefore properties (1) and (3) are true for G. Moreover property (2) is true for G. But $|V(G)| + |E(G)| < |V(H)| + |E(H)|$, contradicting property (4) for H. ∎

Lemma 2. *If v is a vertex of H, then $H \setminus \{v\}$ is total* 8-*colorable.*

Let $G = H \setminus \{v\}$ be obtained from H by deleting vertex v. Clearly $|V(G)| + |E(G)| < |V(H)| + |E(H)|$, therefore one of properties (1), (2), (3) must be false for G. Clearly, G is a planar graph. If G has maximum degree less than 6, then by previous results [4], G is total 7-colorable, and therefore total 8-colorable. So assume property (1) is true for G. To prove that property (3) is false for G, that is G is total 8-colorable, it is enough to prove that property (2) is true for G, which is straightforward, since if G had a vertex v adjacent to cycles of length 3, 4, 5, 6, 7, 8, then adding a vertex and edges to the graph would not change that fact, which contradicts property (2) for H. Note that here we consider cycles, and not faces of H. ∎

Similarly, we have the following lemma:

Lemma 3. *If e is an edge of H, then $H \setminus \{e\}$ is total 8-colorable.*

Lemma 4. *H is 2-connected.*

Assume to the contrary that H is not 2-connected. Let v be a separating vertex of H. Let $H_1, H_2, \ldots H_k$ be the components of $H \setminus \{v\}$, where we assume, without loss of generality, that v has at most 3 neighbors in H_1. Let G_1 be the subgraph of H induced by $V(H_1) \cup \{v\}$ and $G_2 = G \setminus V(H_1)$, such that G_1, G_2 are connected. By assumption, $d_{G_1}(v) \leq 3$. If property (1) is not verified by G_1 (respectively, G_2), then G_1 (respectively, G_2) is total 8-colorable by previous remarks. Property (2) is clearly verified by G_1 and G_2. Therefore, by minimality of H, we can assume that G_1 and G_2 admit total colorings l_1 and l_2 using the colors $1, 2, 3, 4, 5, 6, 7, 8$. Without loss of generality, assume $l_1(v) = l_2(v) = 1$ and the edges incident to v in G_1 are colored by l_1 using (possibly) colors $2, 3, 4$ in that order. Similarly the edges incident to v in G_2 are colored by l_2 using (possibly) colors $8, 7, 6, 5, 4, 3$ in that order. A coloring of H is easily obtained by superimposing the two colorings l_1 and l_2. ∎

Corollary 1. *H has no vertex of degree 1.*

Lemma 5. *H has no vertex of degree 2.*

Assume to the contrary that H has a vertex u of degree 2 and let v be one of its neighbors. Let $G = H \setminus \{uv\}$. By Lemma 3, G admits a total coloring using eight colors. After we have uncolored u, there are at least $8 - 1 - 6 = 1$ colors available for uv. After we have colored uv with an available color, there are at least $8 - 4 = 4$ colors available for u, hence u can be colored too. This gives a total coloring of H with eight colors. ∎

Lemma 6. *If u and v are adjacent vertices of H and $d(u) = 3$ then $d(v) = 6$.*

Assume to the contrary that H has two adjacent vertices u, v with $d(u) = 3$ and $d(v) \leq 5$. By Lemma 3, $H \setminus \{uv\}$ admits a total coloring using eight colors. After we have uncolored u, there are at least $8 - 2 - 5 = 1$ colors available for uv. After we have colored uv with an available color, there are at least $8 - 6 = 2$ colors available for u, hence u can be colored too. This gives a total coloring of H with eight colors. ∎

