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ALTERNATIVE THEOREMS FOR PSEUDO ALMOST AUTOMORPHIC PROBLEMS

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Abstract. This paper gives alternative theorems to investigate the existence of pseudo almost automorphic solutions for semilinear evolution equations. Moreover, we consider a parabolic equation to illustrate the theorem.

1. Introduction

The paper is concerned with the pseudo almost automorphic solutions to the following semilinear evolution equation

(1) $$\frac{du}{dt} = A(t)u(t) + f(t, u) + \lambda g(t, u) \text{ for } t \in \mathbb{R} \text{ and } \lambda \in [0, 1],$$

in a Banach space X. It is assumed that the linear operator $A(t)$ satisfy the "Acquistapace-Terreni" conditions introduced in [2], and $f,\, g : \mathbb{R} \times X \to X$ are pseudo almost automorphic in t uniformly for $x \in X$.

To investigate the existence of solutions of the above mentioned problem, we consider the following auxiliary semilinear evolution equation

(2) $$\frac{du}{dt} = A(t)u + f(t, u) \text{ for } t \in \mathbb{R},$$

where $A(t)$ satisfies the "Acquistapace-Terreni" conditions, and $f : \mathbb{R} \times X \to X$ is pseudo almost automorphic in t uniformly for $x \in X$. We show that if Eq. (2) has

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an s-mild pseudo almost automorphic solution then Eq (1). has an s-mild pseudo almost automorphic solution.

The conception of almost automorphic fuctions more general than almost periodic functions was first introduced by S. Bochner in 1955 ([4]). Veech, Terras, Shen, Yi and so on ([16], [17], [19]) , have given a series of fundamental results in their works.

Pseudo almost automorphic functions are generalizations of asymptotically almost automorphic functions which were defined by N'Guérérkata in 1980s. Generally, consider the following function

$$f = g + \phi,$$

where f is almost automorphic, ϕ is continuous and bounded with $M(\|\phi\|) = 0$. $M(\cdot)$ is the "asymptotic" mean value, defined by

$$M(\phi) = \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \phi(s)ds.$$

In works of Ezzinbi, Liang, Xiao and so on ([7], [11], [20]), they investigated the existence and uniqueness of pseudo almost automorphic solutions for some evolution equations in Banach spaces.

In past decades, many important alternative theorems for Eq. (1) were studied by Li, Mawhin and so on ([10], [14]). But alternative theorems for pseudo almost automorphic problems are rarely studied. In the present paper, we use topological degree methods to investigate them.

This paper is organized as follows: In section 2, we state some facts on pseudo almost automorphic functions, and recall some properties of evolution families. In section 3, we present the alternative theorems, which are applied to investigate the existence of pseudo almost automorphic solutions for Eq. (2) instead of Eq. (1). In section 4, we illustrate the alternative theorem by considering a parabolic equation. The main results of this paper are Theorems 3.1, 3.2 and 4.4.

2. Preliminaries

Let X be a Banach space. We denote $\mathcal{L}(X)$, $C(\mathbb{R}, X)$ the spaces of linear bounded operators on X, all $X-$valued continuous respectively. The notations $BC(\mathbb{R}, X), AP(\mathbb{R}, X)$ will stand for the spaces of bounded continuous functions on $\mathbb{R}$ and its subspace of almost periodic functions (see H. Bohr [5]), respectively. The notion $D(A)$ is the domain of the linear operator A.

Definition 2.1. A function $f \in C(\mathbb{R}, X)$ is said to be almost automorphic (a.a. for short) if for any sequence $\{t'_n\}_{n=1}^{\infty} \subset \mathbb{R}$, there exists a subsequence $\{t_n\}_{n=1}^{\infty}$ and a function $g : \mathbb{R} \to X$ such that

$$f(t+t_n) \to g(t) \qquad \text{and} \qquad g(t-t_n) \to f(t) \tag{3}$$

hold pointwisely.

Denote by $AA(X)$ the set of all a.a. functions.

Theorem 2.2. [9, pp.23]. *If we equip $AA(X)$ with the sup norm*

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|,$$

then $AA(X)$ turns out to be a Banach space.

Denote

$$AA_0(X) = \{\phi \in BC(\mathbb{R}, X) : \lim_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} \|\phi(t)\|dt = 0\}.$$

Definition 2.3. A function $f \in BC(\mathbb{R}, X)$ is called pseudo almost automorphic (p.a.a. for short) if

$$f = g + \phi$$

with $g \in AA(X)$ and $\phi \in AA_0(X)$.

Denote $PAA(X)$ the set of all p.a.a. functions.