Lemma 7. *H has no triangle incident to a vertex of degree 3.*

Assume to the contrary that H has a triangle uvw with $d(v) = 3$. By Lemma 6,

$d(u) = d(w) = 6$. Let $G = H \setminus \{uv\}$. By Lemma 3, G admits a total coloring using colors $1, 2, \ldots, 8$. After we have uncolored v, we put back the edge uv. Suppose no color is available for uv. Then all the colors are used by the incident edges and vertices. Without loss of generality, the coloring is as shown in Figure 1. Suppose we can change the color of the edge vw. Then we can color uv with color 6. If not, then all the colors are used by the vertices and edges incident to vw. Without loss of generality, in Figure 1, $\{a, b, c, d, e\} = \{1, 2, 3, 4, 8\}$. After we have uncolored uw, we color uv with color 5, and uw with color 7. There are now at least $8 - 6 = 2$ colors available for v, hence v can be colored too. This gives a total coloring of H using eight colors.   ∎
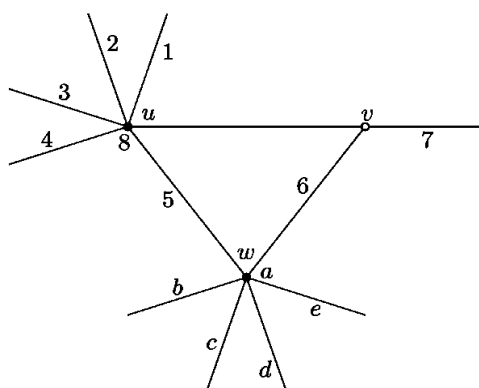

Fig. 1. Illustration for the proof of Lemma 7.

Lemma 8. *H has no triangle incident to two vertices of degree* 4.

Assume to the contrary that H has a triangle uvw with $d(u) = d(v) = 4$. Let $G = H \setminus \{uv\}$. By Lemma 3, G admits a total coloring using colors $1, 2, \ldots, 8$. After we have uncolored v, we put back the edge uv. There are at least $8 - 4 - 3 = 1$ colors available for uv. After we have colored uv, only the vertex v needs to be colored. Suppose no color is available for v. Then all colors are used by the incident vertices and edges. Without loss of generality, the coloring is as shown in Figure 2. If color 8 does not appear on the edges incident to u, then recolor the edge uv with color 8 and color v with color 1. If one of the edges incident to u is colored 8, then uncolor vertex u and color vertex v with color 2. After we have colored v, there are at least $8 - 7 = 1$ colors available for u, hence u can be colored too. This gives a total coloring of H with eight colors.   ∎

Definition 1. A triangle uvw is called an (a, b, c)-triangle if $d(u) = a, d(v) = b, d(w) = c$.

Lemma 9. *H has no* $(4, 5, 5)$-*triangle.*

Assume to the contrary that H has a $(4, 5, 5)$-triangle uvw with $d(u) = 4$. The proof is in two steps. First we show that H has a total 8-coloring with only u uncolored. Then we extend such a coloring to the whole graph.
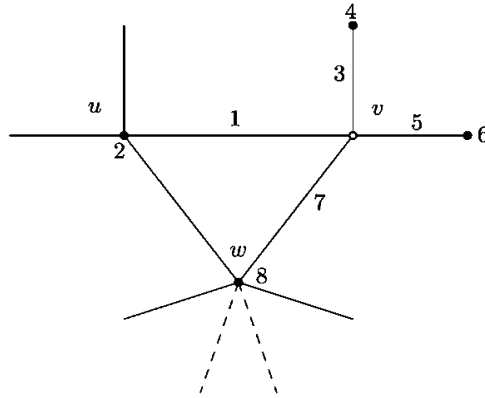

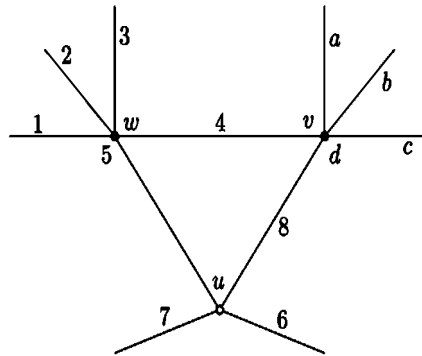Fig. 2. Illustration for the proof of Lemma 8.

Fig. 3. Illustration for the proof of Lemma 9.

Let $G = H \setminus \{uw\}$. By Lemma 3, G admits a total coloring using colors $1, 2, \ldots, 8$. After we have uncolored u, our purpose is to color edge uw. Suppose no color is available for edge uw. Then all colors are used by incident vertices and edges. Without loss of generality, the coloring is as shown in Figure 3. Suppose we can change the color of the edge uv. Then we can color uw with color 8. If not, then all the colors are used by the vertices and edges incident to uv. Without loss of generality, in Figure 3, $\{a, b, c, d\} = \{1, 2, 3, 5\}$. After we have uncolored vw, we color uw with color 4 and finally vw with 6. Now only u is uncolored.

Thus we have a total 8-coloring of H in which only u is uncolored. Suppose

no color is available for vertex u. Then all colors are used by incident vertices and edges. Without loss of generality, the coloring is as shown in Figure 4. Suppose we can change the color of the edge uv or uw. Then we can color u. If not, then all the colors are used by the vertices and edges incident to edges uv and uw. Without loss of generality, in Figure 4, $\{6,7,8\} \subset \{a,b,c,d\}$ and $\{5,7,8\} \subset \{d,e,f,g\}$. If we can change the color of v, then we can color u. So we may assume v has seven forbidden colors.
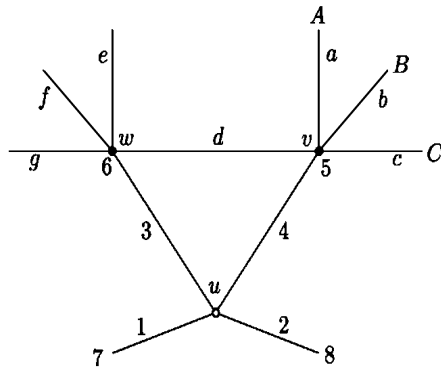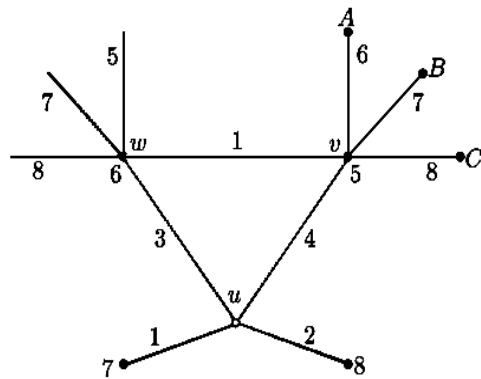

Fig. 4. Illustration for the proof of Lemma 9.

Fig. 5. Illustration for the proof of Lemma 9.

Suppose $d \in \{7,8\}$, say $d = 8$. Then $\{6,7\} \subset \{a,b,c\}$ and $\{5,7\} \subset \{e,f,g\}$. Without loss of generality, we may assume that $\{6,7\} = \{b,c\}$ and $\{5,7\} = \{f,g\}$. If $\{a,e\} = \{1,2\}$ then we can safely interchange the colors of the edges uv and vw and color u with color 4. Now assume $\{a,e\} \neq \{1,2\}$. Then we can recolor vw with a color from $\{1,2\} \setminus \{a,e\}$. From now on we assume $d \in \{1,2\}$, say $d = 1$, so that $\{e,f,g\} = \{5,7,8\}$ and $\{a,b,c\} = \{6,7,8\}$, as shown in Figure 5.

If there exists a color $\alpha \in \{1, 2\} \setminus \{A, B, C\}$, we can recolor vw with a color of $\{1, 2\}$ different from α, recolor v with α and color u with 5. Otherwise, $\{1, 2\} \subset \{A, B, C\}$. Now recall v has 7 forbidden colors, thus $\{A, B, C\} = \{1, 2, 3\}$. Consequently, we can interchange safely the colors of uv and uw, recolor v with color 4 and finally u with color 5. ∎

3. Proof of Theorem 1

For the proof of Theorem 1, we will use a discharging procedure. Assume H is a minimum counterexample. Fix a plane drawing of the graph H and let $F(H)$ be the set of faces of H. The initial charge c_0 of the vertices and faces of H are as following:

- If v is a vertex, then $c_0(v) = d(v) - 4$.
- If f is a face, then $c_0(f) = \ell(f) - 4$.

The discharging process is divided into three rounds.

- First round: Every 6-vertex gives $1/3$ to every adjacent 3-vertex.
- Second round: Every vertex with positive charge after the first round distributes its remaining charge evenly among the incident 3-faces. Every 5^+-face distributes its charge evenly among the adjacent 3-faces and the adjacent 4-faces.
- Third round: Every 3-face and 4-face with positive charge after the second round distributes its remaining charge evenly among the adjacent 3-faces whose charge is negative after the second round.

We now derive a contradiction by showing that the final charge of each vertex and face of H is non-negative (as the total charge is negative by Euler's formula).