Theorem 2.4. [20, Theorem 2.2]. *$PAA(X)$ is a Banach space with the sup norm*

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|.$$

Obviously, we have

$$AA(X) \subset PAA(X) \subset BC(\mathbb{R}, X).$$

Definition 2.5. A function $f : \mathbb{R}\times X \to X$ is called pseudo almost automorphic in t with respect to the second argument x, if there exist functions $g,\ \phi : \mathbb{R}\times X \to X$ with $g(\cdot, x) \in AA(X)$ for each $x \in X$ and $\phi \in AA_0(\mathbb{R} \times X, X)$ such that

$$f = g + \phi,$$

where

$$AA_0(\mathbb{R}\times X, X) = \{\phi : \mathbb{R}\times X \to X : \phi(\cdot, x) \in BC(\mathbb{R}, X) \text{ for all } x \in X$$
$$\text{and } \lim_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} \|\phi(t, x)\|dt = 0 \text{ uniformly in } x \in X\}.$$

Theorem 2.6. [11, Theorem 2.4]. *Let $f = g + \phi \in PAA(\mathbb{R} \times X, X)$ with $g(\cdot, x) \in AA(X)$ for each $x \in X$ and $\phi(t, x) \in AA_0(\mathbb{R} \times X, X)$ satisfying the following conditions:*

$(f1)$ *$g(t, x)$ is uniformly continuous in any bounded subset $K \subset X$ uniformly for $t \in \mathbb{R}$.*

$(f2)$ *$f(t, x)$ is uniformly continuous in each bounded subset $K \subset X$ uniformly for $t \in \mathbb{R}$.*

If $x(t) \in PAA(\mathbb{R}, X)$, then $f(\cdot, x(\cdot)) \in PAA(\mathbb{R}, X)$.

We need the following results.

Throughout this paper, we assume that $A(t)$ satisfies the "Acquistapace-Terreni" conditions, that is,

(H1) There exist constants $\omega \geq 0$, $\phi \in (\frac{\pi}{2}, \pi)$, L, $M \geq 0$, and μ, $\nu \in (0, 1]$ with $\mu + \nu > 1$ such that

$$\Sigma_\phi \cup \{0\} \subset \rho(A(t) - \omega), \quad \|R(\lambda, A(t) - \omega)\| \leq \frac{M}{|\lambda| + 1},$$

and

$$\|(A(t) - \omega)R(\lambda, A(t) - \omega)[R(\lambda, A(t)) - R(\lambda, A(s))]\| \leq L|t - s|^\mu |\lambda|^{-\nu},$$

for s, $t \in \mathbb{R}$, $\lambda \in \Sigma_\phi := \{\lambda \in \mathbb{C} \backslash \{0\} : |\arg \lambda| \leq \phi\}$.

Obviously, if (H1) holds, then there exists a unique evolution family $(U(t, s))_{t \geq s}$ generated by $(A(t))_{t \in \mathbb{R}}$ with the following properties ([1, 6, 17]):

(E1) $U(t, s) = U(t, r)U(r, s)$ and $U(s, s) = I$ for $t \geq r \geq s$;

(E2) $(t, s) \mapsto U(t, s)$ is strongly continuous for $t > s$.

We also make the following assumptions on $(U(t, s))_{t \geq s}$:

(H2) The evolution family $(U(t, s))_{t \geq s}$ has an exponential stable:

$$\|U(t, s)\| \leq Ne^{-\alpha(t-s)}, \quad t \geq s,$$

where N, $\alpha > 0$.

(H3) $R(\omega, A(\cdot)) \in AP(\mathbb{R}, \mathcal{L}(X))$.

Definition 2.7. Consider the following equation

$$\frac{du}{dt} = A(t)u(t) + f(t) \text{ for } t \in \mathbb{R}, \tag{4}$$

where $f \in PAA(X)$. A function $u \in PAA(X)$ is called s-mild solution of Eq. (4) if

$$u(t) = \int_{-\infty}^{t} U(t, \xi)f(\xi)d\xi \text{ for all } t \in \mathbb{R}. \tag{5}$$

If (H1) holds, then Eq. (4) with the initial value $u(a) = \varsigma \in D(A(t))$ has a unique mild solution, namely,

$$u(t) = U(t,a)\varsigma + \int_a^t U(t,\xi)f(\xi)d\xi \text{ for } t \geq a. \tag{6}$$

Furthermore, if f is Hölder continuous, then (6) is a classical solution of (4) with the initial value $u(a) = \varsigma$.

Set $v(a) = \int_{-\infty}^a U(a,\xi)f(\xi)d\xi$, then

$$U(t,a)v(a) = \int_{-\infty}^a U(t,\xi)f(\xi)d\xi.$$

Thus, for $t \geq a$, we have that

$$\begin{aligned}\int_a^t U(t,\xi)f(\xi)d\xi &= \int_{-\infty}^t U(t,\xi)f(\xi)d\xi - \int_a^t U(t,\xi)f(\xi)d\xi \\ &= v(t) - U(t,a)v(a),\end{aligned}$$

namely,

$$v(t) = U(t,a)v(a) + \int_{-\infty}^t U(t,\xi)f(\xi)d\xi \quad \text{ for } t \geq a,$$

which shows that $v(t)$ is the mild solution of Eq. (4) with the initial value $u(a) = v(a)$. So, if the s-mild solution $v(t) = \int_{-\infty}^t U(t,\xi)f(\xi)d\xi$ of Eq. (4) is p.a.a., then the mild solution of Eq. (4) with the initial value $u(a) = \varsigma$ is p.a.a., where $\varsigma \in D(A(t))$ is given. Therefore, in this paper we consider s-mild solutions instead of considering mild solutions.

Theorem 2.8. [20, Theorem 3.3]. *Let (H1)-(H3) hold. Assume that $f \in PAA(\mathbb{R} \times X, X)$ satisfies conditions (f1)-(f2), and it is Lipschitz continuous in x uniformly for all $t \in \mathbb{R}$, i.e.*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\| \textit{ for all } x, y \in X, \tag{7}$$

where $L < \dfrac{\alpha}{N}$. Then Eq. (2) has a unique s-mild p.a.a. solution.

Corollary 2.9. *Suppose that $(H1)$-$(H3)$ hold. If $f \in PAA(X)$, then Eq. (4) admits a unique s-mild p.a.a. solution.*

3. Alternative Theorem

Theorem 3.1. *Consider the following equations:*

$$\frac{du}{dt} = A(t)u(t) + f(t,u) \quad \text{for } t \in \mathbb{R}, \tag{8}$$

and

$$\frac{du}{dt} = A(t)u(t) + f(t,u) + \lambda g(t,u) \quad \text{for } t \in \mathbb{R} \text{ and } \lambda \in [0,1], \tag{9}$$

where $f,\ g \in PAA(\mathbb{R} \times X, X)$ *are satisfied conditions (f1)-(f2). Assume the following conditions hold:*

(i) *There exists an open bounded subset* $\Omega \subset PAA(X)$ *such that every possible solution* u *of Eq. (9) satisfies* $u \notin \partial\Omega$.

(ii)

$$\deg(I - \Phi, \Omega, 0) \neq 0,$$

where $\Phi = \int_{-\infty}^{t} U(t,\xi) f(\xi, u(\xi)) d\xi$ *is the s-mild solution of Eq. (8).*

(iii) *Let (H1)-(H3) hold, and assume that for any* $h > 0$, $(U(t,s))_{t-s \geq h}$ *is compact.*

Then Eq. (9) has an s-mild p.a.a. solution $u \in \Omega$.

Proof. Set

$$H(u,\lambda) = \Phi(u) + \lambda \int_{-\infty}^{\cdot} U(\cdot,\xi) g(\xi, u(\xi)) d\xi \text{ for all } (u,\lambda) \in \Omega \times [0,1]. \tag{10}$$

Obviously, H is a map from $PAA(X) \times [0,1]$ to $PAA(X)$.

First, we prove that $H(\cdot,\lambda) : \Omega \to PAA(X)$ is completely continuous for all $\lambda \in [0,1]$.

Let $u_1,\ u_2 \in \Omega$, for all $t \in \mathbb{R}$. By (f1)-(f2), for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|u_1 - u_2\| < \delta$ then

$$\|f(t,u_1) - f(t,u_2)\| < \varepsilon \text{ and } \|g(t,u_1) - g(t,u_2)\| < \varepsilon.$$

Hence, for $\|u_1 - u_2\| < \delta$, we have that

$$\begin{aligned}
&\Big\|\Phi(u_1)(t) - \Phi(u_2)(t)\Big\| \\
&= \Big\| \int_{-\infty}^{t} U(t,\xi) f(\xi, u_1(\xi)) d\xi - \int_{-\infty}^{t} U(t,\xi) f(\xi, u_2(\xi)) d\xi \Big\| \\
&= \Big\| \int_{-\infty}^{t} U(t,\xi)(f(\xi, u_1(\xi)) - f(\xi, u_2(\xi))) d\xi \Big\| \\
&\leq \int_{-\infty}^{t} \Big\|U(t,\xi)\Big\| \Big\|f(\xi, u_1(\xi)) - f(\xi, u_2(\xi))\Big\| d\xi \\
&\leq \int_{-\infty}^{t} N e^{-\alpha(t-\xi)} \varepsilon d\xi \\
&= \frac{N}{\alpha}\varepsilon.
\end{aligned}$$

Similarly, for $\Big\|u_1 - u_2\Big\| < \delta$, we have that

$$\Big\| \int_{-\infty}^{t} U(t,\xi)g(\xi, u_1(\xi))d\xi - \int_{-\infty}^{t} U(t,\xi)g(\xi, u_2(\xi))d\xi \Big\| \leq \frac{N}{\alpha}\varepsilon.$$