Lemma 10. *Every vertex and 4^+-face in H has non-negative final charge.*

It suffices to show that every vertex has non-negative charge after the first round and that every 4^+-face has non-negative initial charge. By Lemma 6, every vertex of degree 3 is adjacent to three vertices of degree 6, therefore the charge of the vertices of degree 3 is zero after the first round. The 6-vertices are adjacent to at most six 3-vertices. So they have non-negative charge after the first round. All 4^+-faces and 4-vertices and 5-vertices have non-negative initial charge. ∎

It remains to show that 3-faces have non-negative final charges. A 6-vertex v incident to six 3-faces, five 3-faces, or four consecutive 3-faces is called a *big* 6-vertex. Other 6-vertices are called *small* 6-vertices.

Lemma 11. *Let f be a 3-face incident to a 6-vertex v. If v is big, then f has non-negative final charge.*

It suffices to show that f has non-negative charge at some point.

- Suppose v is incident to six 3-faces. By our assumption, for each vertex u, there is an integer $k_u \in \{3, 4, 5, 6, 7, 8\}$ such that u is not incident to any k_u-cycle. If u is a neighbor of v, then u is incident to cycles of length $3, 4, 5, 6, 7$. Therefore $k_u = 8$. This implies that f is adjacent to a 9^+-face. Therefore f receives at least $5/9$ from the 9^+-face. By Lemma 8, f is incident to another 5^+-vertex, say u. Since u is adjacent to at most three 3-vertices, the charge of u after the first round is at least 1. Hence f receives $1/3$ from v and receives at least $1/5$ from u. Therefore the total charge received by f in the first two rounds is at least $5/9 + 1/3 + 1/5 = 49/45$. So f has a non-negative final charge.

- If v is incident to five 3-faces, then the argument is similar and we omit the details.

Suppose v is incident to four consecutive 3-faces f_1, f_2, f_3, f_4. Then every vertex incident to one of these four 3-faces is incident to cycles of length $3, 4, 5, 6$. This implies that $k_v \in \{7, 8\}$.

- Suppose v is incident to four consecutive 3-faces and $k_v = 7$. Then f is adjacent to a 7^+-face and receives at least $3/7$ from that face and receives at least $5/12$ from v since v is adjacent to at most one 3-vertex. By Lemma 8, f is incident to another 5^+-vertex and receives at least $1/5$ from that 5^+-vertex. Therefore the total charge received by f in the first two rounds is at least $3/7 + 5/12 + 1/5 = 439/420$. So f has non-negative final charge.

- Suppose v is incident to four consecutive 3-faces f_1, f_2, f_3, f_4 (in that order) and $k_v \neq 7$, so that $k_v = 8$. Then the two other faces incident to v must be 9^+-faces.

  - If one of f_1, f_4 is adjacent to another 3-face, then each of f_2 and f_3 is adjacent to a 9^+-face, for otherwise one neighbor of v is contained in cycles of lengths $3, 4, 5, 6, 7, 8$, contrary to our assumption. Now each f_i $(i = 1, 2, 3, 4)$ receives at least $5/12$ from v and at least $5/9$ from its adjacent 9^+-face. By Lemma 8, each f_i is incident to another 5^+-vertex and receives at least $1/5$ from that 5^+-vertex. Therefore the total charge received by each f_i in the first two rounds is at least $5/9 + 5/12 + 1/5 = 211/180$. So f has non-negative final charge.

  - If none of f_1, f_4 is adjacent to another 3-face distinct from f_2, f_3, then each of f_1, f_4 is adjacent to two 9^+-faces, for otherwise v or one of its neighbors would be adjacent to cycles of length $3, 4, 5, 6, 7, 8$. These

9^+-faces give at least $10/9$ to f_1, f_4 who also receive at least $5/12$ from v. By Lemma 8, they are incident to another 5^+-vertex and receive at least $1/5$ from it. Therefore the total charge that f_1, f_4 receive in the first two rounds is at least $10/9 + 5/12 + 1/5 = 5/3$. In the third round of discharge they give at least $2/3$ to f_2, f_3 (assuming that the charge of f_2, f_3 after the second round is negative). These faces receive at least $5/12$ from v. Therefore the total charge received by f_2, f_3 is at least $2/3 + 5/12 = 13/12$. Their final charge is non-negative. So f has non-negative final charge. ■

In the remaining, we assume that f is a 3-face which is not incident to any *big* 6-vertex.