Thus

$$\begin{aligned}
&\|H(u_1(t),\lambda) - H(u_2(t),\lambda)\| \\
\leq\ & \|\Phi(u_1)(t) - \Phi(u_2)(t)\| + \lambda \Big\|\int_{-\infty}^{t} U(t,\xi)g(\xi,u_1(\xi))d\xi - \int_{-\infty}^{t} U(t,\xi)g(\xi,u_2(\xi))d\xi\Big\| \\
\leq\ & \frac{N}{\alpha}\varepsilon + \lambda\frac{N}{\alpha}\varepsilon \\
\leq\ & 2\frac{N}{\alpha}\varepsilon,
\end{aligned}$$

which proves that $H(\cdot,\lambda) : \Omega \to PAA(X)$ is continuous for all $\lambda \in [0,1]$.

Note that for $u \in \Omega$, there exists a positive constant S such that $\|f(t,u(t))\| \leq S$. For $t_1 > t_2$, $\varepsilon > 0$ and $u \in \Omega$, we have that

$$\begin{aligned}
&\Phi(u)(t_1) - \Phi(u)(t_2) \\
=\ & \int_{t_2}^{t_1} U(t_1,\xi)f(\xi,u(\xi))d\xi + \int_{-\infty}^{t_2} [U(t_1,\xi) - U(t_2,\xi)]f(\xi,u(\xi))d\xi \\
=\ & \int_{t_2}^{t_1} U(t_1,\xi)f(\xi,u(\xi))d\xi + \int_{t_2-\varepsilon}^{t_2} [U(t_1,\xi) - U(t_2,\xi)]f(\xi,u(\xi))d\xi \\
&+ \int_{-\infty}^{t_2-\varepsilon} [U(t_1,\xi) - U(t_2,\xi)]f(\xi,u(\xi))d\xi \\
=\ & \int_{t_2}^{t_1} U(t_1,\xi)f(\xi,u(\xi))d\xi + \int_{t_2-\varepsilon}^{t_2} [U(t_1,\xi) - U(t_2,\xi)]f(\xi,u(\xi))d\xi \\
&+ \int_{\varepsilon}^{\infty} [U(t_1, t_2 - \frac{\sigma}{2}) - U(t_2, t_2 - \frac{\sigma}{2})]U(t_2 - \frac{\sigma}{2}, t_2 - \sigma) \\
&\times f(t_2 - \sigma, u(t_2 - \sigma))d\sigma.
\end{aligned}$$

Similar to the proof of [15, Theorem 3.2], we can prove that $\{U(\cdot,s) : s < \cdot\}$ is equicontinuous. This implies there exists a $\delta \in (0, \frac{\varepsilon}{2})$ such that if $|t - t_0| < \delta$ and $t,\ t_0 > s$ then

$$\|U(t,s) - U(t_0,s)\| < \varepsilon.$$

Thus

$$\begin{aligned}\|\Phi(u)(t)-\Phi(u)(t_0)\| &\le NS|t-t_0|+2\varepsilon S+\varepsilon NS\int_{\varepsilon}^{\infty}e^{-\frac{\alpha\sigma}{2}}d\sigma\\ &< NS\delta+2\varepsilon S+\frac{2\varepsilon NS}{\alpha}e^{-\frac{\alpha\varepsilon}{2}}\\ &< \frac{NS\varepsilon}{2}+2\varepsilon S+\frac{2\varepsilon NS}{\alpha},\end{aligned}$$

which implies that $\Phi(\Omega)$ is equicontinuous.

On the other hand, we note that

$$\begin{aligned}\|\Phi(u)(t)\| &= \|\int_{-\infty}^{t}U(t,\xi)f(\xi,u(\xi))d\xi\| \le \int_{-\infty}^{t}\|U(t,\xi)\|\|f(\xi,u(\xi))\|d\xi\\ &\le \int_{-\infty}^{t}Ne^{-\alpha(t-\xi)}Sd\xi=\frac{N}{\alpha}S.\end{aligned}$$

By Arzelá-Ascoli Theorem, we know that Φ is compact. Hence, Φ is completely continuous.

Similarly, we can prove that $H(\cdot,\lambda)$ for all $\lambda\in[0,1]$ is completely continuous. Thus

$$\deg(I-H(\cdot,\lambda),\Omega,0)=\deg(I-\Phi,\Omega,0)\neq 0,$$

that is, Eq. (9) has an s-mild p.a.a. solution $u\in\Omega$. ■

Theorem 3.2. *Consider the following problems:*

$$\frac{du}{dt}=A(t)u(t)+f(t),\ t\in\mathbb{R}, \tag{11}$$

and

$$\frac{du}{dt}=A(t)u(t)+f(t)+\lambda g(t,u) \text{ for } t\in\mathbb{R} \text{ and } \lambda\in[0,1], \tag{12}$$

where $f\in PAA(X)$, and $g\in PAA(\mathbb{R}\times X,X)$ satisfies conditions (f1)-(f2). Assume the following conditions hold:

- (i) *There exists an open bounded subset $\Omega\subset PAA(X)$ such that every possible solution u of Eq. (12) satisfies $u\notin\partial\Omega$.*
- (ii) *There exists a unique s-mild p.a.a solution $u\in\Omega$ for Eq. (11).*
- (iii) *Let (H1)-(H3) hold, and assume that for any $h>0$, $(U(t,s))_{t-s\ge h}$ is compact.*

Then Eq. (12) has an s-mild p.a.a. solution $u\in\Omega$.

Proof. Set

$$\tilde{H}(u,\lambda) = u_0 + \lambda \int_{-\infty}^{\cdot} U(\cdot,\xi) g(\xi, u(\xi)) d\xi \text{ for all } (u,\lambda) \in \Omega \times [0,1], \tag{13}$$

where $u_0(t) = \int_{-\infty}^{t} U(t,\xi) f(\xi) d\xi$ is the s-mild p.a.a. solution of Eq. (11). As the proof of Theorem 3.1, $\tilde{H}(\cdot,\lambda) : \Omega \to PAA(X)$ for all $\lambda \in [0,1]$ is completely continuous. Note (see [12])

$$\deg(I, \Omega, u_0) = 1.$$

Therefore,

$$\deg(I - H(\cdot,\lambda), \Omega, 0) = \deg(I - H(\cdot,0), \Omega, 0) = \deg(I, \Omega, u_0) = 1,$$

that is, Eq. (12) has an s-mild p.a.a. solution $u \in \Omega$. ■

Corollary 3.3. *Assume following statements are true:*

(i) *There exists an open bounded subset $\Omega \subset PAA(X)$ such that every possible solution u of Eq. (12) satisfies $u \notin \partial\Omega$.*

(ii) *$f \in PAA(X)$ is Lipschitz continuous, and $g \in PAA(\mathbb{R} \times X, X)$ satisfies (f1)-(f2).*

(iii) *Let (H1)-(H3) hold, and assume that for any $h > 0$, $(U(t,s))_{t-s \geq h}$ is compact.*

Then Eq. (12) has an s-mild p.a.a. solution $u \in \Omega$.

4. Partial Differential Equations

Let $D \subset \mathbb{R}^n$ be a bounded domain of class $C^{2+\eta}$, for some $n \geq 1$ and $\eta \in (0,1)$. Consider the following equation:

$$\begin{aligned} u_t - \Delta u + c(x,t)u &= f(x,t,u,\nabla u), \qquad (x,t) \in D \times \mathbb{R}, \\ u(x,t) &= 0, \qquad (x,t) \in \partial D \times \mathbb{R}. \end{aligned} \tag{14}$$

We set

$$D(A) := \{u \in W_p^2 : \, u(x,t) = 0 \text{ for all } (x,t) \in \partial D \times \mathbb{R}\},$$

and

$$Au := \Delta u \text{ for } u \in D(A).$$

Let $(T(t))_{t \geq 0}$ be the C_0-semigroup generated by A. Then

(i) By [18, Lemma 5.3], the semigroup $(T(t))_{t>0}$ is a compact operator.

(ii) By [3, Proposition 4.1], $\|T(t)\| \le 4e^{-2t}$ for all $t \ge 0$.

Assume the following conditions hold for Eq. (14):

(H4) There exist functions α, $\beta \in C^{2,1}(\bar{D} \times \mathbb{R})$ with $\alpha(x,t) \le \beta(x,t)$ such that $\alpha(x,t)$ and $\beta(x,t)$ are p.a.a in t for each $x \in D$. Furthermore, and α and β are subsolution and supersolution of Eq. (14), respectively, that is,

$$\alpha_t - \Delta\alpha + c(x,t)\alpha \le f(x,t,\alpha,\nabla\alpha) \text{ in } D \times \mathbb{R}, \quad \alpha \le 0 \text{ on } \partial D \times \mathbb{R};$$
$$\beta_t - \Delta\beta + c(x,t)\beta \ge f(x,t,\beta,\nabla\beta) \text{ in } D \times \mathbb{R}, \quad \beta \ge 0 \text{ on } \partial D \times \mathbb{R}.$$

(H5) Nonnegative function $c(x,t) \in C^{\eta,\frac{\eta}{2}}(\bar{D} \times \mathbb{R})$ is p.a.a. in t for all $x \in D$ with $c(x,t) \not\equiv 0$.