Remark 1. Every 5-vertex incident to f sends charge at least $1/5$ to f. Every *small* 6-vertex incident to f sends charge at least $4/9$ to f.

A 5-vertex has initial charge 1. It is clear from the discharging rules that this vertex sends charge at least $1/5$ to every incident 3-face. A 6-vertex v has initial charge 2. If v is incident to k 3-faces, $k = 1, 2, 3$, then it is adjacent to at most $6 - k - 1$ vertices of degree 3. So the charge sent from v to f is at least

$$\min_k (2 - (6 - k - 1)/3)/k = \min_k ((k+1)/3k) = 4/9.$$

If v is incident to four 3-faces, then since the four 3-faces are not consecutive, v is not adjacent to any 3-vertex and gives $1/2$ to each incident 3-face. ■

In particular, we have the following:

Lemma 12. *Every $(5^+, 6, 6)$-face has non-negative final charge.*

By Lemmas 7 and 8, f is incident to no 3-vertex and at most one 4-vertex. By Lemma 9, f is not a $(4, 5, 5)$-face. We finish the proof by considering each of the following cases: f is a $(5, 5, 5)$-face, or a $(5, 5, 6)$-face or a $(4, 5^+, 6)$-face.

Lemma 13. *Every $(5, 5, 5)$-face has non-negative final charge.*

Let $f = uvw$ be a $(5, 5, 5)$-face. In the second round, f receives at least $1/5 + 1/5 + 1/5 = 3/5$ from its incident vertices. If f is adjacent to a 7^+-face, this face gives at least another $3/7$ to f in the second round, and so f has non-negative final charge. We may assume that f is adjacent to three 6^--faces.

- Suppose f is adjacent to three 3-faces. Then at most one of the faces adjacent to f is adjacent to another 3-face (and, in this case, to exactly one), for otherwise u is contained in cycles of lengths $3, 4, 5, 6, 7, 8$. Therefore we can

assume u is incident to at most four 3-faces and v, w to exactly three 3-faces. Let $f' = vwu'$ be the 3-face adjacent to f. We can easily see that f' is adjacent to two 8^+-faces, and receives at least $1/2$ from each of them in the second round. The vertex u' must be a 5^+-vertex by Lemma 9, and it is easy to see that f' receives at least $1/3$ from each of u', v, w during the second round. In the third round, f' gives 1 to f. So f has non-negative final charge. We can now assume that f is adjacent to at most two 3-faces.

- Suppose f is adjacent to two 3-faces f_1, f_2. Let f_3 be the other face adjacent to f.

  – If f_3 is a 4-face, then none of f_1, f_2, f_3 is adjacent to another 3-face. Therefore f receives at least $1/2 + 1/2 + 1/3$ from its incident vertices in the second round. So f has non-negative final charge.
  – If f_3 is not a 4-face, then it is a 7^+-face. Hence f receives at least $3/5$ in the first round and $3/7$ in the second round, so f has non-negative final charge.

- Suppose f is adjacent to exactly one 3-face f_1. Let f_2, f_3 be the other faces adjacent to f.

  – Suppose f_2 is a 4-face and f_3 is a 4-face or 5-face. Then u is incident to cycles of length $3, 4, 5, 6, 7, 8$, contrary to our assumption.
  – Suppose f_2 is a 6^+-face. Then f receives at least $1/3 + 1/4 + 1/4$ from its incident vertices and at least $1/3$ from f_2 in the second round. So f has non-negative final charge.
  – Suppose f_2, f_3 are 5-faces. Then f receives at least $1/3 + 1/4 + 1/4$ from its incident vertices and at least $1/5$ from each of f_2, f_3 in the second round. So f has non-negative final charge.

- Suppose f is not adjacent to any 3-face. Then f receives at least $1/3 + 1/3 + 1/3$ from its incident vertices in the second round. So f has non-negative final charge. ∎

Lemma 14. *Every $(5, 5, 6)$-face has non-negative final charge.*

Let $f = uvw$ be a $(5, 5, 6)$-face with $d(u) = d(v) = 5$. In the second round, f receives at least $1/5 + 1/5 + 4/9 = 38/45$ from its incident vertices. If f is adjacent to a 5^+-face, this face gives another $1/5$ to f in the second round, and so f has non-negative final charge. We may assume that f is adjacent to three 4^--faces.