(H6) $f \in C(\bar{D} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n)$ and $f(\cdot,\cdot,\xi,\zeta) \in C^{\eta,\frac{\eta}{2}}(\bar{D} \times \mathbb{R})$ is uniformly on bounded subsets of $\mathbb{R} \times \mathbb{R}^n$. For every $r > 0$, there exists a positive constant K such that

$$|f(x,t,\xi,\zeta) - f(y,s,\bar{\xi},\bar{\zeta})| < K(|x-y|^\eta + |t-s|^{\frac{\eta}{2}} + |\xi - \bar{\xi}| + |\zeta - \bar{\zeta}|)$$

holds for all $x, y \in \bar{D}$, t, $s \in \mathbb{R}$, ξ, $\bar{\xi} \in [-r, r]$ and $\zeta, \bar{\zeta} \in \mathbb{B}(0,r) = \{x \in \mathbb{R}^n : |x| \le r\}$.

(H7) $f(x,t,\xi,\zeta)$ is p.a.a. in t for all $(x,\xi,\zeta) \in D \times \mathbb{R} \times \mathbb{R}^n$, that is, there exist $g(x,\cdot,\xi,\zeta) \in AA(\mathbb{R})$ and $h(x,\cdot,\xi,\zeta) \in AA_0(\mathbb{R})$ such that

$$f(x,t,\xi,\zeta) = g(x,t,\xi,\zeta) + h(x,t,\xi,\zeta).$$

(H8) $f(D,\mathbb{R},\mathbb{R},\mathbb{R}^n)$ is a bounded subset of $\mathbb{R}$. Moreover, there exist constants M, L such that

$$|f(t,x,u,p)| \le L + M|p| \text{ for } (x,t,u,p) \in D \times \mathbb{R} \times [\alpha,\beta] \times \mathbb{R}^n.$$

Remark 4.4. By (H6)-(H8), we have that $\tilde{f}(x,t) = f(x,t,\xi(x,t),\zeta(x,t))$ is p.a.a. in t for all $x \in D$ if ξ, ζ is p.a.a. in t for all $x \in D$. Furthermore, if $u(x,t)$ is p.a.a. in t for each $x \in D$ then $f(x,t,u(x,t),\nabla u(x,t))$ is p.a.a. in t for each $x \in D$.

Lemma 4.5. *If $f \in AA(\mathbb{R})$, then there exists a $t_0 \in \mathbb{R}$ such that $f(t_0) = \max_{\mathbb{R}} f(t)$.*

Proof. If not, then for any $t \in \mathbb{R}$, there exists a $t_1 \in \mathbb{R}$ such that $f(t) < f(t+t_1)$. And there also exists a $t_2 \in \mathbb{R}$ such that $f(t+t_1) < f(t+t_2)$. Following the above process, we can obtain a sequence $\{t_n\}_{n=1}^{\infty}$ such that

$$f(t) < f(t+t_1) < \cdots < f(t+t_n) < \cdots .$$

Since $f \in AA(\mathbb{R})$, there exists a subsequence $\{t'_n\}_{n=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$ and a function $g \in BC(\mathbb{R}, \mathbb{R})$ such that $\lim_{n\to\infty} f(t+t'_n) = g(t)$. Thus,

$$f(t) < f(t+t'_1) < \cdots < f(t+t'_n) < \cdots \leq g(t).$$

That is, for any $t \in \mathbb{R}$, $f(t) < g(t)$.

Since $g(\mathbb{R}) \subset \overline{f(\mathbb{R})}$ (see [9, Proposition 1.32]), $\inf_{s\in\mathbb{R}}(g(t) - f(s)) = 0$ for all $t \in \mathbb{R}$. Then

$$g(t) - f(t) \leq \inf_{s\in\mathbb{R}}(g(t) - f(s)) = 0,$$

a contradiction. This completes the proof. ■

Lemma 4.6. *Let $g \in AA_0(X)$. Then the following statement is not true.*

- *There exist $l > 0$ and $\alpha > 0$ such that for $|t| > l$*

$$\|g(t)\| \geq \alpha.$$

Proof. If not. Then for any $\varepsilon > 0$, there exists $r_0 > 0$ such that if $r > r_0$ then

$$\frac{1}{2r}\int_{-r}^{r} \|g(t)\| dt < \varepsilon. \tag{15}$$

Set $r > \max\{r_0, l\}$, then

$$\begin{aligned}
\frac{1}{2r}\int_{-r}^{r} \|g(t)\| dt &= \frac{1}{2r}\int_{[-r,r]/[-l,l]} \|g(t)\| dt + \frac{1}{2r}\int_{-l}^{l} \|g(t)\| dt \\
&\geq \frac{1}{2r}\int_{[-r,r]/[-l,l]} \|g(t)\| dt \\
&\geq \frac{2r - 2l}{2r}\alpha,
\end{aligned}$$

which is a contradiction to inequality (15). ■

According to Lemma 4.2 and Lemma 4.3, we conclude that for every $f \in PAA(\mathbb{R})$, there exists $t_0 \in \mathbb{R}$ such that $f'(t_0) = 0$ and $f''(t_0) \geq 0$.

Theorem 4.7. *Assume that conditions (H4)-(H8) hold. Then Eq. (14) admits a p.a.a solution $u \in C^{2,1}(\bar{D} \times \mathbb{R})$ with $\alpha \leq u \leq \beta$.*

Proof. Consider the following equations:

$$\begin{aligned}
&u_t - \Delta u + cu + k(u - \frac{\alpha+\beta}{2}) \\
&= \lambda k(\tilde{r}(u) - \frac{\alpha+\beta}{2}) + \lambda f(x, t, r(u), r_1(\nabla u)) + \lambda r_2(u) \\
&\text{in } D \times \mathbb{R} \text{ and } \lambda \in [0, 1], \\
&u = 0 \qquad \text{on } \partial D \times \mathbb{R} \text{ and } k >> 1.
\end{aligned} \tag{16}$$

Let $\alpha_m = \alpha - \dfrac{1}{m}$ and $\beta_m = \beta + \dfrac{1}{m}$ for $m \in \mathbb{N}$. Obviously, α_m, β_m are p.a.a. in t for all $x \in X$. We set

$$\tilde{r}(u) = \begin{cases} \beta_m, & u > \beta_m, \\ u, & \alpha_m \le u \le \beta_m, \\ \alpha_m, & u < \alpha_m, \end{cases}$$

$$r(u) = \begin{cases} \beta, & u > \beta, \\ u, & \alpha \le u \le \beta, \\ \alpha, & u < \alpha, \end{cases}$$

$$r_1(u) = \begin{cases} \dfrac{p}{|p|}, & |p| > N \\ u, & |u| \le N, \end{cases}$$

where $N = \max\{|\nabla\alpha|+1,\ |\nabla\beta|+1,$ Nagumo constants of f with α and β respectively$\}$. And

$$r_2(u) = \begin{cases} -u + \beta, & u > \beta, \\ 0, & \alpha \le u \le \beta, \\ -u + \alpha, & u < \alpha. \end{cases}$$

Set

$$\begin{aligned} P(u) \;=\; & u_t - \Delta u + c(x,t)u + k\left(u - \frac{\alpha(x,t) + \beta(x,t)}{2}\right) \\ & -\lambda k\left(\tilde{r}(u) - \frac{\alpha(x,t) + \beta(x,t)}{2}\right) \\ & +\lambda f(x,t,r(u), r_1(\nabla u)) + \lambda r_2(u) \\ & \text{for } (x,t) \in D \times \mathbb{R},\ \lambda \in [0,1], \text{ and } k >> 1. \end{aligned}$$

Note there exists $k_1 >> 1$ such that if $k > k_1$ then

$$\beta_t - \Delta\beta + c(x,t)\beta + c(x,t)/m + k\frac{\beta(x,t) - \alpha(x,t)}{2} + \frac{k}{m} \ge 0.$$

Then

$$\begin{aligned}
& P(\beta_m) \\
= \ & \beta_t - \Delta\beta + c(x,t)\beta + c(x,t)/m + k(\frac{\beta(x,t)-\alpha(x,t)}{2} + \frac{1}{m}) \\
& -\lambda k \frac{\beta(x,t)-\alpha(x,t)}{2} - \lambda f(t,x,\beta,\nabla\beta) + \frac{\lambda}{m} \\
= \ & (1-\lambda)(\beta_t - \Delta\beta + c(x,t)\beta + c(x,t)/m + k\frac{\beta(x,t)-\alpha(x,t)}{2}) \\
& +\lambda(\beta_t - \Delta\beta + c(x,t)\beta + c(x,t)/m - f(x,t,\beta,\nabla\beta)) + \frac{k}{m} + \frac{\lambda}{m} \\
\geq \ & (1-\lambda)(\beta_t - \Delta\beta + c(x,t)\beta + c(x,t)/m + k\frac{\beta(x,t)-\alpha(x,t)}{2}) + \frac{k}{m} + \frac{\lambda}{m} \\
\geq \ & (1-\lambda)(\beta_t - \Delta\beta + c(x,t)\beta + c(x,t)/m + k\frac{\beta(x,t)-\alpha(x,t)}{2} + \frac{k}{m}) + \frac{\lambda}{m} \\
\geq \ & 0.
\end{aligned}$$

On other hand,

$$\beta_m|_{\partial D\times\mathbb{R}} = \frac{1}{m} > 0.$$

Thus, for $m \in \mathbb{N}$, β_m are supersolutions of Eq. (16).

Similarly, we can prove that for $m \in \mathbb{N}$, there exists $k_2 >> 1$ such that if $k > k_2$ then α_m are subsolutions of Eq. (16). Set $k := \max\{k_1, k_2\} + 1$. Then α_m and β_m are subsolutions and supersolutions of Eq. (16) for $m \in \mathbb{N}$.