- Suppose f is adjacent to three 3-faces. Then at most one of the faces adjacent to f is adjacent to another 3-face, for otherwise u is contained in cycles of lengths $3, 4, 5, 6, 7, 8$. Therefore we can assume u is incident to at most three 3-faces and v to at most four 3-faces. Therefore f receives at least $1/3$ from u, $1/4$ from v and $4/9$ from w. So f has non-negative final charge.

- Suppose f is adjacent to two 3-faces. Then none of the faces adjacent to f is adjacent to another 3-face. Therefore we can assume u and v are incident to at most three 3-faces. Therefore f receives at least $1/3$ from u, $1/3$ from v and $4/9$ from w. So f has non-negative final charge.

- Suppose f is adjacent to exactly one 3-face. Then u is incident to cycles of length $3, 4, 5, 6, 7, 8$, contrary to our assumption.

- Suppose f is adjacent to no 3-face. Then f receives at least $1/3$ from u, $1/3$ from v, $4/9$ from w. So f has non-negative final charge.   ∎

Lemma 15. *Every $(4, 5^+, 6)$-face has non-negative final charge.*

Suppose $f = uvw$ is a $(4, 5^+, 6)$-face with $d(u) = 4$ and $d(w) = 6$. If f is adjacent to a 7^+-face, then it receives at least $29/45$ from its incident vertices and at least $3/7$ from its incident 7^+-face in the third round. Its final charge is non-negative.

- Suppose f is adjacent to a $(4, 5^+, 6)$-face f_1 and a $(4, 5^+, 6)$-face f_2 and a $(4^+, 5^+, 6^+)$-face f_3. If none of f_1, f_2, f_3 is adjacent to a 3-face g distinct from f, then every face adjacent to f_1, f_2, f_3 and distinct from f is an 8^+-face. At most one of f_1, f_2, f_3 may be adjacent to a 3-face $g \neq f$. Then every face adjacent to f_1, f_2, f_3 and distinct from f and g is a 9^+-face. A $(4^+, 5^+, 6)$-face adjacent to two 3-faces and a 8^+-face receives at least $4/9 + 1/5 + 1/2 = 103/90$ in the second round. A $(4^+, 5^+, 6)$-face adjacent to a 3-face and two 8^+-faces receives at least $4/9 + 1/4 + 1 = 61/36$ in the second round. So at least two faces of f_1, f_2, f_3 receive at least $61/36$ in the second round and give back $25/36$ each to f in the third round. So f has non-negative final charge.

- Suppose f is adjacent to two 3-faces f_1, f_2. Let f_3 be the other face adjacent to f.
  - If f_3 is a 4-face, then f_1, f_2 are both adjacent to two 9^+-faces. They receive $61/36$ in the second round and give back $25/36$ each to f in the third round. So f has non-negative final charge.
  - If f_3 is not a 4-face, then it is a 7^+-face. So f has non-negative final charge.

- Suppose f is adjacent to exactly one 3-face f_1. Let f_2, f_3 be the other faces adjacent to f.
  - If one of f_2, f_3, say f_2 is a 4-face, then f_3 is a 8^+-face. So f has non-negative final charge.
  - If f_2, f_3 are both 5^+-faces, then f receives at least $29/45$ from its incident vertices in the second round and at least $2/5$ from its incident 5^+-faces in the second round. So f has non-negative final charge.

- Suppose f is not adjacent to any 3-face.

  - If f is adjacent to at least two 4-faces, say f_1, f_2, then none of them is adjacent to a 3-face or two 4-faces, for otherwise u is incident to cycles of length $3, 4, 5, 6, 7, 8$, contrary to our assumption. Hence, each of f_1, f_2 is adjacent to two 5^+-faces. Hence each of them receives a minimum charge of $2 \times 1/5 = 2/5$ during the second round. Then each of them gives at least $2/5$ to f in the third round. The face f receives at least $29/45$ during the second round. So f has non-negative final charge.

  - If f is adjacent to at least two 5^+-faces, f receives at least $29/45$ from its incident vertices and at least $2/5$ from its incident 5^+-faces in the second round. So f has non-negative final charge. ∎

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