We claim that if $u(x,t)$ is a p.a.a solution of Eq. (16) then

$$\alpha_m < u < \beta_m \text{ in } D\times\mathbb{R},\ m \in \mathbb{N}. \tag{17}$$

If not, there exists $t_0 \in \mathbb{R}$ and $x_0 \in \bar{D}$ such that

$$0 \leq \delta = u(x_0,t_0) - \beta_m(x_0,t_0) = \max_{D\times\mathbb{R}}(u(x,t) - \beta_m(x,t)). \tag{18}$$

Note

$$u|_{\partial D\times\mathbb{R}} = 0 \text{ and } \beta_m|_{\partial D\times\mathbb{R}} = \frac{1}{m},$$

we have that $x_0 \in D$. Then

$$\Delta u(x_0,t_0) \leq \Delta\beta_m(x_0,t_0) = \Delta\beta(x_0,t_0),$$

$$\nabla u(x_0,t_0) = \nabla\beta_m(x_0,t_0) = \nabla\beta(x_0,t_0).$$

Note

$$\begin{aligned}
&u_t(x_0,t_0)-\Delta u(x_0,t_0)+c(x_0,t_0)u(x_0,t_0)\\
\geq\ &\beta_m(x_0,t_0)-\Delta\beta_m(x_0,t_0)+c(x_0,t_0)\beta_m(x_0,t_0)+c(x_0,t_0)\delta\\
>\ &-k(\frac{\beta(x_0,t_0)-\alpha(x_0,t_0)}{2}+\frac{1}{m})+\lambda k\frac{\beta(x_0,t_0)-\alpha(x_0,t_0)}{2}\\
&+\lambda f(x_0,t_0,\beta,\nabla\beta)+c(x_0,t_0)\delta-\frac{\lambda}{m}.
\end{aligned}$$

On other hand, since $u(x_0,t_0)$ is a p.a.a. solution of Eq. (16), we find that

$$\begin{aligned}
&u_t(x_0,t_0)-\Delta u(x_0,t_0)+c(x_0,t_0)u(x_0,t_0)\\
=\ &-k(\frac{\beta(x_0,t_0)-\alpha(x_0,t_0)}{2}+\frac{1}{m})+\lambda k\frac{\beta(x_0,t_0)-\alpha(x_0,t_0)}{2}\\
&+\lambda f(x_0,t_0,\beta,\nabla\beta)-\lambda\delta-\frac{\lambda}{m}.
\end{aligned}$$

This is a contradiction.

Set

$$\Omega_m=\{u(\cdot,x)\in PAA(\mathbb{R}):\alpha_m<u(x,t)<\beta_m,\ |\nabla u|<N\},\ m=1,2,\cdots.$$

For $\lambda=0$, we claim that Eq. (16) has a unique solution $u\in\Omega_m$. In fact, if there exists a p.a.a. $u\not\equiv 0$ such that

$$u_t-\Delta u+cu+ku=0 \text{ in } D\times\mathbb{R},\qquad u=0 \text{ on } \partial D\times\mathbb{R}. \tag{19}$$

Since $u(x,t)$ is p.a.a., there exist $x_0\in D$ and $t_0\in\mathbb{R}$ such that

$$u(x_0,t_0)=\max_{D\times\mathbb{R}}u(x,t).$$

Then

$$u_t(x_0,t_0)=0 \text{ and } \Delta u(x_0,t_0)\leq 0.$$

Thus

$$u_t(x_0,t_0)-\Delta u(x_0,t_0)+cu(x_0,t_0)+ku(x_0,t_0)>0.$$

This is a contradiction. Therefore, Eq. (19) admits a unique p.a.a solution in Ω_m. This yields that For $\lambda=0$, Eq. (16) has a unique solution $u\in\Omega_m$.

By conditions (H4)-(H8) and Theorem 3.2, Eq. (16) has a p.a.a. solution $u_m\in\Omega_m$, $m\in\mathbb{N}$.

In view of the proof in Theorem 3.1, $\{u_m\}_{m=1}^{\infty}$ is equicontinuous in $D\times\mathbb{R}$, and it is uniformly bounded. Applying Arzelá-Ascoli Theorem, there exists a subsequence of $\{u_m(x,t)\}_{m=1}^{\infty}$ and a function $u_*(x,t) \in \cap_{m=1}^{\infty} \Omega_m =: \Omega$ such that $\lim_{m\to\infty} u_m(x,t) = u_*(x,t)$, for all $(x,t) \in \bar{D} \times \mathbb{R}$. Thus Eq. (14) admits a p.a.a solution $u_*(x,t)$ satisfying

$$\alpha(x,t) \leq u_*(x,t) \leq \beta(x,t). \qquad \blacksquare$$

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